Introduction to A-hypergeometric functions

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The concept of hypergeometric function was introduced as a concept to encompass many of the existing classical functions such as arcsin, arctan, log etc. However, they also extend the classical functions in a natural way. For example, Riemann discovered their analytic continuation and monodromy group. Equally important, hypergeometric functions occur at many places in mathematics and mathematical physics (both classical and modern). They have been generalised in many senses : the order of the differential equation, the number of variables and q-analogues have been introduced. One important line of generalisation was introduced around 1988 by Gel'fand, Kapranov and Zelevinski. They called their functions A-hypergeometric functions and nowadays they are sometimes referred to as GKZ-hypergeometric functions. We summarise their definition here to emphasize that the underlying data are very combinatorial in nature. Start with a finite set $A \subset \mathbb{Z}^r$. We assume

1. The $\mathbb{Z}$-span of A is $\mathbb{Z}^r$
2. There is a linear form h such that $h(a) = 1$ for all $a \in A$.

Their definition begins with a finite subset $A \subset \mathbb{Z}^r$ consisting of N vectors $\mathbf{a}_1, \ldots, \mathbf{a}_N$ forming a set of rank r and such that

i) The $\mathbb{Z}$-span of $\mathbf{a}_1, \ldots, \mathbf{a}_N$ equals $\mathbb{Z}^r$.

ii) There exists a linear form h on $\mathbb{R}^r$ such that $h(\mathbf{a}_i) = 1$ for all i.

The latter condition ensures that we shall be working in the case of so-called Fuchsian systems. We are also given a vector of parameters $\alpha = (\alpha_1, \ldots, \alpha_r)$ which could be chosen in $\mathbb{C}^r$, but we shall restrict to $\alpha \in \mathbb{R}^r$. The lattice $L \in \mathbb{Z}^N$ of relations consists of all $(l_1, \ldots, l_N) \in \mathbb{Z}^N$ such that $\sum_{i=1}^N l_i \mathbf{a}_i = 0$. The A-hypergeometric equations are a set partial differential equations with independent variables $v_1, \ldots, v_N$. This set consists of two groups. The first are the structure equations

$$\Box_{\mathbf{l}} \Phi := \prod_{l_i > 0} \partial_i^{l_i} \Phi - \prod_{l_i < 0} \partial_i^{|l_i|} \Phi = 0$$

for all $\mathbf{l} = (l_1, \ldots, l_N) \in L$.
The second groups consists of the homogeneity equations.

$$Z(\mathbf{m})\Phi := (m(\mathbf{a}_1)v_1\partial_1 + m(\mathbf{a}_2)v_2\partial_2 + \cdots + m(\mathbf{a}_N)v_N\partial_N - m(\alpha))\,\Phi = 0$$

for all linear forms m on $\mathbb{R}^r$.
In general the A-hypergeometric system is a holonomic system of dimension equal to the $r-1$-dimensional volume of the so-called A-polytope $Q(A)$, which is the convex hull of the endpoints of the $\mathbf{a}_i$. The volume-measure is normalised to 1 for a fundamental $r-1$-simplex of lattice-points in the plane $h(\mathbf{x}) = 1$.
A formal explicit solution can be given quite easily. Choose $\gamma = (\gamma_1, \ldots, \gamma_N)$ such that $\alpha = \gamma_1\mathbf{a}_1 + \cdots + \gamma_N\mathbf{a}_N$. Then a formal solution of the A-hypergeometric system can be given by

$$\Phi_{L,\gamma}(v_1, \ldots, v_N) = \sum_{\mathbf{l}\in L} \frac{\mathbf{v}^{\mathbf{l}+\gamma}}{\Gamma(\mathbf{l}+\gamma+\mathbf{1})}$$

where we use the short-hand notation

$$\frac{\mathbf{v}^{\mathbf{l}+\gamma}}{\Gamma(\mathbf{l}+\gamma+\mathbf{1})} = \frac{v_1^{l_1+\gamma_1} \cdots v_N^{l_N+\gamma_N}}{\Gamma(l_1+\gamma_1+1)\cdots\Gamma(l_N+\gamma_N+1)}.$$

Note that γ is determined modulo $L \otimes \mathbb{R}$.
The function $\Phi_{L,\gamma}$ is a template for all classical hypergeometric functions (Gaussian, Appell, Lauricella, Horn). The study of convergent power series solutions is associated with triangulations of the polytope A and the so-called secondary polytope. Another important feature are the monodromy group of the GKZ-system and the arithmetic properties of the solutions. Remarkably enough the latter question has been addressed by B. Dwork in his book Generalised hypergeometric Functions from about 1990. There he independently developed a general theory for hypergeometric functions which is completely parallel to the development of A-hypergeometric functions. Dwork's arithmetic study is of importance for transcendence and irrationality since all known Siegel G-functions are in fact restrictions of A-hypergeometric functions. The question of monodromy for A-hypergeometric functions has not been solved yet. The case of finite monodromy has been the subject of a recent study by F. Beukers. In the lectures we give an introduction to the basic theory of A-hypergeometric functions and cover as many of the above aspects as possible.

References

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[2] J. Stienstra, GKZ-Hypergeometric Structures, Proceedings of the Summer School Algebraic Geometry and Hypergeometric Functions, Istanbul, June 2005, preprint 2005 ; available at http ://arxiv.org/abs/0511.5351.