LINEAR FORMS IN ZETA VALUES ARISING FROM CERTAIN SOROKIN-TYPE INTEGRALS

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1. Statement of the result

For any fixed integers $n \geq 0$ and $s \geq 3$, let us define the Sorokin-type integral

$$I_n(s) := \int_{[0,1]^s} \frac{\prod_{j=1}^{s} z_j^n (1-z_j)^n}{\prod_{j=2}^{s}(1-z_1z_2\cdots z_j)^{n+1}}\, \mathrm{d}z_1 \cdots \mathrm{d}z_s.$$

Sorokin proved in [21] that

$$I_n(3) = a_n\zeta(3) + b_n \in \mathbb{Q}\zeta(3) + \mathbb{Q}, \tag{1.1}$$

where a_n and b_n are exactly the sequences originally found by Apéry [1] in his proof of the irrationality of $\zeta(3)$. His method consists of solving a suitable Padé approximation problem, a method which is not easily generalisable.

In this note, we will prove the following generalisation of (1.1) by a completely different method. A usual, we set $\mathrm{d}_n := \mathrm{lcm}\{1, 2, \ldots, n\}$.

Theorem 1. *For any fixed integers $n \geq 0$ and $s \geq 3$, there exist $s-1$ sequences of rational $(p_{j,n,s})_{n\geq 0}$, $j = 0, 3, \ldots, s$, such that*

$$I_n(s) = p_{0,n,s} + \sum_{j=3}^{s} p_{j,n,s}\, \zeta(j) \in \mathbb{Q} + \mathbb{Q}\zeta(3) + \mathbb{Q}\zeta(4) + \cdots + \mathbb{Q}\zeta(s).$$

Furthermore, we have $\mathrm{d}_n^{s-j} p_{j,n,s} \in \mathbb{Z}$ for all j.

We remark that the coefficient of $\zeta(2)$ is *a priori* 0: though this is a non-trivial fact, it is not at all clear what new diophantine result could be obtained from this theorem. Nonetheless, it has some interest for the following reasons. It is proved in [5] and [23] that a Sorokin-type integral can be expanded in a rational linear form in multiple zeta values (MZV). Conjecturally this form cannot usually be reduced to a linear form in zeta values. For example, consider the Sorokin-type integral

$$U_n = \int_{[0,1]^5} \frac{\prod_{j=1}^{5} z_j^n (1-z_j)^n}{(1-z_1z_2z_3)^{n+1}(1-z_1z_2z_3z_4z_5)^{n+1}}\, \mathrm{d}z_1 \cdots \mathrm{d}z_5.$$

For $n = 0$, $U_0 = \zeta(3, 2) = -11\zeta(5)/2 + 3\zeta(2)\zeta(3)$ and for $n = 1, 2$ and 3, U_n is a rational linear form in $\zeta(2)$, $\zeta(3)$, $\zeta(4)$, $\zeta(5)$, $\zeta(2, 2)$ and $\zeta(3, 2)$. But according to the Goncharov-Zagier conjecture, $\zeta(3, 2)$ and U_n are unlikely to be rational linear forms in zeta values.

Thus, $I_n(s)$ is in some sense accidental but this is not an isolated accident. Indeed, Vasilyev's generalisation [22] of Beukers' famous integrals [3] can be expressed as certain Sorokin-type integrals and, thanks to a theorem of Zudilin [25], they are also linear in zeta values of a very special form: only odd or even zeta values can occur, according to the parity of the dimension of the integral. These "dichotomic" linear forms coincid with those constructed in [2, 18] for proving that infinitely many odd zeta values are linearly independent. Thus, such integrals are very useful for studying the diophantine nature of MZV and it would be very interesting to find other families of integrals whose evaluations as linear forms in MZV have special properties. The effective implementation [7] under PARI/GP of the algorithm described in [5] could be usefull for this: indeed, Theorem 1 was literally guessed by applying the algorithm to the series $S_n(s)$ in (1.2) below for a few values of s and n. See [6, 9] for other examples of multiple series whose properties where initially detected experimentally.

The proof of Theorem 1 is not straightforward and will be obtained by a certain number of successive reductions. By definition, $(\alpha)_0 := 1$ and $(\alpha)_m := \alpha(\alpha+1)\cdots(\alpha+m-1)$ for $m \geq 1$. From now on and unless otherwise specified, we assume that the integers n and s satisfy $n \geq 0$ and $s \geq 3$.

