A Roth-type theorem for values of E-functions

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Irrationality exponent, Liouville numbers

Once we have proved our favorite real number ξ to be irrational, we may wonder how to measure its distance to a rational number p/q in terms of q, ie to obtain *an irrationality measure* for ξ.

For instance, consider $\xi = \sqrt{2}$: for any $(p, q) \in \mathbb{Z} \times \mathbb{N}^*$, we have $|p^2 - 2q^2| \geq 1$,

$$\left| \sqrt{2} - \frac{p}{q} \right| \geq \frac{1}{q|q\sqrt{2} + p|} > \frac{c}{q^2} \tag{1}$$

for some absolute constant $c > 0$.

In Eq. (1), the 2 in q^2 is called *an irrationality exponent* for $\sqrt{2}$.

Definition 1

Given $\xi \in \mathbb{R} \setminus \mathbb{Q}$, *set*

$$E(\xi) := \left\{ \mu \in \mathbb{R} : \exists \infty \ (p, q) \in \mathbb{Z} \times \mathbb{N}^* \text{ s.t. } \left| \xi - \frac{p}{q} \right| < \frac{1}{q^\mu} \right\}$$

and $\mu(\xi) := \sup E(\xi)$ *is* the irrationality exponent *of* ξ.

• Eq. (1) shows that $\mu(\sqrt{2}) \leq 2$.

• Dirichlet: $\forall \xi \in \mathbb{R} \setminus \mathbb{Q}$, $2 \in E(\xi)$ so that $\mu(\xi) \geq 2$.

• If ξ is a real algebraic number of degree $d \geq 2$, we have $\mu(\xi) \leq d$ (Liouville 1844) and in fact $\mu(\xi) \leq 2$ (Roth 1955).

• For almost all real numbers ξ (in Lebesgue' sense), $\mu(\xi) = 2$. Because of this, a folklore belief is that $\mu(\xi) = 2$ for any classical constant ξ of analysis.

Definition 2

$\xi \in \mathbb{R} \setminus \mathbb{Q}$ is said to be a Liouville number if $\mu(\xi) = +\infty$.

Equivalently, there exist two sequences $(p_n, q_n) \in \mathbb{Z} \times \mathbb{N}^*$ such that

$$0 < \left| \xi - \frac{p_n}{q_n} \right| < \frac{1}{q_n^n}, \quad \forall n \geq 0.$$

$\xi := \sum_{k \geq 0} 10^{-k!}$ is a Liouville number

$$0 < \left| \xi - \frac{\sum_{k=0}^{n} 10^{n!-k!}}{10^{n!}} \right| < \frac{1}{(10^{n!})^n}, \quad n \geq 0.$$

How to obtain an irrationality measure?

To prove the irrationality of some number ξ, a standard method is to construct two sequences of *integers* p_n and $q_n \geq 1$ such that

$$0 < \varepsilon_n := |q_n\xi - p_n| \to 0, \quad n \to +\infty.$$

An irrationality measure for ξ is obtained as follows: for $a/b \neq p_n/q_n$ with $b > 0$, we have

$$\left|\xi - \frac{a}{b}\right| \geq \left|\frac{p_n}{q_n} - \frac{a}{b}\right| - \frac{\varepsilon_n}{q_n} \geq \frac{1}{bq_n} - \frac{\varepsilon_n}{q_n} \geq \frac{1}{2bq_n}$$

provided $2b\varepsilon_n \leq 1$. With $n = n(b)$ minimal satisfying this condition:

$$\left|\xi - \frac{a}{b}\right| \geq \frac{1}{b^{\omega(b)}} \quad \text{where} \quad \omega(b) := \frac{\ln(2q_{n(b)})}{\ln(b)} + 1.$$

When the behaviors of q_n and ε_n are known, the value of $\omega(b)$ can be simplified, and we can also consider the case when $a/b = p_n/q_n$.

