SOME REMARKS ON CLASSICAL MODULAR FORMS AND HYPERGEOMETRIC SERIES

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1. Diophantine results for modular forms

Let $z, q \in \mathbb{C}$ be such that $\text{Im}(z) > 0$ and $|q| < 1$.
We consider Ramanujan’s $q$-series

$$P(q) = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n}, \quad Q(q) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3q^n}{1 - q^n}, \quad R(q) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5q^n}{1 - q^n},$$

and the function

$$J(q) = 1728 \frac{Q(q)^3}{Q(q)^3 - R(q)^2}.$$ 

We also define the Fourier series $E_2(z) = P(e^{2i\pi z})$, $E_4(z) = Q(e^{2i\pi z})$, $E_6(z) = R(e^{2i\pi z})$ and the modular invariant $j(z) = J(e^{2i\pi z})$, where $\text{Im}(z) > 0$: the functions $E_4, E_6$ and $j$ are modular forms with respect to $\text{SL}_2(\mathbb{Z})$, while $E_2$ is only quasi-modular with respect to $\text{SL}_2(\mathbb{Z})$. Sometimes $E_2(q)$, $E_4(q)$ and $E_6(q)$ are just alternative notations for $P(q), Q(q)$ and $R(q)$ (see [4, p. 159]); the choice of the variable $z$ vs. $q$ makes this clear in the sequel. These functions satisfy the non-linear differential system (as functions of $q$)

$$q \frac{dE_2}{dq} = \frac{E_2^2 - E_4}{12}, \quad q \frac{dE_4}{dq} = \frac{E_2 E_4 - E_6}{3}, \quad q \frac{dE_6}{dq} = \frac{E_2 E_6 - E_4^2}{2}. \quad (1.1)$$

The Diophantine study of the values taken by modular forms and functions has a long history, see [4]. We quote the following results.

**Theorem 1** (Bertrand [3]). For any $q$ such that $0 < |q| < 1$, the numbers $E_4(q)$ and $E_6(q)$ can not be simultaneously algebraic. Equivalently, for any $q$ such that $0 < |q| < 1$ et $J(q) \notin \{0, 1728\}$, the numbers $J(q)$ and $qJ'(q)$ can not be simultaneously algebraic.

This is in fact a consequence of a more general result of Schneider, expressed in a different language. When $q$ is algebraic, Bertrand’s theorem is also a consequence of the more recent

**Theorem 2** (Théorème stéphanois [2]). For any $q$ such that $0 < |q| < 1$, the numbers $q$ and $J(q)$ can not be simultaneously algebraic.

Both theorems are now consequences of

**Theorem 3** (Nesterenko [6]). For any $q$ such that $0 < |q| < 1$, at least three of the numbers $q, E_2(q), E_4(q)$ and $E_6(q)$ are algebraically independent over $\mathbb{Q}$.

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André obtained a particular case of Bertrand’s theorem in [1], though the result is stated and proven for certain hypergeometric functions. His method of simultaneous adelic uniformization is very general in principle but it has been applied so far essentially only to those he dealt with. In particular, his hypergeometric functions have a special connection to the modular world, which is not universal amongst hypergeometric functions.

2. HYPERGEOMETRIC FUNCTIONS AND MODULAR FORMS

The generalized hypergeometric function is defined as

\[ p+1F_p \left[ \begin{array}{c} a_0, a_1, \cdots, a_p \\ b_1, \cdots, b_p \end{array} ; x \right] = \sum_{n=0}^{\infty} \frac{(a_0)_n(a_1)\cdots(a_p)_n}{(1)_n(b_1)\cdots(b_p)_n} x^n, \]

where \(|x| < 1\), \((r)_0 = 1\) and \((r)_m = r(r + 1)\cdots(r + m - 1)\) for \(m \geq 1\), and the parameters \(a_j, b_j\) are suitable complex numbers.

In the sequel, we will consider the function

\[ F(t) = 2F_1 \left[ \begin{array}{c} -1/12, 5/12 \\ 1/1, 5/6 \\ 1/6, 1/1 \end{array} ; t \right]. \quad (2.1) \]

The second equality is a particular case of Clausen’s identity [7, p. 75, eq. (2.5.7)].

