A new proof of Galochkin’s characterization of hypergeometric $G$-functions

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Abstract

$G$-functions are power series in $\mathbb{Q}[[z]]$ solutions of linear differential equations, and whose Taylor coefficients satisfy certain (non)-archimedean growth conditions. In 1929, Siegel proved that every generalized hypergeometric series $q_{+1}F_q$ with rational parameters are $G$-functions, but rationality of parameters is in fact not necessary for an hypergeometric series to be a $G$-function. In 1981, Galochkin found necessary and sufficient conditions on the parameters of a $q_{+1}F_q$ series to be a non polynomial $G$-function. His proof used specific tools in algebraic number theory to estimate the growth of the denominators of the Taylor coefficients of hypergeometric series with algebraic parameters. In this paper, we give a different proof using methods from the theory of arithmetic differential equations, in particular the André-Chudnovskii-Katz Theorem on the structure of the non-zero minimal differential equation satisfied by any given $G$-function, which is Fuchsian with rational exponents.

1 Introduction

Siegel [8] defined a $G$-function as any power series $F(z) = \sum_{n=0}^{\infty} A_n z^n \in \mathbb{Q}[[z]]$ such that

(i) $F(z)$ is solution of a linear differential equation with coefficients in $\mathbb{Q}(z)$;

(ii) there exists $C > 0$ such that for all $n \geq 0$, $|A_n| \leq C^{n+1}$.

(iii) there exists $D > 0$ such that for all $n \geq 0$, $\text{den}(A_0, A_1, \ldots, A_n) \leq D^{n+1}$.

Here, $\text{den}$ denotes the maximum modulus of the Galoisian conjugates of a non-zero algebraic $x$, and given $m$ algebraic numbers $x_1, \ldots, x_m$, $\text{den}(x_1, \ldots, x_m)$ is the smallest integer $\geq 1$ such that $\text{den}(x_1, \ldots, x_m) x_j$ is an algebraic integer for all $j \in \{1, \ldots, m\}$. Siegel also defined an $E$-function as a power series $\sum_{n=0}^{\infty} \frac{A_n}{n!} z^n \in \mathbb{Q}[[z]]$ such that $\sum_{n=0}^{\infty} A_n z^n$ is a $G$-function. In fact, for $E$-functions, he considered weaker assumptions, with $C^{n+1}$ and $D^{n+1}$ replaced by $n!^\varepsilon$ for any fixed $\varepsilon > 0$ provided $n$ is large enough with respect to $\varepsilon$. It is believed that these two possible classes of $E$-functions are identical; see [2, p. 715].
G-functions form a subring of $\mathbb{C}[[z]]$, stable by $\frac{d}{dz}$ and $\int_0^z$, and in fact a differential $\overline{\mathbb{Q}}$-algebra. The first interesting examples of G-functions are algebraic functions over $\overline{\mathbb{Q}}(z)$, holomorphic at $z = 0$. Other important G-functions, like polylogarithms $\sum_{n=1}^{\infty} \frac{z^n}{n^s}$, $s \in \mathbb{Z}$, are obtained as specializations of the generalized hypergeometric series

$$\,{}_{p}F_{q}\left(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z\right) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(1)_n (\beta_1)_n \cdots (\beta_q)_n} z^n$$  \hspace{1cm} (1.1)

where $p, q \geq 0$, $\alpha_1, \ldots, \alpha_p \in \mathbb{C}$ and $\beta_1, \ldots, \beta_q \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. We also assume without loss of generality that $\alpha_j \neq \beta_k$ for all $j, k$, because if $\alpha_j = \beta_k$ we can simply replace $(\alpha_j)_n/(\beta_k)_n$ by 1; in turn, this assumption is important in Theorem 1 below, that would be false without it. Under these conditions, Siegel proved that if $p = q + 1$ and if the $\alpha$'s and $\beta$'s are all rational numbers, then the hypergeometric series (1.1) is a G-function. Still when $p = q + 1$, the converse is not true as the following example shows: for every $\alpha \in \overline{\mathbb{Q}} \setminus \mathbb{Z}_{\leq 0}$,

$$\,{}_{2}F_{1}\left(\alpha, 1; \alpha + 1; z\right) = \sum_{n=0}^{\infty} \frac{(\alpha + 1)_n (1)_n}{(1)_n (\alpha)_n} z^n = \sum_{n=0}^{\infty} \frac{\alpha + n}{\alpha} z^n = \frac{\alpha(1-z) + z}{\alpha(1-z)^2} \hspace{1cm} (1.2)$$

is a G-function. The following characterization of non polynomial hypergeometric G-functions was obtained by Galochkin in [6, p. 8], and the goal of this paper is to give a new proof of his result \footnote{Galochkin also assumed that $\alpha_1 = 1$ (in our notations) but this is not less general because he can recover the generalized hypergeometric series by taking $b_1 = 0$ (in his notations). In fact, he first characterized non polynomial hypergeometric E-functions in the case $p = q$, for which he obtained a result formally similar to Theorem 1, \textit{mutatis mutandis}. His method is strong enough to apply to hypergeometric E-functions with Siegel’s original definition.} See §4 for some remarks in the polynomial case.

