Arithmetic theory of $E$ and $G$-operators

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We fix an embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}$.

**Definition 1**

A G-function is a formal power series $G(z) = \sum_{n=0}^{\infty} a_n z^n$ such that $a_n \in \overline{\mathbb{Q}}$ and there exists $C > 0$ such that:

(i) the maximum of the moduli of the conjugates of $a_n$ is $\leq C^{n+1}$ for any $n$.

(ii) there exists a sequence of rational integers $d_n \neq 0$, with $|d_n| \leq C^{n+1}$, such that $d_n a_m$ is an algebraic integer for all $m \leq n$.

(iii) $G(z)$ satisfies a homogeneous linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$.

An E-function $E(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$ is defined similarly.
Properties of $E$ and $G$-functions

A $G$-function is not entire, unless it is a polynomial, but it is always holomorphic at $z = 0$. The set of $G$-functions is a ring (for the Cauchy product), stable by derivation and integration, it contains algebraic functions (over $\overline{\mathbb{Q}}(z)$) holomorphic at $z = 0$ and $\log(1 - z)$ for instance. Its group of units is formed by the algebraic functions holomorphic and non zero at $z = 0$ (André).

An $E$-function is an entire function. The set of $E$-functions is a ring (for the Cauchy product), stable by derivation and integration, it contains the exponential function and the Bessel functions for instance. Its units are of the form $\alpha \exp(\beta z)$, where $\alpha \in \overline{\mathbb{Q}}^*$ and $\beta \in \overline{\mathbb{Q}}$ (André).

The intersection of both classes is reduced to polynomial functions.
Three sets of numbers related to $E$ and $G$-functions

Definition 2

(i) The set $E$ is the set of all the values taken at algebraic points by $E$-functions.
   It is a ring. Its group of units contains $\mathbb{Q}^\ast \exp(\mathbb{Q})$.

(ii) The set $G$ is the set of all the values taken at algebraic points by (analytic continuation of) $G$-functions.
    It is a ring. Its group of units contains $\mathbb{Q}^\ast$ and the Beta values $B(\mathbb{Q}, \mathbb{Q})$.

(iii) The set $S$ is the module generated over $G$ by all the values of derivatives of the Gamma function at rational points.
    It is also the module generated over $G[\gamma]$ by all the values of $\Gamma$ at rational points, where $\gamma$ is Euler’s constant.
    It is a ring.
André-Chudnovski-Katz Theorem, G-operator

Given a $G$-function $G(z)$, consider the minimal linear differential equation $My = 0$ of order $\eta$ and with coefficients in $\mathbb{Q}[z]$, of which $G(z)$ is a solution. Let $\xi_1, \ldots, \xi_p$ denote the singularities of the operator $M$ at finite distance. Then,

- $M$ is globally fuchsian, with rational exponents at each $\xi_j$ and at $\infty$.

- In $\mathbb{C}$ minus (fixed) cuts with the $\xi_j's$ for origin, $M$ has a local basis of solutions $F_1(z), \ldots, F_\eta(z)$ at $z = \xi \in \mathbb{Q}$ such that

$$F_k(z) = \sum_{s \in S_k} \sum_{t \in T_k} \alpha_{s,t,k} \log (z - \xi)^s (z - \xi)^t G_{s,t,k}(z - \xi)$$

where $S_k \subset \mathbb{N}$ and $T_k \subset \mathbb{Q}$ are finite, $\alpha_{s,t,k} \in \mathbb{Q}$, and if $\xi \neq \xi_k$, $S_k = T_k = \{0\}$.

$G_{s,t,k}(z)$ are holomorphic at $z = 0$; and they are $G$-functions.

- If $\xi = \infty$, the same result holds provided we replace $z - \xi$ by $1/z$ everywhere.

$M$ is called a $G$-operator.
Let $G(z)$ be a $G$-function solution of the minimal differential equation $My(z) = 0$ of order $\eta$.

Locally around $\alpha \in \overline{\mathbb{Q}} \cup \{\infty\}$, we have

$$G(z) = \omega_1 F_1(z) + \cdots + \omega_\eta F_\eta(z).$$

where $F_1(z), \ldots, F_\eta(z)$ are given by the André-Chudnovski-Katz theorem, and $\omega_1, \ldots, \omega_\eta$ are certain complex numbers.