Examples

- For $\xi := \sum_{n=0}^{\infty} 10^{-n!}$, with $q_n = 10^{n!}$ and $p_n = \sum_{m=0}^{n} 10^{n!-m!}$, we get

$$\left| \xi - \frac{p}{q} \right| > \frac{c}{q^{b \ln(q)/ \ln \ln(q)}}$$

for some absolute constants $b, c > 0$.

- For b, k integers ≥ 2, consider $\xi := \sum_{n=0}^{\infty} 1/b^{k^n}$: with $q_n = b^{k^n}$ and $p_n = \sum_{m=0}^{n} b^{k^n - k^m}$, we get

$$\left| \xi - \frac{p}{q} \right| > \frac{c}{q^{k^2/(k-1)}}$$

and also $\mu(\xi) \geq k$.

More generally, let $F(z) \in \mathbb{Z}[[z]]$ be a Mahler function. For any integer $b \geq 2$, Bell-Bugeaud-Coons proved in 2015 that $F(1/b)$ cannot be a Liouville number (when it is defined).

The above example shows that we don't have $\mu(F(1/b)) = 2$ in general.

Other examples

Let $d_n := \mathrm{lcm}(1, 2, \ldots, n) = e^{n+o(n)}$.

- Alladi-Robinson 1980. $\exists p_n, q_n \in \mathbb{Z}^*$ such that

$$q_n \ln(2) - p_n = d_n \int_0^1 \frac{x^n(1-x)^n}{(1+x)^{n+1}} dx = \left(e(\sqrt{2}-1)^2\right)^{n+o(n)}$$

and $q_n = (e(\sqrt{2}+1)^2)^{n+o(n)}$. Hence, $\mu(\ln(2)) \le 4.6221$.

Best known record: $\mu(\ln(2)) \le 3.5746$ by Marcovecchio in 2008.

- Beukers 2000. $\exists p_n \in \mathbb{Q}, q_n \in \mathbb{Z}^*$ such that

$$q_n \pi - p_n = \int_{-1}^1 \frac{x^{2n}(1-x^2)^{2n}}{(1+ix)^{3n+1}} dx.$$

Hence, $\mu(\pi) \le 23.271$.

Best known record: $\mu(\pi) \le 7.1033$ by Zeilberger-Zudilin in 2020.

Irrationality measure of e

- We first seek good sequences of functional approximations of $\exp(z)$: there exist $A_n, B_n \in \mathbb{Z}[z]$ not both zero, of degree $\leq n$ and such that

$$\mathrm{ord}_{z=0}\big(B_n(z)\exp(z) - A_n(z)\big) \geq 2n+1.$$

A_n/B_n is unique and is called the n-th diagonal Padé approximant of the exponential.

- We have

$$B_n(z) = \sum_{k=0}^{n} k! \binom{n}{k} \binom{n+k}{k} (-z)^{n-k}, \quad A_n(z) = B_n(-z)$$

and

$$B_n(z)e^z - A_n(z) = \frac{z^{2n+1}}{n!} \int_0^1 x^n (x-1)^n e^{zx}\, dx.$$

- We get $\mu(e) = 2$ because

$$|B_n(1)| \asymp n^{\alpha} a^n n!, \qquad \big|B_n(1)e - A_n(1)\big| \asymp \frac{n^{\beta} b^n}{n!}.$$

Irrationality measure of e, continued

The irrationality measure of a real number ξ is deduced from the sequence of convergents p_n/q_n of its the continued fraction:

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, \ldots] = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}$$

- $p_{3m-2} = A_m(1)$, $q_{3m-2} = B_m(1)$, and when $n \equiv 1 \mod 3$:

$$\left| e - \frac{p_n}{q_n} \right| \sim \frac{\ln\ln(q_n)}{2q_n^2 \ln(q_n)}.$$

- Davis (1978): for any $\varepsilon > 0$ the inequation

$$\left| e - \frac{p}{q} \right| < (0.5 + \varepsilon) \frac{\ln\ln(q)}{q^2 \ln(q)} \qquad (2)$$

has infinitely many solutions $(p, q) \in \mathbb{Z} \times \mathbb{N}$ while for all $(p, q) \in \mathbb{Z} \times \mathbb{N}$ with $q \geq q_0(\varepsilon)$, we have the irrationality measure

$$\left| e - \frac{p}{q} \right| > (0.5 - \varepsilon) \frac{\ln\ln(q)}{q^2 \ln(q)}. \qquad (3)$$

(For $\varepsilon = 0$, Eq. (3) holds infinitely often, and it also seems to be the case of Eq. (2).)