**Theorem 1** (Galochkin). Let $p = q + 1$, $q \geq 0$, $\alpha := (\alpha_1, \ldots, \alpha_{q+1}) \in (\mathbb{C} \setminus \mathbb{Z}_{\leq 0})^{q+1}$ and $\beta := (\beta_1, \ldots, \beta_q) \in (\mathbb{C} \setminus \mathbb{Z}_{\leq 0})^q$ be such that $\alpha_i \neq \beta_j$ for all $i, j$.

Then, the hypergeometric series (1.1) with parameters $\alpha$ and $\beta$ is a G-function if and only if the following two conditions hold:

(i) The $\alpha$'s and $\beta$'s are all in $\overline{\mathbb{Q}}$;

(ii) The $\alpha$'s and $\beta$'s which are not rational (if any) can be grouped in $k \leq q$ pairs $(\alpha_{j_1}, \beta_{j_1}), \ldots, (\alpha_{j_k}, \beta_{j_k})$ such that $\alpha_{j_i} - \beta_{j_i} \in \mathbb{N}$.

It follows that if $\alpha \notin \mathbb{Z}$, then

$$
\,{}_{2}F_{1}\left(\frac{\alpha}{\alpha + 1}; z\right) = \sum_{n=0}^{\infty} \frac{\alpha}{n + \alpha} z^n
$$

is not a G-function, to be compared with (1.2).

Galochkin’s proof of Theorem 1 is \textit{ad hoc} and not easy. He used some results from [5] and in particular the “prime number theorem” for prime ideals in number fields to obtain
precise estimates on the factors of the denominators of Taylor coefficients of hypergeometric series with algebraic parameters. We shall give a new proof of Theorem 1 using the theory of arithmetic differential equations. Our approach is quite different, hence it might be an alternative in other situations where his method would be difficult to implement. But it is not fundamentally simpler as it uses a result of Katz on the rationality of exponents of globally nilpotent differential operators in $\mathbb{Q}(z)\left[\frac{d}{dz}\right]$. The latter result uses a consequence of Chebotarev’s density theorem.

The paper is organized as follows: the proof of Theorem 1 is given in §3, after some preliminary results are stated in §2. We conclude in §4 with some considerations on $G$-functions of a different hypergeometric nature than in Theorem 1.

## 2 Preliminary results

Our method is based on the following theorem, due to the (independent) works of André, Chudnovskii and Katz. See [2, pp. 717-720] for stronger statements, and [1] or [3] for proofs.

**Theorem 2.** Let $F(z)$ be a $G$-function and $L \in \mathbb{Q}(z)[\frac{d}{dz}] \setminus \{0\}$ be such that the differential equation $LF(z) = 0$ is of minimal order for $F(z)$. Then $L$ is Fuchsian with rational exponents.

We shall also use two lemmas, the proofs of which are included for the reader’s convenience.

**Lemma 1.** Let $L, M, N \in \mathbb{C}(z)[\frac{d}{dz}]$ be such that $L = MN$. We assume that $L$ is Fuchsian. Then,

(i) $M$ and $N$ are Fuchsian.

(ii) Let us fix $\xi \in \mathbb{C} \cup \{\infty\}$. Then the indicial polynomial of $N$ at $\xi$ divides the indicial polynomial of $L$ at $\xi$.

(iii) Let us assume that the differential equation $Ly(z) = 0$ has a power series solution $y(z) = \sum_{n=0}^{\infty} U_n z^n$. Then there exist an integer $\ell \geq 0$ and some polynomials $Q_j(X) \in \mathbb{C}[X]$, $j = 0, \ldots, \ell$, such that for any $n \geq \ell$,

$$\sum_{j=0}^{\ell} Q_j(n) U_{n-j} = 0,$$

where $Q_0(X) \neq 0$ is the indicial polynomial of $L$ at $0$, and $Q_\ell(\ell-X) \neq 0$ is the indicial polynomial of $L$ at $\infty$. 

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Proof. (i) A differential operator in \(\mathbb{C}(z)[\frac{d}{dz}]\) is Fuchsian at a given point \(\xi \in \mathbb{C} \cup \{\infty\}\) if and only if 0 is its only slope at \(\xi\). Moreover, the set of slopes of \(L\) at \(\xi\) is the union of the set of slopes of \(M\) and that of \(N\) at \(\xi\) (see [7, p. 92, Lemma 3.45]). The statement follows.

(ii) It is enough to prove the result for \(\xi = 0\), because for \(\xi \neq 0\), we can return to the case \(\xi = 0\) by changing \(z\) to \(z - \xi\) (if \(\xi\) is finite) and to \(1/z\) (if \(\xi = \infty\)) in \(L, M, N\).