**Theorem 1 (Fischler-R, 2012)**

(i) The connection constants $\omega_1, \ldots, \omega_\eta$ belong to $G$.

(ii) A number $\xi$ is in $G$ if and only if $\xi = G(1)$, where $G$ is a $G$-function with coefficients in $\mathbb{Q}(i)$, whose radius of convergence can be as large as a priori wished.

**Corollary 1**

$G$ is a ring.
Theorem 1(\textit{ii}) for $E$-functions?

Given $\xi \in E$, can we always find an $E$-function $E(z)$ with coefficients in $\mathbb{Q}(i)$ such that $\xi = E(1)$?

No.

\textbf{Theorem 2}

An $E$-function with coefficients in a number field $K$ takes at an algebraic point $\alpha$ either a transcendental value or a value in $K(\alpha)$.

In particular, there is no $E$-function $E(z) \in \mathbb{Q}[[z]]$ such that $E(1) = \sqrt{2}$.

This theorem is due to the referee of our 2012 paper in the case $K = \mathbb{Q}(i)$ and $\alpha = 1$, but his proof can be easily generalized. It is based on Beukers’ refinement of the Siegel-Shidlovskii theorem.
Let $Y(z) = ^t(E_1(z), \ldots, E_n(z))$ be a vector of $E$-functions solution of a differential system $Y'(z) = M(z)Y(z)$ where $M(z) \in M_n(\overline{\mathbb{Q}}(z))$. Let $T(z)$ be the least common denominator of the entries of $M(z)$.

- **Siegel-Shidlovskii** (1929, 1956). For any $\alpha \in \overline{\mathbb{Q}}$ such that $\alpha T(\alpha) \neq 0$

  $$\degtr_{\overline{\mathbb{Q}}(z)}(E_1(z), \ldots, E_n(z)) = \degtr_{\overline{\mathbb{Q}}}(E_1(\alpha), \ldots, E_n(\alpha)).$$

- **Nesterenko-Shidlovskii** (1996). There exists a finite set $S$ such that for any $\alpha \in \overline{\mathbb{Q}}$, $\alpha \not\in S$, the following holds. For any $P \in \overline{\mathbb{Q}}[X_1, \ldots, X_n]$ such that $P(E_1(\alpha), \ldots, E_n(\alpha)) = 0$, there exists $Q \in \overline{\mathbb{Q}}[Z, X_1, \ldots, X_n]$ such that $Q(\alpha, X_1, \ldots, X_n) = P(X_1, \ldots, X_n)$ and $Q(z, E_1(z), \ldots, E_n(z)) = 0$.

- **Beukers** (2006). We have $S \subset \{\alpha \in \overline{\mathbb{Q}} : \alpha T(\alpha) \neq 0\}$.

- The analogue of the Siegel-Shidlovskii theorem for $G$-functions is false in general (André, Beukers, $n = 2$). It is believed that the polynomial relations between values of $G$-functions are described by the “Period Conjecture” of Grothendieck, through the Bombieri-Dwork Conjecture (ie, $G$-functions come from geometry).
E-operators

Definition 3 (André, 2000)

A differential operator \( L \in \overline{Q}[x, \frac{d}{dx}] \) is an \( E \)-operator if the operator \( M \in \overline{Q}[z, \frac{d}{dz}] \) obtained from \( L \) by formally changing

\[
x \rightarrow -\frac{d}{dz}, \quad \frac{d}{dx} \rightarrow z \quad \text{(Fourier-Laplace transform of } L)\]

is a \( G \)-operator, i.e. \( My(z) = 0 \) has at least one \( G \)-function solution for which it is minimal.

Motivation: Given an \( E \)-function \( E(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n \), there exists an \( E \)-operator \( L \), of order \( \mu \) say, such that \( LE(x) = 0 \). Moreover, let

\[
g(z) = \int_{0}^{\infty} E(x) e^{-xz} dx = \sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}} \quad \text{(Laplace transform of } E)\).