E-functions

Siegel defined E-functions to generalize the Lindemann-Weierstrass Theorem: *Given any pairwise distinct algebraic numbers $\alpha_1, \ldots, \alpha_n$, the numbers $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are $\overline{\mathbb{Q}}$-linearly independent.* His program culminated with the Siegel-Shidlovskii Theorem (1929-1956).

Definition 3

A power series $F(z) = \sum_{n=0}^{\infty} a_n z^n/n! \in \overline{\mathbb{Q}}[[z]]$ is a (strict) E-function if

(i) $F(z)$ is solution of a non-zero linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$.

(ii) There exists $C > 0$ such that $\overline{|a_n|} \leq C^{n+1}$ for all $n \geq 0$.

(iii) There exists a sequence of positive integers d_n, with $d_n \leq C^{n+1}$, such that $d_n a_m$ are algebraic integers for all $m \leq n$.

If $a_n \in \mathbb{Q}$, (ii) and (iii) read $|a_n| \leq C^{n+1}$ and $d_n a_m \in \mathbb{Z}$.

Siegel's definition is more general: the two bounds $(\cdots) \leq C^{n+1}$ are replaced by: for all $\varepsilon > 0$, $(\cdots) \leq n!^{\varepsilon}$ for all $n \geq N(\varepsilon)$.

Examples

Polynomials in $\overline{\mathbb{Q}}[z]$, hypergeometric functions:

$$ {}_pF_q\left[\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; z^{q-p+1}\right] := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{n!(b_1)_n \cdots (b_q)_n} z^{n(q-p+1)}, $$

when $q \geq p \geq 1$, $a_j \in \mathbb{Q}$ and $b_j \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ for all j. For instance $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ and Bessel's function

$$ J_0(z) := \sum_{n=0}^{\infty} (-1)^n \frac{(z/2)^{2n}}{n!^2} = {}_0F_1\left[\begin{matrix} \cdot \\ 1 \end{matrix}; -(z/2)^2\right]. $$

$$ \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{n} \right) \frac{z^n}{n!} = e^{(3-2\sqrt{2})z} \cdot {}_1F_1\left[\begin{matrix} 1/2 \\ 1 \end{matrix}; 4\sqrt{2}z\right], $$

$$ \sum_{n=0}^{\infty} \left(\sum_{k=1}^{n} \frac{1}{k} \right) \frac{z^n}{n!} = ze^z \cdot {}_2F_2\left[\begin{matrix} 1, 1 \\ 2, 2 \end{matrix}; -z\right]. $$

E-functions are not all polynomials in hypergeometric functions (Fresán-Jossen 2021).

Bessel's function J_0

The E-functions J_0 et J_0' are $\overline{\mathbb{Q}}(z)$-algebraically independent and

$$\begin{pmatrix} J_0 \\ J_0' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & -\frac{1}{z} \end{pmatrix} \begin{pmatrix} J_0 \\ J_0' \end{pmatrix}.$$

• Siegel 1929: For any $P \in \mathbb{Z}[X_1, X_2] \setminus \{0\}$ of degree δ, any $\varepsilon > 0$ and any $\alpha \in \overline{\mathbb{Q}}^*$ of degree d, $\exists c = c(\alpha, \delta, \varepsilon) > 0$ such that

$$\left| P(J_0(\alpha), J_0'(\alpha)) \right| > \frac{c}{H(P)^{123\delta^2 d^3 + \varepsilon}}.$$

For any $r \in \mathbb{Q}^*$ and any $\varepsilon > 0$, $\exists c = c(r, \varepsilon) > 0$ such that for all $(u, v, w) \in \mathbb{Z}^3 \setminus \{0\}$,

$$\left| u + vJ_0(r) + wJ_0'(r) \right| > \frac{c}{\max(|u|, |v|, |w|)^{2+\varepsilon}}, \qquad (4)$$

• The exponent 2 is optimal in Eq. (4). In particular, $\mu(J_0(r)) \leq 3$ and Lang asked in 1965 if $\mu(J_0(r)) \leq 2$.