We first recall some general facts pertaining to any Fuchsian differential operator \(L \in \mathbb{C}(z)[\frac{d}{dz}]\), of order \(\mu\). Let \(A(X) = X^a A_0(X)\) be any polynomial in \(\mathbb{C}[X] \setminus \{0\}\) such that \(A_0(0) \neq 0\), \(a \in \mathbb{N}\) and \(AL = \sum_{j=0}^{\mu} P_j(z)(\frac{d}{dz})^j \in \mathbb{C}[z][\frac{d}{dz}]\). Let \(\alpha \neq 0\) be the leading coefficient of \(A(X)\). We let \(\delta = \max_j(\deg(P_j))\), \(\omega = \ord_0(P_\mu)\) and \(\ell = \delta - \omega\). Then,

\[
AL = \sum_{j=0}^{\ell} z^{j+\omega-\mu} Q_j(\theta + j) \tag{2.2}
\]

where \(\theta = z \frac{d}{dz}\), \(Q_j(X) \in \mathbb{C}[X]\) for every \(j\) and \(\deg(Q_0) = \deg(Q_\ell) = \mu\) (because \(AL\) is Fuchsian, see [4, Lemma 1]). Given \(A\) and \(L\), the representation (2.2) is unique. Moreover, \(Q_0(X)\) depends on \(A\) only by the multiplicative factor \(A_0(0) \neq 0\), while \(Q_\ell(X)\) depends on \(A\) only by the multiplicative factor \(\alpha \neq 0\). Hence, up to non-zero multiplicative constants, \(Q_0(X)\) and \(Q_\ell(X)\) depend uniquely on \(L\). By definition, \(Q_0(X)\) is the indicial polynomial of \(L\) at 0, while \(Q_\ell(\ell - X)\) is the indicial polynomial of \(L\) at \(\infty\).

We now come back to the setting of the lemma, with \(L, M, N \in \mathbb{C}(z)[\frac{d}{dz}]\) such that \(L = MN\). Let \(B(X) \in \mathbb{C}[X] \setminus \{0\}\) be such that \(\tilde{N} := BN \in \mathbb{C}[z][\frac{d}{dz}]\). The differential operator \(\frac{1}{A} M\) can be written \(\frac{1}{A} \tilde{M}\) with \(A(X) \in \mathbb{C}[X] \setminus \{0\}\) and \(\tilde{M} \in \mathbb{C}[z][\frac{d}{dz}]\). Therefore, \(AL = \tilde{M} \tilde{N}\) and by (i) both \(\tilde{M}\) and \(\tilde{N}\) are Fuchsian because \(AL\) is. Moreover, by the discussion above, the indicial polynomial of \(L\), respectively \(\tilde{N}\), at 0 is the same as that of \(AL\), respectively \(\tilde{N}\), at 0. Using obvious notations coherent with (2.2), we set

\[
AL = \sum_{j=0}^{\ell} z^{j+\omega-\mu} Q_j(\theta + j), \quad \tilde{M} = \sum_{i=0}^{m} z^{i+\bar{\omega}-\bar{\mu}} W_i(\theta + i), \quad \tilde{N} = \sum_{k=0}^{n} z^{k+\bar{\omega}-\bar{\mu}} V_k(\theta + k)
\]

where the \(Q\)'s, \(V\)'s and \(W\)'s are all in \(\mathbb{C}[X]\).

Let \(s\) be any integer. We shall apply the various differential operators to the function \(z \mapsto z^s\). Since \(\theta^s(z^s) = s^r z^s\), we have \(AL(z^s) = \sum_{j=0}^{\ell} \tilde{M} \tilde{N}(z^s)\) and

\[
AL(z^s) = \sum_{i=0}^{m} \sum_{k=0}^{n} W_i(s + i + k + \bar{\omega} - \bar{\mu}) V_k(s + k) z^{i+k+\bar{\omega}-\bar{\mu}+\omega+\mu+s}.
\]

The equality \(AL(z^s) = \tilde{M} \tilde{N}(z^s)\) is thus a Laurent polynomial identity in \(z\) (that depends on \(s\)). We now take \(s\) large enough so that it is a root of neither \(Q_0(X)\) nor \(V_0(X)\). The monomials in \(z\) of lowest degree on both sides of \(AL(z^s) = \tilde{M} \tilde{N}(z^s)\) (at \(j = 0\) and \(i = k = 0\) respectively) are thus \(Q_0(s)^{\omega-\mu+s}\) and \(V_0(s + \bar{\omega} - \bar{\mu}) V_0(s)^{\bar{\omega}-\bar{\mu}+\omega+\mu+s} \)
respectively. It follows that \( \omega - \mu = \tilde{\omega} - \tilde{\mu} + \omega - \mu \) and that \( Q_0(s) = W_0(s + \tilde{\omega} - \tilde{\mu})V_0(s) \). Since the integer \( s \) can be taken arbitrarily large, we must have the polynomial identity

\[
Q_0(X) = W_0(X + \tilde{\omega} - \tilde{\mu})V_0(X).
\]

This proves the claimed divisibility because \( Q_0(X) \) and \( V_0(X) \) are the indicial polynomials at 0 of \( L \) and \( N \) respectively.