There exists a well-known connection between modular forms and hypergeometric series, going back to Klein at least. More recently, Stiller obtained a very elegant formulation.

Let us define \( t(q) = \frac{1728}{J(q)} \in \mathbb{Q}[[q]] \). It is an holomorphic function at \( q = 0 \), and \(|t(q)| < 1\) iff \(|J(q)| > 1728\) (i.e., if \(|q|\) is sufficiently close to 0).

**Theorem 4** (Stiller [8]). For any \( q \in \mathbb{C} \) such that \(|q| < 1\) and \(|J(q)| > 1728\), we have

\[ E_4(q) = F(t(q))^2 \quad \text{and} \quad E_6(q) = (1 - t(q))^{1/2} F(t(q))^3, \quad (2.2) \]

where the principal branch of the logarithm is chosen to define the square root.

It is natural question to wonder if \( E_2(q) \) can also be expressed with similar hypergeometric functions. We prove the following result.

**Theorem 5.** For any \( q \in \mathbb{C} \) such that \(|q| < 1\) and \(|J(q)| > 1728\), we have

\[ E_2(q) = (1 - t(q))^{1/2} \left( F(t(q)) + 6t(q)F'(t(q)) \right) \quad (2.3) \]

\[ = 2F_1 \left[ \begin{array}{c} 1/12, 5/12 \\ 1/1, 5/6 \\ 1/6, 1/1 \end{array} ; t(q) \right] \cdot 2F_1 \left[ \begin{array}{c} -1/12, 7/12 \\ 1/1, 5/6 \\ 1/6, 1/1 \end{array} ; t(q) \right]. \quad (2.4) \]

**Proof.** The hypergeometric function \( F(t) \) is solution of the differential equation

\[ \left[ \theta^3 - t(\theta + \frac{1}{2})(\theta + \frac{5}{6}) \right] y(t) = 0, \quad \theta = t \frac{d}{dt}. \quad (2.5) \]

Moreover,

\[ J'(q) = \frac{E_6(q)}{qE_4(q)} J(q) \]
from which we deduce that

$$t'(q) = \frac{E_6(q)}{qE_4(q)}t(q).$$  \hspace{1cm} (2.6)

Let $D(q)$ denote the function on the right-hand side of (2.3), which is holomorphic at $q = 0$. Using (2.2), (2.5) and (2.6), some tedious computations show that $D(q)$, $E_4(q)$ and $E_6(q)$ satisfy

$$q\frac{dD}{dq} = \frac{D^2 - E_4}{12}, \quad q\frac{dE_4}{dq} = \frac{DE_4 - E_6}{3}, \quad q\frac{dE_6}{dq} = \frac{DE_6 - E_4^2}{2}. \hspace{1cm} (2.7)$$

This system is formally the same as (1.1) with $E_2(q)$ replaced by $D(q)$, and we shall now prove that $D(q) = E_2(q)$.

For this, let us consider the differential equation (extracted from (2.7)):

$$12q\frac{dY}{dq} = Y^2 - E_4 \hspace{1cm} (2.8)$$

where $Y(q) = \sum_{n=0}^{\infty} y_n q^n \in \mathbb{C}[[q]]$ is an unknown function. For simplicity, we set $E_4(q) = \sum_{n=0}^{\infty} e_n q^n$. Eq. (2.8) implies that

$$12ny_n = -e_n + \sum_{j=0}^{n} y_j y_{n-j}, \quad n \geq 0.$$ 

For $n = 0$, we deduce that $y_0^2 = 1$. Moreover, for $n \geq 1$, we have

$$(12n - 2y_0)y_n = -e_n + \sum_{j=1}^{n-1} y_j y_{n-j},$$

which determines each $y_n$ uniquely once the value of $y_0$ is fixed. Hence, (2.8) has exactly two solutions $Y(q) \in \mathbb{C}[[q]]$, one for $y_0 = 1$ and the other one for $y_0 = -1$. Now, $D(q)$ and $E_2(q)$ are both solutions of (2.8) and $D(0) = E_2(0) = 1$. Hence $D(q) = E_2(q)$ as expected. Eq. (2.4) follows from (2.1) and (2.3) by Euler’s hypergeometric identity [7, p. 10, eq. (1.3.15)].