Shidlovskii's measure

Theorem 1 (Shidlovskii 1966)

$Y = {}^t(F_1, \dots, F_N)$ a vector of E-functions in $\mathbb{Q}[[z]]$ and $A \in M_N(\mathbb{Q}(z))$ such that $Y' = AY$. Let $T \in \mathbb{Q}[z] \setminus \{0\}$ be a common denominator of the entries of A.

If $F_1, \dots, F_N$ are linearly independent over $\mathbb{Q}(z)$, then for all $r \in \mathbb{Q}$ such that $rT(r) \neq 0$, for any $\varepsilon > 0$, $\exists c > 0$ such that

$$\forall (a_1, \dots, a_N) \in \mathbb{Z}^N \setminus \{0\}, \quad \left| \sum_{j=1}^{N} a_j F_j(r) \right| > \frac{c}{(\max |a_j|)^{N-1+\varepsilon}}. \qquad (5)$$

The exponent $N - 1$ is optimal.

• When r is not a singularity of the minimal inhomogeneous equation $\mathcal{M}_F$ of order $m \geq 1$ satisfied by a transcendental E-function $F \in \mathbb{Q}[[z]]$, the value $F(r)$ is not a Liouville number: we have $F(r) \in \mathbb{R} \setminus \mathbb{Q}$ and $\mu(F(r)) \leq m + 1$.

• His proof does not work with $\mathbb{Q}$ replaced by a number field $\mathbb{K}$.The qualitative part $\sum_{j=1}^{N} a_j F_j(r) \neq 0$ (with $a_j, r \in \mathbb{K}$) was proved by Beukers in 2006.

Measure over a number field $\mathbb{K}$ of degree d

Theorem 2 (Fischler-R, 2023)

$Y = {}^t(F_1, \ldots, F_N)$ a vector of E-functions in $\mathbb{K}[[z]]$, solution of $Y' = AY$ with $A \in M_N(\mathbb{K}(z))$. For all $\alpha \in \mathbb{K}$, for any $\varepsilon > 0$, $\exists c > 0$ such that $\forall (a_1, \ldots, a_N) \in \mathcal{O}_{\mathbb{K}}^N \setminus \{0\}$, either

$$L := \sum_{j=1}^{N} a_j F_j(\alpha) = 0 \quad \text{or} \quad |L| > \frac{c}{(\max \overline{|a_j|})^{dN^d - 1 + \varepsilon}}. \tag{6}$$

• If $F_1, \ldots, F_N$ are linearly independent over $\mathbb{K}(z)$ and $\alpha T(\alpha) \neq 0$, Beukers' theorem (2006) implies that $L \neq 0$.

• André & Beukers ensure 1) that $L^\sigma := \sum_{j=1}^{N} \sigma(a_j) F_j^\sigma(\sigma(\alpha)) \neq 0$ for all embedding σ of $\mathbb{K}$ into $\mathbb{C}$ if $L \neq 0$, and 2) enable to deal with singular α's. Then (when $\mathbb{K}$ is Galoisian)

$$0 \neq \mathcal{L} := \prod_{\sigma} L^\sigma = \sum_{j=0}^{N^d} A_j \Phi_j(1), \quad A_j \in \mathbb{Z}$$

where Φ_j are independent E-functions in $\mathbb{Q}[[z]]$ solutions of a differential system not singular at 1. To get (6), we apply Shidlovskii's lower bound (5) to $\mathcal{L}$ and trivial upper bounds for L^σ when $\sigma \neq id$.

Measure over a number field $\mathbb{K}$, continued

Corollary 1

For any E-function F and any $\alpha \in \overline{\mathbb{Q}}$, the number $F(\alpha)$ is not a Liouville number.