(iii) Let \( y(z) = \sum_{k=0}^{\infty} U_k z^k \) be such that \( Ly(z) = 0 \). We use the same notations as in (ii). Since \( \theta^r(z^s) = s^r z^s \), we deduce from (2.2) that

\[
0 = z^{\mu-\omega}A(z)Ly(z) = \sum_{j=0}^{\ell} z^j Q_j(\theta + j)y(z)
\]

\[
= \sum_{k=0}^{\infty} \sum_{j=0}^{\ell} Q_j(k + j)U_k z^{k+j} = \sum_{n=0}^{\infty} z^n \sum_{k+j=n}^{\ell} Q_j(k + j)U_k.
\]

Thus, for every \( n \geq \ell \),

\[
0 = \sum_{k+j=n}^{\ell} Q_j(k + j)U_k = \sum_{j=0}^{\ell} Q_j(n)U_{n-j}.
\]

This concludes the proof of the lemma. \( \square \)

**Lemma 2.** Let the integers \( p, q \geq 0 \) be such that \( p + q \geq 1 \), and let \( \alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0} \) be such that \( \alpha_i \neq \beta_j \) for all \( i, j \). Assume that for infinitely many \( n \geq 0 \),

\[
\frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \in \overline{\mathbb{Q}}.
\]

Then the \( \alpha \)'s and \( \beta \)'s are in \( \overline{\mathbb{Q}} \).

If \( p = 0 \), resp. \( q = 0 \), its must be understood that the lemma applies to

\[
\frac{1}{(\beta_1)_n \cdots (\beta_q)_n}, \quad \text{resp.} \quad (\alpha_1)_n \cdots (\alpha_p)_n,
\]

and the condition “\( \alpha_i \neq \beta_j \) for all \( i, j \)” is dropped. The examples given at the beginning of §4 below show that the conclusion does not necessarily hold if one of the \( \alpha \)'s is in \( \mathbb{Z}_{\leq 0} \).

**Proof.** The proof is a slight generalization of that of [6, Lemma 1], where the case \( p \leq q \) was considered. Let

\[
V_n := \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n}.
\]
Since the $\alpha$’s and $\beta$’s are in $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, $V_n \neq 0$ for all $n \geq 0$ and $V_n/V_{n-1}$ is a well defined non-zero algebraic number for all $n \geq 1$. Let $R(X) := \prod_{j=1}^{p} (X + \alpha_j - 1) = \sum_{\ell=0}^{p} r_{\ell} X^{\ell}$ and $S(X) := \prod_{j=1}^{q} (X + \beta_j - 1) = \sum_{\ell=0}^{q} s_{\ell} X^{\ell}$, with $r_p = s_q = 1$. Let $\omega_1, \omega_2, \ldots, \omega_t$ be a basis of the $\mathbb{Q}$-vector space generated by the $r_{\ell}$’s and $s_{\ell}$’s, with $\omega_1 := 1$ and $t \leq p + q + 2$. We write

$$R(X) = \sum_{\ell=1}^{t} R_{\ell}(X) \omega_{\ell}, \quad S(X) = \sum_{\ell=1}^{t} S_{\ell}(X) \omega_{\ell}$$

where $R_{\ell}(X), S_{\ell}(X) \in \mathbb{Q}[X]$ for each $\ell \in \{1, 2, \ldots, t\}$. Since $\omega_1 = 1$ and $r_p = s_q = 1$, we have deg($R_1$) = $p$ and deg($S_1$) = $q$, and both polynomials have leading coefficient 1. The identity $S(n) \frac{V_n}{V_{n-1}} - R(n) = 0$ (for all $n \in \mathcal{N}$, an infinite set by assumption) becomes

$$\sum_{\ell=1}^{t} \left( S_{\ell}(n) \frac{V_n}{V_{n-1}} - R_{\ell}(n) \right) \omega_{\ell} = 0, \quad \forall n \in \mathcal{N}.$$  

Since $S_1(X) \neq 0$, we have $S_1(n) \neq 0$ if $n \in \mathcal{N}$ is large enough, say $n \geq N$. By independence of the $\omega$’s, it follows that $\frac{R(n)}{S(n)} = \frac{R_1(n)}{S_1(n)}$ for every $n \in \mathcal{N}$ such that $n \geq N$. This must then be an equality of rational fractions, i.e. $\frac{R(X)}{S(X)} \equiv \frac{R_1(X)}{S_1(X)}$. Now the assumption that $\alpha_j \neq \beta_k$ for all $j, k$ implies that $R$ and $S$ are coprime. Hence, comparing the degrees and leading coefficients, it follows that $R(X) \equiv R_1(X)$ and $S(X) \equiv S_1(X)$. 

\[\square\]