1) Firstly, using the general decomposition of Sorokin-type integrals in multiple series proved in [5, p.11-12, Proposition 1], we have that

$$I_n(s) = n! \sum_{k_1 \geq k_2 \geq \cdots \geq k_{s-1} \geq 1} \frac{(k_1 - k_2 + 1)_n}{(k_1)_{n+1}^2} \prod_{j=2}^{s-1} \frac{(k_j - k_{j+1} + 1)_n}{(k_j)_{n+1}} =: S_n(s), \tag{1.2}$$

(where $k_s = n+1$ by convention).

2) Less easy is the next fact: we also have that

$$S_n(s) = \int_{[0,1]^s} \frac{\prod_{j=1}^{s} z_j^n (1-z_j)^n}{(1-(1-z_1\cdots z_{s-1})z_s)^{n+1}} \,\mathrm{d}z_1 \cdots \mathrm{d}z_s =: J_n(s). \tag{1.3}$$

Hence, we have $I_n(s) = J_n(s)$.

3) By a change of variable and n-fold integrations by parts, we will then prove that

$$J_n(s) = -\int_{[0,1]^{s-1}} \frac{P_n(z_1)P_n(z_2)\prod_{j=3}^{s-1}(1-z_j)^n}{1 - z_1\cdots z_{s-1}} \log(z_1\cdots z_{s-1}) \,\mathrm{d}z_1\cdots \mathrm{d}z_{s-1} =: B_n(s), \tag{1.4}$$

where $P_n(x) = (x^n(1-x)^n)^{(n)/n!}$ denotes the n-th Legendre polynomial on $[0,1]$ and where for $s = 3$ the value of the empty product is set to 1.

4) From $B_n(s)$, we will obtain the final reduction

$$B_n(s) = -n!^{s-3} \sum_{k=1}^{\infty} \frac{\partial}{\partial k}\left(\frac{(k-n)_n^2}{(k)_{n+1}^{s-1}}\right) =: D_n(s), \tag{1.5}$$

by the process used by the author (in [20]) to prove that (1.5) holds for $s = 3$.

5) Finally, $D_n(s)$ can be expressed as a linear form in zeta values, say $Z_n(s)$, which is of the form announced in Theorem 1.

We thus have obtained the chain of non-trivial equalities

$$I_n(s) = S_n(s) = J_n(s) = B_n(s) = D_n(s) = Z_n(s) \tag{1.6}$$

and $I_n(s) = Z_n(s)$ is exactly the content of Theorem 1. We note that for $s = 3$, the chain (1.6) is well-known : the equalities $I_n(3) = Z_n(3)$, $J_n(3) = B_n(3) = Z_n(3)$ and $D_n(3) = Z_n(3)$ are proved more or less explicitly in [21], [3] and [4, 10, 13] respectively. Proofs of $I_n(3) = J_n(3)$ were obtained in [8] and [24] independently.

2. Nesterenko and Rhin-Viola generalisations and further problems

It would be interesting to obtain a more direct proof that $I_n(s) = Z_n(s)$ and to generalise it to different exponents, i.e., to consider the integrals

$$I_{\underline{\ell},\underline{m},\underline{n}}(s) := \int_{[0,1]^s} \frac{\prod_{j=1}^s z_j^{n_j}(1-z_j)^{m_j}}{\prod_{j=2}^s (1-z_1z_2\cdots z_j)^{\ell_j+1}}\,\mathrm{d}z_1\cdots\mathrm{d}z_s$$

for suitable integers $\ell_j, m_j, n_j \geq 0$. Independently of the present work, Rhin-Viola [17] recently managed by ingenious changes of variables to prove that under certain conditions we have ([1])

$$I_{\underline{\ell},\underline{m},\underline{n}}(s) = J_{\ell',\underline{m}',\underline{n}'}(s) \in \mathbb{Q} + \mathbb{Q}\zeta(2) + \mathbb{Q}\zeta(3) + \mathbb{Q}\zeta(4) + \cdots + \mathbb{Q}\zeta(s).$$

where

$$J_{\ell',\underline{m}',\underline{n}'}(s) := \int_{[0,1]^s} \frac{\prod_{j=1}^s z_j^{n'_j}(1-z_j)^{m'_j}}{(1-(1-z_1\cdots z_{s-1})z_s)^{\ell'+1}}\,\mathrm{d}z_1\cdots\mathrm{d}z_s \tag{2.1}$$