• Take $F_1 = 1$, $F_2 = F$ in Theorem 2 and $\alpha \in \overline{\mathbb{Q}}$. If $F(\alpha) \in \mathbb{Q}$, then $F(\alpha)$ is not a Liouville number. If $F(\alpha) \notin \mathbb{Q}$, then $a_1 + a_2 F(\alpha) \neq 0$ for all $a_1, a_2 \in \mathbb{Z}$ not both 0, and (6) implies the result.

• When $F(\alpha) \notin \mathbb{Q}$, $\mu(F(\alpha)) \leq d(m+1)^d$ where m is the order of $\mathcal{M}_F$. In particular, for all $\alpha \in \overline{\mathbb{Q}}^*$ of degree $d \geq 1$,

$$\left| e^{\alpha} - \frac{p}{q} \right| > \frac{c}{q^{d2^d+\varepsilon}} \quad (m=1), \qquad \left| J_0(\alpha) - \frac{p}{q} \right| > \frac{c}{q^{d3^d+\varepsilon}} \quad (m=2). \quad (7)$$

e^{α}: Lang-Galochkin $4d^2+1$, Kappe $4d^2-2d$. Eq. (7) is better for $d \in \{2,3\}$.

$J_0(\alpha)$: Siegel $123d^3+1$ and 3 for $d=1$, Lang-Galochkin $16d^3+1$ and Zudilin 2 for $d=1$. Eq. (7) is better for $d \in \{2,3,4,5\}$.

Roth-type measure in the rational case

Theorem 3 (Fischler-R, 2023)

Let F be an E-function in $\mathbb{Q}[[z]]$ and $r \in \mathbb{Q}^*$. Then either $F(r) \in \mathbb{Q}$ or $\mu(F(r)) = 2$.

• Announced in 1984 by Chudnovsky but there were gaps in the proof.

• Zudilin (1995) filled in these gaps under other assumptions: F is a strict E-function, r is not a singularity of $\mathcal{M}_F$ of order $m \geq 1$, and either $m \leq 2$ or $F, F', \ldots, F^{(m-1)}$ are algebraically independent. He obtained

$$\left| F(r) - \frac{p}{q} \right| > \frac{c}{q^{2+a/\ln\ln(q)^b}}. \tag{8}$$

His assumptions apply to J_0 for all $r \in \mathbb{Q}^*$, answering Lang's 1965 question. Eq. (8) is not known for E-functions in Siegel's sense.

• The hypergeometric E-function

$$g(z) := {}_1F_2\left[\begin{matrix} 1/2 \\ 1/3, 2/3 \end{matrix}; z^2\right].$$

does not satisfy Zudilin's assumptions because $m = 3$ and

$$4g(z)^2 - g'(z)^2 + 9z^2\big(4g(z) - g''(z)\big)^2 = 4.$$

• The possibility that $F(r) \in \mathbb{Q}$ can not be dropped even when F is transcendental: consider the trivial example $(z-1)e^z$ at $z=1$.

• Non-trivial exotic hypergeometric rational evaluations (Bostan-R-Salvy 2024):

$$ {}_1F_1\left[\begin{matrix}1\\7/3\end{matrix};-\frac{2}{3}\right]=\frac{5}{27}, \quad {}_1F_1\left[\begin{matrix}6\\-2/5\end{matrix};-\frac{12}{5}\right]=\frac{1309}{625}, $$

$$ {}_2F_2\left[\begin{matrix}1/4,3/4\\5/4,-9/4\end{matrix};-\frac{9}{4}\right]=0. $$

• If r is not a singularity of $\mathcal{M}_F$ of order ≥ 1, then $F(r) \notin \mathbb{Q}$, by Beukers' theorem (2006).

If r is a singularity of $\mathcal{M}_F$, Adamczewski-R's algorithm (2018), refined and implemented by Bostan-R-Salvy (2024), enables to decide weither $F(r) \in \mathbb{Q}$ or not.