Then

\[
\left( \left( \frac{d}{dz} \right)^\mu \circ M \right) g(z) = 0.
\]
Basis of solutions of $L$ at $z = 0$

Theorem 3 (André, 2000)

(i) An $E$-operator has at most 0 and $\infty$ as singularities: 0 is always a regular singularity, while $\infty$ is an irregular one in general.

(ii) An $E$-operator $L$ of order $\mu$ has a basis of solutions at $z = 0$ of the form

$$(E_1(z), \ldots, E_\mu(z)) \cdot z^M$$

where $M$ is an upper triangular $\mu \times \mu$ matrix with coefficients in $\mathbb{Q}$ and the $E_j(z)$ are $E$-functions.

Any local solution $F(z)$ of $Ly(z) = 0$ at $z = 0$ is of the form

$$F(z) = \sum_{j=1}^{\mu} \left( \sum_{s \in S_j} \sum_{k \in K_j} \phi_{j,s,k} z^s \log(z)^k \right) E_j(z)$$

where $S_j \subset \mathbb{Q}$, $K_j \subset \mathbb{N}$ are finite and $\phi_{j,s,k} \in \mathbb{C}$.

**Interesting case for us:** $\phi_{j,s,k} \in \overline{\mathbb{Q}}$. 
Connection constants at finite distance

Let $F(z)$ be a local solution of $Ly(z) = 0$ at $z = 0$, of the form given in (1).

Any point $\alpha \in \overline{\mathbb{Q}} \setminus \{0\}$ is a regular point of $L$.

There exists a basis of local solutions $F_1(z), \ldots, F_\mu(z) \in \overline{\mathbb{Q}}[[z - \alpha]]$, holomorphic around $z = \alpha$, such that

$$F(z) = \omega_1 F_1(z) + \cdots + \omega_\mu F_\mu(z) \quad (2)$$

where $\omega_1, \ldots, \omega_\mu$ are connection constants.

**Theorem 4 (F-R, 2014)**

*If $\phi_{j,s,k} \in \overline{\mathbb{Q}}$ in (1), then $\omega_1, \ldots, \omega_\mu$ belong to $\mathbb{E}[\log \alpha]$, and even to $\mathbb{E}$ if $F(z)$ is an $E$-function.*

Proof: Differentiate $\mu - 1$ times (2) to construct a $\mu \times \mu$ linear system with the $\omega_j$'s as unknown. Solve it at $z = \alpha$ using the wronskian built on the $F_j$'s (Cramer's rule). Use in particular the fact that, by André's result on singularities of $E$-operators, the wronskian $= cz^\rho e^{\beta z}$ with $c \in \overline{\mathbb{Q}}^*$, $\rho \in \mathbb{Q}$ and $\beta \in \overline{\mathbb{Q}}$. 
Basis of solutions of $L$ at $z = \infty$

The situation is more complicated because of divergent asymptotic series and of Stokes’ phenomenon.

Let $\theta \in [0, 2\pi)$ not in some explicit finite set which contains the anti-Stokes directions. We have a generalized asymptotic expansion

$$E(z) \sim \sum_{\rho \in \Sigma} e^{\rho z} \sum_{\alpha \in S} z^{\alpha} \sum_{i \in T} \log(z)^i \sum_{n=0}^{\infty} \frac{c_{\theta, \rho, \alpha, i}(n)}{z^n}$$

as $|z| \to \infty$ in a large angular sector bisected by $\{z : \arg(z) = \theta\}$.

The sets $\Sigma \subset \overline{Q}$, $S \subset Q$ and $T \subset \mathbb{N}$ are finite, and $c_{\theta, \rho, \alpha, i}(n) \in \mathbb{C}$.

We have found a new explicit construction of (3) by deforming the integral

$$E(x) = \frac{1}{2i\pi} \int_L g(z)e^{zx}\,dz \quad (L \text{ “vertical”}).$$
The series \( \sum_{n=0}^{\infty} c_{\theta, \rho, \alpha, i}(n)z^{-n} \) in (3) are divergent, but

\[
\sum_{n=0}^{\infty} \frac{1}{n!} c_{\theta, \rho, \alpha, i}(n)z^n
\]

are finite linear combinations of \( G \)-functions.