is a suitable generalisation of our $J_n(s)$ and the ℓ', m'_j, n'_j are related to the ℓ_j, m_j, n_j. It is not clear when $\zeta(2)$ can be removed (even for $s = 3$, this is not always possible for general exponents, see [16]): in general, step 3) above can not be done because the "n-fold integrations by parts" trick does not always work for a general integral $J_{\ell',\underline{m}',\underline{n}'}(s)$. Hence, the *ad hoc* methods of Rhin-Viola developped in [15, 16] to avoid this difficulty could be useful to provide natural conditions on the ℓ_j, m_j, n_j under which certain coefficients of the zeta values are 0 in the expansion of $I_{\underline{\ell},\underline{m},\underline{n}}(s)$.

We also mention that a functional version (the variable being x, say) of the integral (2.1) was first studied by Nesterenko [14]. He proved that this integral is equal to a complex Barnes type integral (see Theorem 2, page 547 of [14]) under suitable conditions on the coefficients, from which he deduced a representation as a linear form in polylogarithms in x. Specialising x to 1, he obtained a linear form in $1, \zeta(3), \zeta(4)$, etc. Hence, once we have proved that "our" integrals $I_n(s)$ and $J_n(s)$ are equal, we could apply Nesterenko's theorems to conclude the proof. However, we feel that our approach is of interest, particularly because it is more elementary than Nesterenko's method and because it might shed new light on these problems.

[1]Our proof that $I_n(s) = J_n(s)$ does not use changes of variable. Instead, we follow Zlobin's method [24]. We did not try to see if the general Rhin-Viola's identity $I_{\underline{\ell},\underline{m},\underline{n}}(s) = J_{\ell',\underline{m}',\underline{n}'}(s)$ can also be proved this way but in principle this should not be difficult to check.

Another approach to the evaluation of $I_{\underline{\ell},\underline{m},\underline{n}}(s)$ in terms of zeta values could be the following. We note that the identity $S_n(s) = D_n(s)$ relates a multiple hypergeometric series and a "differenciated" hypergeometric series. It is thus similar to the limiting cases of Andrews identity which were proved in [12] to give a new proof of the theorem of Zudilin mentioned above. These identities relate a multiple hypergeometric series to a a very-well-poised hypergeometric series. Although strictly speaking $D_n(s)$ is not a hypergeometric series, one can also find in [11, Chapitre 16] a trick enabling us to see $D_n(3)$ as a limiting case of a linear combination of two hypergeometric series. Therefore, it seems reasonable to expect the existence of an identity relating $S_{\underline{\ell},\underline{m},\underline{n}}(s)$ (= the multiple series "trivially" equal to $I_{\underline{\ell},\underline{m},\underline{n}}(s)$) and suitable hypergeometric series, of which $D_n(s)$ would be a suitable limiting case.

3. Proof of (1.3): $S_n(s) = J_n(s)$

The proof given below is somewhat long but it is not difficult. It is an adaptation of Zlobin's original method [24], see also [12, p. 215, Proposition 2]. Set $Q_s(z_1, z_2, \ldots, z_s) = 1-(1-z_s z_{s-1}\cdots z_2)z_1$ for $s \geq 2$. One checks immediately that for all $s \geq 3$, we have

$$\begin{aligned} Q_s(z_1, z_2, \ldots, z_s) &= Q_{s-1}(z_1, \ldots, z_{s-1}) - (1-z_s)z_{s-1}\cdots z_1 \\ &= Q_{s-1}(z_1, \ldots, z_{s-1})\left(1-\frac{(1-z_s)z_{s-1}\cdots z_1}{Q_{s-1}(z_1, \ldots, z_{s-1})}\right). \end{aligned}$$

We remark that $0 \leq (1-z_s)z_{s-1}\cdots z_1 \leq Q_{s-1}(z_1, z_2, \ldots, z_{s-1})$ and the second equality holds if and only if $z_1 = 1$ and $z_2 = z_3 = \cdots = z_s$, which is a set A of measure 0 in $[0,1]^s$. On the set $[0,1]^s \setminus A$, we can expand

$$\frac{1}{\left(1-\frac{(1-z_s)z_{s-1}\cdots z_1}{Q_{s-1}(z_1, \ldots, z_{s-1})}\right)^{n+1}} = \sum_{\ell_s=0}^{\infty}\binom{n+\ell_s}{n}\frac{(1-z_s)^{\ell_s} z_{s-1}^{\ell_s}\cdots z_1^{\ell_s}}{Q_{s-1}(z_1, \ldots, z_{s-1})^{\ell_s}}.$$