### 3 Proof of Theorem 1

#### 3.1 Sufficiency

We first prove that every hypergeometric series with parameters not in $\mathbb{Z}_{\leq 0}$ and satisfying (i) and (ii) of Theorem 1 is a $G$-function. If $k = 0$, then all the $\alpha$’s and $\beta$’s are rational numbers, and Siegel’s result applies directly. We now assume that $k \geq 1$. Reordering the parameters if necessary, we assume without loss of generality that $j_\ell = \ell$ for $\ell = 1, \ldots, k$, and let $m_\ell := \alpha_\ell - \beta_\ell$, which is in $\mathbb{N}$ by assumption. We then have for every $n \geq 0$:

$$\frac{(\alpha_{j_\ell})_n}{(\beta_{j_\ell})_n} = \frac{(\beta_\ell + m_\ell)_n}{(\beta_\ell)_n} = \frac{(\beta_\ell + n)_{m_\ell}}{(\beta_\ell)_{m_\ell}} =: P_\ell(n) \in \mathbb{Q}[n].$$

Hence,

$$F(z) := \sum_{n=0}^{\infty} \frac{(\alpha_{j_1} \ldots, \alpha_{j_{q+1}}; z)}{(\beta_1 \ldots, \beta_q; z)} = \sum_{n=0}^{\infty} \left( \prod_{\ell=1}^{k} P_\ell(n) \right) \frac{(\alpha_{k+1})_n \cdots (\alpha_{q+1})_n}{(1)_n (\beta_{k+1})_n \cdots (\beta_q)_n} z^n.$$ 

Writing $\prod_{\ell=1}^{k} P_\ell(n) = \sum_{j=0}^{d} q_j n^j$ with $q_j \in \mathbb{Q}$, we have

$$F(z) = \sum_{j=0}^{d} q_j z^j \left( \sum_{k=0}^{\infty} \frac{(\alpha_{k+1} \ldots, \alpha_{q+1}; z) q_{j+k} z^k}{(\beta_{k+1} \ldots, \beta_q; z)} \right) = \sum_{j=0}^{d} q_j z^j \left( q_{j+k} z^k \right).$$

(3.1)
where \( \theta := z \frac{d}{dz} \). Since \( \alpha_j, \beta_j \in \mathbb{Q} \) for every \( j \geq k + 1 \), each hypergeometric function on the right-hand side of (3.1) is a \( G \)-function (again, by Siegel). Thus \( F(z) \) is a \( G \)-function.

### 3.2 Necessity

We set

\[
U_n := \frac{(\alpha_1)_n \cdots (\alpha_{q+1})_n}{(1)_n (\beta_1)_n \cdots (\beta_q)_n},
\]

the \( n \)-th Taylor coefficient of the hypergeometric series. The \( U_n \)'s are defined and not equal to 0 for every \( n \geq 0 \) because the \( \alpha \)'s and \( \beta \)'s are not in \( \mathbb{Z}_{\leq 0} \). Since \( F(z) := \sum_{n=0}^{\infty} U_n z^n \) is a \( G \)-function, we also have that \( U_n \in \overline{\mathbb{Q}} \) for all \( n \geq 0 \). This is equivalent to the requirement that \( \frac{(\alpha_1)_n \cdots (\alpha_{q+1})_n}{(\beta_1)_n \cdots (\beta_q)_n} \in \overline{\mathbb{Q}} \) for all \( n \geq 0 \). The assumptions of Theorem 1 enable us to apply Lemma 3 with \( p = q + 1 \), so that the \( \alpha \)'s and \( \beta \)'s are in fact algebraic numbers, that is (i) in Theorem 1 holds.

We now turn our attention to the proof of (ii). The classical differential equation satisfied by the hypergeometric series (1.1) when \( p = q + 1 \) is \( Ly(z) = 0 \) with

\[
L := \theta(\theta + \beta_1 - 1) \cdots (\theta + \beta_q - 1) - z(\theta + \alpha_1) \cdots (\theta + \alpha_{q+1}) \in \mathbb{C}[z] \left[ \frac{d}{dz} \right].
\]

It reflects the fact that the sequence \( (U_n)_{n \geq 0} \) satisfies the linear recurrence \( B(n) U_n - A(n) U_{n-1} = 0 \) for \( n \geq 1 \), where

\[
A(X) = \prod_{j=1}^{q+1} (X + \alpha_j - 1), \quad B(X) = \prod_{j=1}^{q+1} (X + \beta_j - 1)
\]

are both in \( \overline{\mathbb{Q}}[X] \), with \( \beta_{q+1} := 1 \). In particular, the indicial polynomial of \( L \) at 0 is \( B(X) \) and the indicial polynomial of \( L \) at \( \infty \) is \( A(1 - X) \); their roots are in \( \overline{\mathbb{Q}} \).

Let \( N \in \mathbb{C}[z][\frac{d}{dz}] \setminus \{0\} \) be such that \( NF(z) = 0 \) and is of minimal order for \( F(z) \). Then \( N \) is a right factor of \( L \). Since \( NF(z) = 0 \), by Lemma 1 (iii), there exist an integer \( \ell \geq 1 \) and some polynomials \( C_j(X) \), \( j = 1, \ldots, \ell \), such that

\[
\sum_{j=0}^{\ell} C_j(n) U_{n-j} = 0 \tag{3.2}
\]

for all \( n \geq \ell \), and \( C_0(X) \) and \( C_{\ell}(X - \ell) \) are the respective indicial polynomials of \( N \) at 0 and \( \infty \).