Hermite-Padé approximants

- For any integer $n \geq 0$, there exist $P_{1,n}, \ldots, P_{N,n} \in \mathbb{Z}[z]$ not all zero, of degree $\leq n$ such that

$$\mathrm{ord}_{z=0}\Big(\sum_{j=1}^{N} P_{j,n}(z)F_j(z)\Big) \geq N(n+1) - 1.$$

- Shidlovskii constructed N "independent" functions

$$R_{k,n}(z) := \sum_{j=1}^{N} P_{j,k,n}(z)F_j(z), \quad k = 1, ..., N$$

using the differential system $Y' = AY$, where $\deg(P_{j,k,n}) \leq n + c$ and $\mathrm{ord}_{z=0}(R_{k,n}) \geq Nn - [\varepsilon n]$. When $rT(r) \neq 0$,

$$P_{j,k,n}(r) \ll a^n n!^{1+\varepsilon}, \quad R_{k,n}(r) \ll b^n / n!^{N(1-\varepsilon)}, \quad \det(P_{j,k,n}(r)) \neq 0.$$

Shidlovskii's linear independence measure "follows".

Graded Padé approximants for F with $\mathcal{M}_F$ of order 2

- With $F_1 = 1$ and $F_2 = F \in \mathbb{Q}[[z]]$, Shidlovskii gives $\mu(F(r)) \leq 3$ because we also have to consider $F_3 = F'$.

- We construct $2M+1$ polynomials $A_{j,n}$ and $B_{j,n}$ in $\mathbb{Z}[z]$ not all zero of degree $\leq n$ such that $B_{-1,n} = B_{M,n} \equiv 0$, and for $j = 0, \ldots, M$:

$$\mathrm{ord}_{z=0}\Big(A_{j,n}(z)+B_{j-1,n}(z)F(z)+B_{j,n}(z)F'(z)\Big) \geq (2-\varepsilon_M)n, \quad \varepsilon_M \asymp \frac{1}{M}.$$

- Setting $R_{M,n}(r) := A_{M,n}(r) + B_{M-1,n}(r)F(r)$, we have

$$A_{M,n}(r) \ll a^n n!^{1+\varepsilon_M} \quad \text{and} \quad R_{M,n}(r) \ll b^n/n!^{1-\varepsilon_M}.$$

If we could prove $|R_{M,n}(r)| \gg c^n/n!^{1-\varepsilon_M}$, $\mu(F(r)) \leq 2$ would follow by taking n, then M, large enough (as for e). But we can't prove that.

- We then proceed as Siegel and Shidlovskii, and construct other "independent" approximations, using the differential system satisfied by ${}^t(1, F, F')$.

- Crucial, and very difficult, is the proof that a certain matrix has maximal rank (Shidlovskii-type lemma).We use our generalization of Bertrand-Beukers' 1985 multiplicity estimate to Nilsson-Gevrey series.

Beyond Theorem 3

• The graded Padé construction can be carried over a number field $\mathbb{K}$ in a straightforward way. But we cannot prove that $\mu(F(\alpha)) = 2$ when $\alpha \in \mathbb{K}$, $F \in \mathbb{K}[[z]]$ and $F(\alpha) \notin \mathbb{Q}$. It is not even possible to deduce that $\mu(F(\alpha))$ is finite (but it is a consequence of our other result); this is the same difficulty as with Shidlovskii's construction.

• We proved in 2016 that if $e^{\alpha} = F(r)$ for $r \in \mathbb{Q}$ and an E-function $F \in \mathbb{Q}[[z]]$, then $\alpha \in \mathbb{Q}$.

Hence, Theorem 3 can not be applied directly to prove that $\mu(e^{\sqrt{2}}) = 2$, which remains conjectural.

• Nonetheless:

Kappe with $d = 2$: $\mu(e^{\sqrt{2}}) \leq 12$.

Eq. (7) with $d = 2$: $\mu(e^{\sqrt{2}}) \leq 8$.

Zudilin with $F(z) := e^{\sqrt{2}z} + e^{-\sqrt{2}z} \in \mathbb{Q}[[z]]$: $\mu(F(1)) = 2$ hence $\mu(e^{\sqrt{2}}) \leq 4$.