After multiplication of this series by $\frac{\prod_{j=1}^s z_j^n(1-z_j)^n}{Q_{s-1}(z_1,\ldots,z_{s-1})^{n+1}}$ and integration over $[0,1]^s$, positivity ensures that we can exchange the $\sum$ and $\int$ signs. This yields that, for $s \geq 3$, we have

$$\begin{aligned} &J_n(s) \\ &= \int_{[0,1]^s}\frac{\prod_{j=1}^s z_j^n(1-z_j)^n}{Q_{s-1}(z_1, \ldots, z_{s-1})^{n+1}\left(1-\frac{(1-z_s)z_{s-1}\cdots z_1}{Q_{s-1}(z_1,\ldots,z_{s-1})}\right)^{n+1}}\,\mathrm{d}z_1\cdots\mathrm{d}z_s \\ &= \sum_{\ell_s=0}^{\infty}\binom{n+\ell_s}{n}\int_0^1 z_s^n(1-z_s)^{n+\ell_s}\mathrm{d}z_s\cdot\int_{[0,1]^{s-1}}\frac{\prod_{j=1}^{s-1} z_j^{n+\ell_s}(1-z_j)^n}{Q_{s-1}(z_1, \ldots, z_{s-1})^{n+\ell_s+1}}\,\mathrm{d}z_1\cdots\mathrm{d}z_{s-1} \\ &= \sum_{\ell_s=0}^{\infty}\frac{\binom{n+\ell_s}{n}}{\binom{2n+\ell_s}{n+\ell_s}(2n+\ell_s+1)}\int_{[0,1]^{s-1}}\frac{\prod_{j=1}^{s-1} z_j^{n+\ell_s}(1-z_j)^n}{Q_{s-1}(z_1, \ldots, z_{s-1})^{n+\ell_s+1}}\mathrm{d}z_1\cdots\mathrm{d}z_{s-1}. \end{aligned}$$

Clearly, we can apply this process to the last integral and by iteration we obtain that

$$J_n(s) = \sum_{\ell_3,\ldots,\ell_s \geq 0} \frac{\binom{n+\ell_s}{\ell_s}\binom{n+\ell_s+\ell_{s-1}}{\ell_{s-1}}\cdots\binom{n+\ell_s+\ell_{s-1}+\cdots+\ell_3}{\ell_3}}{\binom{2n+\ell_s}{n+\ell_s}\binom{2n+\ell_s+\ell_{s-1}}{n+\ell_{s-1}}\cdots\binom{2n+\ell_s+\ell_{s-1}+\cdots+\ell_3}{n+\ell_3}}$$
$$\cdot \frac{1}{(2n+\ell_s+1)(2n+\ell_s+\ell_{s-1}+1)\cdots(2n+\ell_s+\ell_{s-1}+\cdot+\ell_3+1)}$$
$$\cdot \int_{[0,1]^2} \frac{z_1^{n+\ell_s+\cdots+\ell_3}(1-z_1)^n z_2^{n+\ell_s+\cdots+\ell_3}(1-z_2)^n}{Q_2(z_1,z_2)^{n+\ell_s\cdots+\ell_3+1}}\,\mathrm{d}z_1\mathrm{d}z_2.$$

It remains to evaluate the double integral. To simplify, let $m = \ell_s + \cdots + \ell_3$. Since $Q_2(z_1, z_2) = 1 - (1 - z_1)z_2$, by changing z_1 in $1 - z_1$, we see that