Below, we shall consider the multiset \( R(P) \) of the roots of a polynomial \( P \); each element of \( R(P) \) appears as many times as its multiplicity as a root of \( P \). We denote a multiset by \( \{\cdot\} \) to distinguish it from a set \( \{\cdot\} \).

**Informations coming from the indicial polynomials of \( L \) and \( N \) at 0.** Recall that the indicial polynomial of \( L \) at 0 is \( B(X) \). By Lemma 1 (ii), there exists \( D_0(X) \in \mathbb{C}[X] \) such that

\[
B(X) = C_0(X) D_0(X).
\]
By Theorem 2, the roots of \( C_0 \) are in \( \mathbb{Q} \), so that those of \( D_0 \) are in \( \overline{\mathbb{Q}} \). Since \( U_n \neq 0 \) for all \( n \geq 0 \), we can rewrite the recurrence (3.2) as

\[
C_0(n) \frac{U_n}{U_{n-\ell}} = -\sum_{j=1}^{\ell} C_j(n) \frac{U_{n-j}}{U_{n-\ell}}, \quad \forall n \geq \ell. \tag{3.3}
\]

Now,

\[
\frac{U_{n-j}}{U_{n-\ell}} = \prod_{k=j}^{\ell-1} \frac{A(n-k)}{B(n-k)}
\]

so that after clearing the denominators, (3.3) yields

\[
C_0(n) \prod_{k=0}^{\ell-1} A(n-k) = -\sum_{j=1}^{\ell} C_j(n) \prod_{k=0}^{j-1} B(n-k) \prod_{k=j}^{\ell-1} A(n-k), \quad \forall n \geq \ell.
\]

This is a polynomial identity for infinitely many values of the integer \( n \), hence a genuine polynomial identity

\[
C_0(X) \prod_{k=0}^{\ell-1} A(X-k) = -\sum_{j=1}^{\ell} C_j(X) \prod_{k=0}^{j-1} B(X-k) \prod_{k=j}^{\ell-1} A(X-k).
\]

We observe that \( B(X) \) is a factor of each summand on the right hand side. Hence \( B(X) \) divides \( C_0(X) \prod_{k=0}^{\ell-1} A(X-k) \), so that \( D_0(X) \) divides \( \prod_{k=0}^{\ell-1} A(X-k) \).

Since \( B(X) = C_0(X)D_0(X) \), we have

\[
R(C_0) = \{1 - \beta_{j_1}, \ldots, 1 - \beta_{j_k}\}, \quad R(D_0) = \{1 - \beta_{j_{k+1}}, \ldots, 1 - \beta_{j_{q+1}}\}
\]

for some \( \kappa \in \{0, \ldots, q+1\} \), and where \( \{j_m : m = 1, \ldots, q+1\} = \{1, 2, \ldots, q+1\} \). With \( \tilde{A}(X) := \prod_{k=0}^{\ell-1} A(X-k) \), we have

\[
R(\tilde{A}) = \{1 - \alpha_1, \ldots, 1 - \alpha_{q+1}, 2 - \alpha_1, \ldots, 2 - \alpha_{q+1}, \ldots, \ell - \alpha_1, \ldots, \ell - \alpha_{q+1}\}.
\]

and \( R(D_0) \subset R(\tilde{A}) \).

Now, let \( 1 - \beta_j \in R(B) \).

- If \( 1 - \beta_j \in R(C_0) \), then \( \beta_j \in \mathbb{Q} \).
- If \( 1 - \beta_j \in R(D_0) \), then \( 1 - \beta_j \in R(\tilde{A}) \). Hence, \( 1 - \beta_j = k - \alpha_i \) for some integers \( k, i \) such that \( 1 \leq k \leq \ell \) and \( 1 \leq i \leq q+1 \). It follows that \( \alpha_i - \beta_j = k - 1 \geq 0 \). (If \( \beta_j \in \mathbb{Q} \), \( \alpha_i \in \mathbb{Q} \) as well).

This completely determines the nature of the parameters \( \beta \) in accordance with \((ii)\) in Theorem 1. However, if \( \kappa \geq 1 \), then \( \deg(A) > \deg(D_0) \) and thus there exists at least one parameter \( \alpha \) which is not associated to a parameter \( \beta \) such that \( 1 - \beta_j \in R(D_0) \). It might even be the case that \( \kappa = q+1 \) and \( \deg(D_0) = 0 \), so that this argument says in fact nothing on the \( \alpha \)'s. We shall now give explain how to determine the nature ot the \( \alpha \)'s.
Informations coming from the indicial polynomials of $L$ and $N$ at $\infty$. Recall that the indicial polynomial of $L$ at $\infty$ is $A(1 - X)$. By Lemma 1(ii), there exists $D_\ell(X) \in \mathbb{C}[X]$ such that

$$A(X - \ell + 1) = C_\ell(X)D_\ell(X).$$

By Theorem 2, the roots of $C_\ell$ are in $\mathbb{Q}$, so that those of $D_\ell$ are in $\overline{\mathbb{Q}}$. We can rewrite the recurrence (3.2) as

$$C_\ell(n)\frac{U_{n-\ell}}{U_n} = -\sum_{j=0}^{\ell-1} C_j(n)\frac{U_{n-j}}{U_n}, \quad \forall n \geq \ell.$$  \hspace{1cm} (3.4)