\textbf{André (2000):} Construction of a special basis \( H_1(z), \ldots, H_\mu(z) \) of formal solutions at infinity of the \( E \)-operator \( L \) that annihilates \( E(z) \). Each \( H_k \) involves series like in (3) but with coefficients in \( c_k \overline{\mathbb{Q}} \) for some \( c_k \).

The asymptotic expansion (3) of \( E(z) \) in a large sector bisected by \( \{ z : \arg(z) = \theta \} \) can be rewritten with this basis as

\[
\omega_{\theta, 1} H_1(z) + \cdots + \omega_{\theta, \mu} H_\mu(z) \tag{4}
\]

with \textbf{Stokes’ constants} \( \omega_{\theta, k} \).

When \( \theta \) “crosses” one of the anti-Stokes directions, the values of the \( \omega_{\theta, k} \) may change. This is the Stokes phenomenon.
Stokes’ constants at infinity

Setting:

\[ E(z) \sim \omega_{\theta,1} H_1(z) + \cdots + \omega_{\theta,\mu} H_{\mu}(z) \]

\[ \sim \sum_{\rho \in \Sigma} e^{\rho z} \sum_{\alpha \in S} z^\alpha \sum_{i \in T} \log(z)^i \sum_{n=0}^{\infty} \frac{c_{\theta,\rho,\alpha,i}(n)}{z^n}. \]

**S** is the module generated over \( \mathbb{G}[\gamma] \) by all the values of \( \Gamma \) at rational points.

**Theorem 5 (F-R, 2014)**

Let \( \theta \in [0, 2\pi) \) be a direction not in some explicit finite set. Then:

(i) The Stokes constants \( \omega_{\theta,k} \) belong to \( \mathbb{S} \).

(ii) All the coefficients \( c_{\theta,\rho,\alpha,i}(n) \) belong to \( \mathbb{S} \).

(iii) Let \( F(z) \) be a local solution at \( z = 0 \) of \( L \), with \( \phi_{j,s,k} \in \overline{\mathbb{Q}} \) in (1). Then Assertions (i) and (ii) hold with \( F(z) \) instead of \( E(z) \).
**G-approximations**

**Definition 4**
Sequences \((P_n)\) and \((Q_n)\) of algebraic numbers are said to form G-approximations of \(\alpha \in \mathbb{C}\) if

\[
\lim_{n \to +\infty} \frac{P_n}{Q_n} = \alpha
\]

and the generating functions \(\sum_{n=0}^{\infty} P_n z^n\) and \(\sum_{n=0}^{\infty} Q_n z^n\) are both G-functions.

**Diophantine motivation:** Many sequences of algebraic approximations of classical numbers are G-approximations. For instance, Apéry’s approximations to \(\zeta(2)\) and \(\zeta(3)\).

**Theorem 6 (F-R, 2012)**
The set of numbers having G-approximations is \(\text{Frac } \mathbf{G}\).

Proof: We first show that a number \(\alpha\) having G-approximations is the quotient of two connection constants of the G-operators related to the generating functions, and then we use Theorem 1\((i)\).
**E-approximations**

**Definition 5**

Sequences \((P_n)\) and \((Q_n)\) of algebraic numbers are said to form E-approximations of \(\alpha \in \mathbb{C}\) if

\[
\lim_{n \to +\infty} \frac{P_n}{Q_n} = \alpha
\]

and

\[
\sum_{n=0}^{\infty} P_n z^n = a(z) \cdot E(b(z)) , \quad \sum_{n=0}^{\infty} Q_n z^n = c(z) \cdot F(d(z))
\]

where \(E\) and \(F\) are E-functions, and \(a, b, c, d\) are algebraic functions in \(\overline{\mathbb{Q}}[[z]]\) with \(b(0) = d(0) = 0\).