$$\begin{aligned}
&\int_{[0,1]^2} \frac{z_1^{n+m}(1-z_1)^n z_2^{n+m}(1-z_2)^n}{Q_2(z_1,z_2)^{n+m+1}}\,\mathrm{d}z_1\mathrm{d}z_2 \\
&\quad = \int_{[0,1]^2} \frac{z_1^{n}(1-z_1)^{n+m} z_2^{n+m}(1-z_2)^n}{(1-z_1z_2)^{n+m+1}}\,\mathrm{d}z_1\mathrm{d}z_2 \\
&\quad = \sum_{\ell_2=0}^{\infty} \binom{n+m+\ell_2}{\ell_2} \int_0^1 z_1^{\ell_2+n}(1-z_1)^{n+m}\mathrm{d}z_1 \int_0^1 z_2^{n+m+\ell_2}(1-z_1)^n\mathrm{d}z_2 \\
&\quad = \sum_{\ell_2=0}^{\infty} \frac{\binom{n+m+\ell_2}{\ell_2}}{\binom{2n+m+\ell_2}{n+\ell_2}\binom{2n+m+\ell_2}{n}} \cdot \frac{1}{(2n+m+\ell_2+1)^2}.
\end{aligned}$$

Therefore,

$$J_n(s) = \sum_{\ell_2,\ldots,\ell_s \geq 0} \frac{\binom{n+\ell_s}{\ell_s}\binom{n+\ell_s+\ell_{s-1}}{\ell_{s-1}}\cdots\binom{n+\ell_s+\ell_{s-1}+\cdots+\ell_2}{\ell_2}}{\binom{2n+\ell_s}{n+\ell_s}\binom{2n+\ell_s+\ell_{s-1}}{n+\ell_{s-1}}\cdots\binom{2n+\ell_s+\ell_{s-1}+\cdots+\ell_2}{n+\ell_2}}$$
$$\cdot \frac{1}{(2n+\ell_s+1)(2n+\ell_s+\ell_{s-1}+1)\cdots(2n+\ell_s+\ell_{s-1}+\cdots+\ell_2+1)}$$
$$\cdot \frac{1}{\binom{n+\ell_s+\ell_{s-1}+\cdots+\ell_2}{n}(2n+\ell_s+\ell_{s-1}+\cdots+\ell_2+1)},$$

which is a perfectly convergent series.

We now make a change of indices $K_j = \ell_{j+1} + \ell_{j+2} + \cdots + \ell_s$ for $j = 1, \ldots, s-1$ and obtain

$$\begin{aligned}
J_n(s) &= \sum_{1\le K_{s-1}\le \cdots \le K_1} \frac{\binom{n+K_{s-1}}{K_{s-1}}\binom{n+K_{s-2}}{K_{s-2}-K_{s-1}}\cdots\binom{n+K_1}{K_1-K_2}}{\binom{2n+K_{s-1}}{n+K_{s-1}}\binom{2n+K_{s-2}}{n+K_{s-2}-K_{s-1}}\cdots\binom{2n+K_1}{n+K_1-K_2}} \\
&\quad\cdot \frac{1}{(2n+K_{s-1}+1)(2n+K_{s-2}+1)\cdots(2n+K_1+1)} \cdot \frac{1}{\binom{n+K_1}{n}(2n+K_1+1)} \\
&= n! \sum_{1\le K_{s-1}\le \cdots \le K_1} \frac{(K_{s-1}+1)_n (K_{s-2}-K_{s-1}+1)_n \cdots (K_1-K_2+1)_n}{(K_e+n+1)_{n+1}\cdots(K_2+n+1)_{n+1}(K_1+n+1)_{n+1}^2} \\
&= n! \sum_{1\le k_{s-1}\le \cdots \le k_1} \frac{(k_{s-1}-n)_n (k_{s-2}-k_{s-1}+1)_n \cdots (k_1-k_2+1)_n}{(k_{s-1})_{n+1}\cdots(k_2)_{n+1}(k_1)_{n+1}^2} \\
&=: S_n(s).
\end{aligned}$$

In the last equality, we set $k_j = K_j + n + 1$ for $j = 1, \ldots, s-1$.