Now,

$$\frac{U_{n-j}}{U_n} = \prod_{k=0}^{j-1} \frac{B(n-k)}{A(n-k)}$$

so that after clearing the denominators, (3.4) yields

$$C_\ell(n) \prod_{k=0}^{\ell-1} B(n-k) = -\sum_{j=0}^{\ell-1} C_j(n) \prod_{k=0}^{j-1} B(n-k) \prod_{k=j}^{\ell-1} A(n-k), \quad \forall n \geq \ell.$$  

This is a polynomial identity for infinitely many values of the integer $n$, hence a genuine polynomial identity

$$C_\ell(X) \prod_{k=0}^{\ell-1} B(X-k) = -\sum_{j=0}^{\ell-1} C_j(X) \prod_{k=0}^{j-1} B(X-k) \prod_{k=j}^{\ell-1} A(X-k).$$

We observe that $A(X) := A(X - \ell - 1)$ is a factor of each summand on the right hand side. Hence $A(X)$ divides $C_\ell(X) \prod_{k=0}^{\ell-1} B(X-k)$, so that $D_\ell(X)$ divides $\prod_{k=0}^{\ell-1} B(X-k)$. Since $A(X) = C_\ell(X)D_\ell(X)$, we have

$$R(C_\ell) = \{\ell - \alpha_{j_1}, \ldots, \ell - \alpha_{j_q}\}, \quad R(D_\ell) = \{\ell - \alpha_{j_{q+1}}, \ldots, \ell - \alpha_{k_{\ell+1}}\}$$

for some $\omega \in \{0, \ldots, q + 1\}$, and where $\{j_m : m = 1, \ldots, q + 1\} = \{1, 2, \ldots, q + 1\}$. With $B(X) := \prod_{k=0}^{\ell-1} B(X-k)$, we have

$$R(B) = \{1 - \beta_1, \ldots, 1 - \beta_{q+1}, 2 - \beta_1, \ldots, 2 - \beta_{q+1}, \ldots, \ell - \beta_1, \ldots, \ell - \beta_{q+1}\}$$

and $R(D_\ell) \subset R(B)$.

Now, let $\ell - \alpha_i \in R(A)$.

- If $\ell - \alpha_i \in R(C_\ell)$, then $\alpha_i \in \mathbb{Q}$.
- If $\ell - \alpha_i \in R(D_\ell)$, then $\ell - \alpha_i \in R(B)$. Hence, $\ell - \alpha_i = k - \beta_j$ for some integers $k, j$ such that $1 \leq k \leq \ell$ and $1 \leq j \leq q + 1$. It follows that $\alpha_i - \beta_j = \ell - k \geq 0$. (If $\alpha_i \in \mathbb{Q}$, $\beta_j \in \mathbb{Q}$ as well.)

This completely determines the nature of the parameters $\alpha$ in accordance with (ii) in Theorem 1, the proof of which is now complete.
4 Other types of hypergeometric $G$-functions

If we keep all the assumptions of Theorem 1, except that we enable one of the $\alpha$’s to be in $\mathbb{Z}_{\leq 0}$ (in which case the hypergeometric series reduces to a polynomial), then it is no longer true that the other parameters $\alpha$’s and $\beta$’s are necessarily in $\overline{\mathbb{Q}}$ for this hypergeometric polynomial to be a $G$-function. For instance, if $\alpha_1, \beta_1 \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, then

$$2F_1 \left( -1, \alpha_1 \beta_1; z \right) = 1 - \frac{\alpha_1}{\beta_1} z,$$

is a $G$-function if and only if $\frac{\alpha_1}{\beta_1} \in \overline{\mathbb{Q}}$. On the other hand, if $\alpha_1, \beta_1 \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, then

$$2F_1 \left( -2, \alpha_1 \beta_1; z \right) = 1 - 2 \frac{\alpha_1}{\beta_1} z + \frac{\alpha_1(\alpha_1 + 1)}{\beta_1(\beta_1 + 1)} z^2,$$

is a $G$-function if and only if $\frac{\alpha_1}{\beta_1}$ and $\frac{\alpha_1(\alpha_1 + 1)}{\beta_1(\beta_1 + 1)}$ are both in $\overline{\mathbb{Q}}$; this forces $\alpha_1$ and $\beta_1$ to be in $\overline{\mathbb{Q}}$. It would be interesting to determine for any given $q \geq 1$ the smallest integer $\eta(q)$ such that if (i) $m \in \mathbb{N}_{\geq \eta(q)}$,

(ii) $(\alpha_1, \ldots, \alpha_q, \beta_1, \ldots, \beta_q) \in (\mathbb{C} \setminus \mathbb{Z}_{\leq 0})^{2q}$,

(iii) the hypergeometric polynomial

$$\begin{aligned}
\sum_{n=0}^{m} \frac{(-m)_{n}(\alpha_1)_{n} \cdots (\alpha_q)_{n}}{(1)_{n}(\beta_1)_{n} \cdots (\beta_q)_{n}} z^{n}
\end{aligned}$$

is a $G$-function,

then the $\alpha$’s and $\beta$’s are necessarily in $\overline{\mathbb{Q}}$. The above examples show that $\eta(2) = 2$.