**Diophantine motivation:** Many sequences of algebraic approximations of classical numbers are E-approximations. For instance diagonal Padé approximants to \(\exp(z)\) evaluated at \(z\) algebraic, and in particular the convergents to \(e\).
The set of $E$-approximable numbers

Given two subsets $X$ and $Y$ of $\mathbb{C}$, let

$$X \cdot Y = \{xy \mid x \in X, y \in Y\}, \quad \frac{X}{Y} = \left\{\frac{x}{y} \mid x \in X, y \in Y \setminus \{0\}\right\}.$$ 

Theorem 7 (F-R, 2014)

The set of numbers having $E$-approximations contains

$$E \cup \Gamma(\mathbb{Q}) \cup \text{Frac } G$$

and it is contained in

$$E \cup (\Gamma(\mathbb{Q}) \cdot G) \cup \left(\Gamma(\mathbb{Q}) \cdot \exp(\overline{\mathbb{Q}}) \cdot \text{Frac } G\right).$$

Proof of (5): Explicit constructions.

Proof of (6): Saddle point method, singularity analysis, and Theorems 4 and 5 because $E$-approximable numbers appear either as connection constants or as Stokes' constants.
\( E \)-approximations of Gamma values

Let

\[
E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!(n+\alpha)}, \quad \alpha \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}
\]

and define \( P_n(\alpha) \) by

\[
\frac{1}{(1-z)^{\alpha+1}} E_\alpha \left( -\frac{z}{1-z} \right) = \sum_{n=0}^{\infty} P_n(\alpha)z^n \in \mathbb{Q}[[z]].
\]

Then,

\[
P_n(\alpha) = \sum_{k=0}^{n} \binom{n+\alpha}{k+\alpha} \frac{(-1)^k}{k!(k+\alpha)} \rightarrow \Gamma(\alpha) \quad \text{if} \quad \alpha < 1.
\]

The number \( \Gamma(\alpha) \) appears as a Stokes constant in the expansion

\[
E_\alpha(-z) \sim \frac{\Gamma(\alpha)}{z^\alpha} - e^{-z} \sum_{n=0}^{\infty} (-1)^n \frac{(1-\alpha)^n}{z^{n+1}}.
\]
What about Euler’s constant $\gamma = -\Gamma'(1)$?

We conjecture that $\gamma$ does not have $E$-approximations, nor $G$-approximations. However, let

$$E(z) = \sum_{n=1}^{\infty} \frac{z^n}{n!n}$$

and define the sequence $(P_n)$ by

$$-\frac{1}{1-z}E\left(-\frac{z}{1-z}\right) + \frac{\log(1-z)}{1-z} = \sum_{n=0}^{\infty} P_n z^n \in \mathbb{Q}[\lbrack z \rbrack].$$

Then

$$P_n = \sum_{k=1}^{n} (-1)^k \binom{n}{k} \frac{1}{k} \left(1 - \frac{1}{k!}\right) \longrightarrow \gamma.$$

Again, $\gamma$ appears as a Stokes’ constant in the asymptotic expansion

$$E(-z) \sim -\gamma - \log(z) - e^{-z} \sum_{n=0}^{\infty} (-1)^n \frac{n!}{z^{n+1}}.$$
Linear recurrences related to $\Gamma(\alpha)$ and $\gamma$

$$(n + 3)(n + 3 + \alpha)P_{n+3}(\alpha)$$
$$- (3n^2 + 4n\alpha + 14n + \alpha^2 + 9\alpha + 17)P_{n+2}(\alpha)$$
$$+ (3n + 5 + 2\alpha)(n + 2 + \alpha)P_{n+1}(\alpha)$$
$$- (n + 2 + \alpha)(n + 1 + \alpha)P_n(\alpha) = 0$$

with $P_0(\alpha) = \frac{1}{\alpha}$, $P_1(\alpha) = \frac{1+\alpha+\alpha^2}{\alpha(\alpha+1)}$ and $P_2(\alpha) = \frac{4+5\alpha+6\alpha^2+4\alpha^3+\alpha^4}{2\alpha(\alpha+1)(\alpha+2)}$.

$$(n + 3)^2 P_{n+3} - (3n^2 + 14n + 17)P_{n+2}$$
$$+ (3n + 5)(n + 2)P_{n+1} - (n + 2)(n + 1)P_n = 0$$

with $P_0 = 0$, $P_1 = 0$ and $P_2 = \frac{1}{4}$. 