4. Proof of (1.4): $J_n(s) = B_n(s)$

We follow closely Beukers' method (case $s = 3$, see [3]). We have

$$\frac{\log(z_1\cdots z_{s-1})}{1 - z_1\cdots z_{s-1}} = -\int_0^1 \frac{\mathrm{d}w}{(1 - z_1\cdots z_{s-1})w}$$

and therefore

$$\begin{aligned}
B_n(s) &:= -\int_{[0,1]^s} \frac{P_n(z_1)P_n(z_2)\prod_{j=3}^{s-1}(1-z_j)^n}{1 - z_1\cdots z_{s-1}} \log(z_1\cdots z_{s-1})\,\mathrm{d}z_1\cdots\mathrm{d}z_{s-1} \\
&= \int_{[0,1]^s} \frac{P_n(z_1)P_n(z_2)\prod_{j=3}^{s-1}(1-z_j)^n}{1-(1 - z_1\cdots z_{s-1})w}\,\mathrm{d}z_1\cdots\mathrm{d}z_{s-1}\mathrm{d}w.
\end{aligned}$$

We now integrate n times by parts in the last integral with respect to z_1: we obtain that

$$B_n(s) = (-1)^n \int_{[0,1]^s} \frac{z_1^n(1-z_1)^n z_2^n P_n(z_2)\prod_{j=3}^{s-1} z_j^n(1-z_j)^n}{(1-(1 - z_1\cdots z_{s-1})w)^{n+1}}\,\mathrm{d}z_1\cdots\mathrm{d}z_{s-1}\mathrm{d}w.$$

The change of variable $z_s = (1-w)/(1-(1-z_1\cdots z_{s-1})w)$ yields that

$$B_n(s) = (-1)^n \int_{[0,1]^s} \frac{(1-z_1)^n P_n(z_2)\prod_{j=3}^{s-1}(1-z_j)^n}{1-(1 - z_1\cdots z_{s-1})z_s}\,\mathrm{d}z_1\cdots\mathrm{d}z_{s-1}\mathrm{d}z_s.$$

Finally, integrating n times by parts with respect to z_2, we get

$$B_n(s) = \int_{[0,1]^s} \frac{\prod_{j=1}^{s} z_j^n(1-z_j)^n}{(1-(1 - z_1\cdots z_{s-1})z_s)^{n+1}}\,\mathrm{d}z_1\cdots\mathrm{d}z_s =: J_n(s)$$

as claimed.

5. Proof of (1.5): $B_n(s) = D_n(s)$

As in the previous section, we start with an alternative expression for $\log(z_1 \cdots z_{s-1})/(1-z_1 \cdots z_{s-1})$. However, to avoid technicalities, we introduce a complex parameter x such that $|x| < 1$ and we will let $x \to 1$ in the end. The alternative expression we will use is:

$$\frac{\log(z_1 \cdots z_{s-1})}{1 - xz_1 \cdots z_{s-1}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{(z_1 \cdots z_{s-1})^t}{1 - xz_1 \cdots z_{s-1}}\right)_{|t=0}.$$

Furthermore, provided all $z_j \in [0,1]$, we can expand

$$\frac{1}{1 - xz_1 \cdots z_{s-1}} = \sum_{k=1}^{\infty} (xz_1 \cdots z_{s-1})^{k-1}$$

and we obtain that

$$\frac{\log(z_1 \cdots z_{s-1})}{1 - xz_1 \cdots z_{s-1}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{k=1}^{\infty} x^{k-1}(z_1 \cdots z_{s-1})^{k+t-1}\right)_{|t=0} \tag{5.1}$$

We now define the integral

$$B_n(s,x) := -\int_{[0,1]^s} \frac{P_n(z_1)P_n(z_2)\prod_{j=3}^{s-1}(1-z_j)^n}{1 - xz_1 \cdots z_{s-1}} \log(z_1 \cdots z_{s-1})\,\mathrm{d}z_1 \cdots \mathrm{d}z_{s-1},$$

which is such that $\lim_{x\to 1} B_n(s,x) = B_n(s,1) = B_n(s)$. From (5.1), we deduce that

$$B_n(s,x) =$$
$$-\sum_{k=1}^{\infty} x^{k-1}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 z_1^{k+t-1}P_n(z_1)\,\mathrm{d}z_1 \cdot \int_0^1 z_2^{k+t-1}P_n(z_2)\,\mathrm{d}z_2 \cdot \prod_{j=3}^{s-1}\int_0^1 z_j^{k+t-1}(1-z_j)^n\,\mathrm{d}z_j\right)_{|t=0}.$$

The various exchanges of integrals, summations and derivations are all justified because $|x| < 1$ and the integrals are bounded independently of k.