Galochkin only delt with the case $p = q + 1$, and did not consider the possibility that there might exist hypergeometric $G$-functions with $p \neq q + 1$. There exist indeed such functions but they are essentially trivial.

**Proposition 1.** Let the integers $p, q \geq 0$ be such that $p + q \geq 0$ and $p \neq q + 1$, and let $\alpha := (\alpha_1, \ldots, \alpha_p) \in \mathbb{C}^p$, $\beta := (\beta_1, \ldots, \beta_q) \in (\mathbb{C} \setminus \mathbb{Z}_{\leq 0})^q$ be such that $\alpha_i \neq \beta_j$ for all $i, j$. If the hypergeometric series (1.1) with parameters $\alpha$ and $\beta$ is a $G$-function, then at least one of the $\alpha$’s is in $\mathbb{Z}_{\leq 0}$, i.e. the hypergeometric series reduces to a polynomial.

A remark similar to that above can be made here: the parameters of the hypergeometric polynomials in Proposition 1 may or may not be algebraic. To prove the proposition, we need the following lemma.

**Lemma 3.** A $G$-function is holomorphic at $z = 0$. An entire $G$-function reduces to a polynomial.
Proof. Let $F(z) = \sum_{n=0}^{\infty} A_n z^n$ be a $G$-function. Since $|A_n| \leq |A_n| \leq C^n + 1$ for some $C > 0$ and all $n \geq 0$, $F(z)$ has positive radius of convergence.

The differential equation for $F(z)$ implies that the $A_n$’s all lie in a certain number field of degree $h$. Recall that $1 \leq D_n := \text{den}(A_0, A_1, \ldots, A_n) \leq D^{n+1}$ for some $D > 0$ and all $n \geq 0$. Hence, for every $n \geq 0$, we have either $A_n = 0$ or

$$|A_n| \geq \frac{1}{|A_n|^{h-1}} D_n^{h-1} \geq \frac{1}{(CD)^{h(n+1)}}.$$  

Moreover, the sequence $(A_n)_{n \geq 0}$ satisfies a linear recurrence of finite order $\ell$ with polynomial coefficients (see Lemma 1(iii)). Assuming that $A_n \neq 0$ for infinitely many $n$, it follows that for every $n$ large enough, there exists $k_n \in \{n, n+1, \ldots, n+\ell\}$ such that $A_{k_n} \neq 0$ and thus that

$$|A_{k_n}| \geq \frac{1}{(CD)^{h(k_n+1)}}.$$  

Since $k_n \ll n$, we deduce that $F(z)$ has finite radius of convergence.

Let us now prove Theorem 1. For any $x_1, \ldots, x_k, y_1, \ldots, y_k \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, Stirling’s formula implies that as $n \to +\infty$

$$\left| \frac{(x_1)_n \cdots (x_k)_n}{(y_1)_n \cdots (y_k)_n} \right| \sim n^{\Re(\sum_{j=1}^{k}(x_j-y_j))}.$$  

We shall distinguish two cases: $p \geq q + 2$ or $p \leq q$.

If $p \geq q + 2$, and $(\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q) \in (\mathbb{C} \setminus \mathbb{Z}_{\leq 0})^{p+q}$, we write

$$\frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(1)_n (\beta_1)_n \cdots (\beta_q)_n} = \frac{(\alpha_1)_n \cdots (\alpha_{q+1})_n}{(1)_n (\beta_1)_n \cdots (\beta_q)_n} \cdot \frac{(\alpha_{q+2})_n \cdots (\alpha_p)_n}{(1)_n (\beta_1)_n \cdots (\beta_q)_n},$$  

whose modulus is $\sim n^\delta |(\alpha_{q+2})_n \cdots (\alpha_p)_n|$ for some $\delta \in \mathbb{R}$. The hypergeometric series is not holomorphic at 0, hence not a $G$-function.

If $p \leq q$, and $\alpha := (\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q) \in (\mathbb{C} \setminus \mathbb{Z}_{\leq 0})^{p+q}$, we write

$$\frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(1)_n (\beta_1)_n \cdots (\beta_q)_n} = \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(1)_n (\beta_1)_n \cdots (\beta_{p-1})_n} \cdot \frac{1}{(\beta_p)_n \cdots (\beta_q)_n},$$  

whose modulus is $\sim n^\delta \frac{1}{|(\beta_p)_n \cdots (\beta_q)_n|}$ for some $\delta \in \mathbb{R}$. The hypergeometric series is entire and not a polynomial, hence not a $G$-function.

References


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