\[ \square \]

3. Further remarks

Let us consider the three series in $\mathbb{Q}[[t]]$, of hypergeometric type, defined by

$$A(t) := (1 - t)^{1/2}(F(t) + 6tF'(t)), \quad B(t) := F(t)^2, \quad C(t) := (1 - t)^{1/2}F(t)^3.$$ 

Using Ramanujan’s system for $E_2(q)$, $E_4(q)$, $E_6(q)$ and Eq. (2.6), we get
Proposition 1. We have
\[
\begin{align*}
\frac{t}{dt}A(t) &= \frac{B(t)}{12C(t)}(A(t)^2 - B(t)) \\
\frac{t}{dt}B(t) &= \frac{B(t)}{3C(t)}(A(t)B(t) - C(t)) \\
\frac{t}{dt}C(t) &= \frac{B(t)}{2C(t)}(A(t)C(t) - B(t)^2).
\end{align*}
\]

Up to the multiplicative factor \(B(t)/C(t)\), this is the same differential system as the one satisfied by \(E_2(q), E_4(q), E_6(q)\). Hence, the ring \(\mathbb{C}[A(t), B(t), C(t)]\) is left stable by the differential operator \(\frac{tC(t)\,d}{B(t)\,dt}\).

As we now explain, the functions \(A, B, C\) are “universal” in some sense. Let \(T(q)\) be an hauptmodul holomorphic at 0 and vanishing at \(q = 0\). We can use \(T(q)\) instead of \(t(q) = \frac{1728}{J(q)}\). Since \(J(q)\) generates over \(\mathbb{C}\) the field of modular functions (over \(SL_2(\mathbb{Z})\)), there exists \(\varphi(X) \in \mathbb{C}(X)\) such that \(T(q) = \varphi(t(q))\).

Given such a \(T\), consider \(a(t), b(t), c(t)\) the functions (holomorphic at \(t = 0\)) such that
\[
E_2(q) = a(T(q)), \quad E_4(q) = b(T(q)), \quad E_6(q) = c(T(q)).
\]
We thus have
\[
A(t(q)) = a(T(q)), \quad B(t(q)) = b(T(q)), \quad C(t(q)) = c(T(q))
\]
in a neighborhood of \(q = 0\), so that
\[
A(t) = a(\varphi(t)), \quad B(t) = B(\varphi(t)), \quad c(t) = C(\varphi(t)),
\]
in a neighborhood of \(t = 0\).

Thus, we see that \(a, b, c\) are not really different from \(A, B, C\) as they can expressed in terms of the hypergeometric function \(F\) and the reciprocal of \(\varphi\). We remark that the functions in the sets \(\{a(t), b(t), c(t)\}\), \(\{a(t), a'(t), a''(t)\}\), \(\{b(t), b'(t), b''(t)\}\) et \(\{c(t), c'(t), c''(t)\}\) are algebraically dependent over \(\mathbb{C}(t)\). It is thus not clear if one can get Nesterenko’s theorem with such functions by means of [1, p. 119, Theorem 4], where algebraic independence is an hypothesis.

André did not use Stillier’s hypergeometric \(F(t)\), but instead \(2F_1\left[\frac{1}{2}, \frac{1}{2}; 1; t\right]\), which satisfies
\[
2F_1\left[\frac{1}{2}, \frac{1}{2}; 1; \left(\frac{\theta_2(q)}{\theta_3(q)}\right)^4\right] = \theta_3(q)^8
\]
where
\[
\theta_2(q) = 2 \sum_{n=1}^{\infty} q^{(n-1/2)^2}, \quad \theta_3(q) = 1 + 2 \sum_{n=1}^{\infty} q^n.
\]
Note that \(\left(\frac{\theta_2(q)}{\theta_3(q)}\right)^4\) is a modular function and that
\[
E_4(q^2) = \theta_2(q)^8 + \theta_3(q)^8 - \theta_2(q)^4\theta_3(q)^4.
\]
Bibliography