We now compute the two types of integrals occuring. Firstly, we have

$$\int_0^1 z^{k+t-1}(1-z)^n\,\mathrm{d}z = \frac{n!}{(k+t)(k+t+1)\cdots(k+t+n)}. \tag{5.2}$$

Secondly, integrating n times by parts and then using (5.2), we have

$$\begin{aligned}\int_0^1 z^{k+t-1}P_n(z)\,\mathrm{d}z &= \frac{n!}{(k+t-1)\cdots(k+t-n)}\int_0^1 z^{k+t-1}(1-z)^n\,\mathrm{d}z \\ &= \frac{(k+t-1)\cdots(k+t-n)}{(k+t)(k+t+1)\cdots(k+t+n)}.\end{aligned}$$

Therefore, we have

$$\begin{aligned} B_n(s,x) &= -\sum_{k=1}^{\infty} x^{k-1} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{(k+t-1)^2\cdots(k+t-n)^2}{(k+t)^2\cdots(k+t+n)^2} \cdot \frac{n!^{s-3}}{(k+t)^{s-3}\cdots(k+t+n)^{s-3}} \right)_{|t=0} \\ &= -n^{s-3}\sum_{k=1}^{\infty} x^{k-1} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{(k+t-1)^2\cdots(k+t-n)^2}{(k+t)^{s-1}\cdots(k+t+n)^{s-1}} \right)_{|t=0}. \end{aligned}$$

Using Abel's theorem, we can now let $x \to 1$ and we finally obtain that $B_n(s) = D_n(s)$.

6. Proof of Theorem 1

So far, we have proved that $I_n(s) = D_n(s)$. It is now fairly easy to complete the proof of Theorem 1 by standard arguments. We will prove a little more than needed and, for this, we define the polylogarithmic functions by $\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} z^n/n^s$ for $|z| \le 1$, $s \ge 1$ and $(z,s) \neq (z,1)$.

Set

$$R(k) := n!^{s-3}\frac{(k-1)^2\cdots(k-n)^2}{k^{s-1}(k+1)^{s-1}\cdots(k+n)^{s-1}},$$

which is a rational function of k. The partial fraction expansion of $R(k)$ reads

$$R(k) = \sum_{j=0}^{n}\sum_{t=1}^{s-1}\frac{C(j,t)}{(k+j)^t},$$

where the coefficients $C(j,t) \in \mathbb{Q}$ depend on s,n and could in principle made explicit. Therefore, we have

$$\frac{\partial}{\partial k}R(k) = -\sum_{j=0}^{n}\sum_{t=2}^{s}\frac{(t-1)C(j,t-1)}{(k+j)^t}.$$

Consider now the series $V(z) = -\sum_{k=1}^{\infty} R^{(1)}(k)z^{-k}$, which converges absolutely for $|z| \ge 1$; in particular for $z=1$, we have $V(1) = D_n(s)$. For $|z| \ge 1$, we have

$$\begin{aligned} V(z) &= \sum_{j=0}^{n}\sum_{t=2}^{s}(t-1)C(j,t-1)\sum_{k=1}^{\infty}\frac{z^{-k}}{(k+j)^t} \\ &= \sum_{j=0}^{n}\sum_{t=2}^{s}(t-1)C(j,t-1)\left(z^j \mathrm{Li}_t(1/z) - \sum_{k=1}^{j}\frac{z^{j-k}}{k^t} \right) \\ &= P_0(z) + \sum_{t=2}^{s} P_t(z)\mathrm{Li}_s(1/z) \end{aligned}$$

where, for $t \geq 2$, $P_t(z) = \sum_{j=0}^{n} (t-1)C(j,t-1)z^j \in \mathbb{Q}[z]$ and

$$P_0(z) = -\sum_{j=0}^{n}\sum_{t=2}^{s}(t-1)C(j,t-1)\sum_{k=1}^{j}\frac{z^{j-k}}{k^t} \in \mathbb{Q}[z].$$

We now remark $P_2(1) = \sum_{j=0}^{n} C(j,1) = 0$ because the total degree of $R(k)$ is ≤ -2. Therefore, we have

$$D_n(s) = V(1) = P_0(1) + \sum_{t=3}^{s} P_t(1)\mathrm{Li}_t(1) = P_0(1) + \sum_{t=3}^{s} P_t(1)\zeta(t).$$

We don't prove the last assertion of the theorem, i.e. that $\mathrm{d}_n^{s-t}P_t(1) \in \mathbb{Z}$: see [2, 18, 19, 26, 25] for similar proofs.

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