Padé type approximants of Hurwitz zeta function $\zeta(4, x)$

Tanguy Rivoal

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1 Introduction

The Hurwitz zeta function is defined by

$$\zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}$$

for any s, x such that $\mathrm{Re}(s) > 1$ and $x \in \mathbb{R} \setminus \mathbb{Z}_{\le 0}$. For any fixed integer s, $\zeta(s, x)$ is meromorphic in $\mathbb{C} \setminus \mathbb{Z}_{\le 0}$, with poles of order s at each non-negative integer. For any fixed $\varepsilon > 0$, $\zeta(s, x)$ has an asymptotic expansion when $x \to \infty$ in the angular sector $|\arg(x)| < \pi - \varepsilon$:

$$\zeta(s, x) \sim \frac{x^{1-s}}{s-1} + \sum_{k=1}^{\infty} \frac{(s)_{k-1}}{k!} \frac{B_k}{x^{k+s-1}},$$

where $(B_k)_{k \ge 0}$ is the sequence of Bernoulli numbers.

In this paper, we address the following problem: *Given two integers $m, n \ge 0$, determine two polynomials $A(x)$ and $B(x) \in \mathbb{Q}[x]$, of degree $\le n$, such that*

$$A(x)\zeta(4, x) + B(x) = \mathcal{O}\Big(\frac{1}{x^{m+1}}\Big) \tag{1}$$

as $x \to \infty$ in any angular sector $|\arg(x)| < \pi - \varepsilon$.

Stated in this form, this is an analytic problem. However, using the asymptotic expansion of $\zeta(s, x)$ at $x = \infty$, Eq. (1) can also be interpreted as a Padé type problem at $x = \infty$ for the formal series

$$2x^{-3} + \sum_{k=0}^{\infty} (k+2)(k+3) B_{k+1}\, x^{-k-4}.$$

See [10, Sec. 2] for details. This Padé problem amounts to solving a linear system with $2n+2$ indeterminates (the polynomial coefficients) and $m+n+1$ equations (the vanishing conditions): provided $m \le n$, this system has at least one non-zero solution. The case

$m = n$ corresponds to the usual diagonal Padé approximation. The explicit polynomials obtained below are automatically solutions of the associated Padé type problem. Our main result is the explicit determination of a non-zero analytic solution of (1) when $m \leq n/2$ (essentially). Unicity of the solution is obviously not guaranteed.

Theorem 1. *For any integer $n \geq 0$, consider the following Padé type problem: determine two polynomials $Q_{0,n}(x)$ and $Q_{2,n}(x) \in \mathbb{Q}[x]$, of degree $\leq 4n$, such that*

$$S_n(x) := 3Q_{0,n}(x)\zeta(4,x) + Q_{2,n}(x) = \mathcal{O}\Big(\frac{1}{x^{2n+3}}\Big) \tag{2}$$

Problem (2) admits the following solution:

$$S_n(x) = -\sum_{k=0}^{\infty} \frac{\partial}{\partial k}\left(\Big(k+x+\frac{n}{2}\Big)\frac{(k+1)_n^2(k+2x)_n^2}{(k+x)_{n+1}^4}\right). \tag{3}$$

The series converges for all $x \in \mathbb{C} \setminus \{0,-1,-2,-3,\ldots\}$.

Moreover, for the series on the right-hand side of (3), we have

$$Q_{0,n}(x) = \sum_{j=0}^{n} \frac{\partial}{\partial \varepsilon}\left(\Big(\frac{n}{2}-j+\varepsilon\Big)\frac{(x+j-n-\varepsilon)_n^2(x-j+\varepsilon)_n^2}{(1-\varepsilon)_j^4(1+\varepsilon)_{n-j}^4}\right)_{|\varepsilon=0}$$

and

$$Q_{2,n}(x) = \\ -\frac{1}{6}\sum_{j=0}^{n}\Big(\frac{\partial}{\partial\varepsilon}\Big)^3\left(\Big(\frac{n}{2}-j+\varepsilon\Big)\frac{(x+j-n-\varepsilon)_n^2(x-j+\varepsilon)_n^2}{(1-\varepsilon)_j^4(1+\varepsilon)_{n-j}^4}\sum_{k=0}^{j-1}\frac{1}{(x+k-\varepsilon)^2}\right)_{|\varepsilon=0}.$$

Diagonal Padé approximants are known for $\zeta(2,x)$ and $\zeta(3,x)$: the formulas are given in [8] and [10, Theorem 2]. However diagonal Padé approximants are not explicitely known for $\zeta(4,x)$ and Theorem 1 offers a weaker alternative. The polynomial $Q_{0,n}(x)$ can also be written more symbolically in the form

$$Q_{0,n}(x) = \sum_{j=0}^{n}\frac{\partial}{\partial j}\left(\Big(\frac{n}{2}-j\Big)\binom{n}{j}^4\binom{x+j-1}{n}^2\binom{x+n-j-1}{n}^2\right).$$

We now let

$$Q_{1,n}(x) = \\ -\frac{1}{6}\sum_{j=0}^{n}\Big(\frac{\partial}{\partial\varepsilon}\Big)^3\left(\Big(\frac{n}{2}-j+\varepsilon\Big)\frac{(x+j-n-\varepsilon)_n^2(x-j+\varepsilon)_n^2}{(1-\varepsilon)_j^4(1+\varepsilon)_{n-j}^4}\sum_{k=0}^{j-1}\frac{1}{x+k-\varepsilon}\right)_{|\varepsilon=0}$$

which is also a polynomial of degree $\leq 4n$. Then, our proof will also show that

$$R_n(x) := Q_{0,n}(x)\zeta(3,x) + Q_{1,n}(x) = \sum_{k=0}^{\infty} \Big(k + x + \frac{n}{2}\Big) \frac{(k+1)_n^2 (k+2x)_n^2}{(k+x)_{n+1}^4}.$$

As $x \to \infty$,

$$\Big(k + x + \frac{n}{2}\Big) \frac{(k+1)_n^2 (k+2x)_n^2}{(k+x)_{n+1}^4} \sim \frac{4^n (k+1)_n^2}{x^{2n+3}} \tag{4}$$

and this suggests that $R_n(x) = \mathcal{O}(\frac{1}{x^{2n+3}})$ as $S_n(x)$. However, this is false and in fact one can not prove anything better than $R_n(x) = \mathcal{O}(\frac{1}{x^2})$. Therefore, the property $S_n(x) = \mathcal{O}(\frac{1}{x^{2n+3}})$ is a non-trivial property, which is not a simple consequence of (4).

We give the proof of Theorem 1 in Section 2 while we make connections with other results in the literature in Section 3.

I warmly thank Pierre Bel for his careful reading of a previous version of this paper.

2 Proof of Theorem 1

We follow the method used in [10] and split the proof in three parts. We also include the case of $\zeta(3,x)$ in the first part of the proof.

2.1 Linear forms in $1, \zeta(3,x)$, respectively $1, \zeta(4,x)$

We define the rational function

$$\rho(t) = \Big(t + x + \frac{n}{2}\Big) \frac{(t+1)_n^2 (t+2x)_n^2}{(t+x)_{n+1}^4}.$$

By partial fraction expansion, we have

$$\rho(t) = \sum_{s=1}^{4} \sum_{j=0}^{n} \frac{E_{j,n,s}(x)}{(t+x+j)^s}$$

with

$$E_{j,n,s}(x) = \frac{1}{(4-s)!} \Big(\frac{\partial}{\partial t}\Big)^{4-s} \big(\rho(t)\,(t+x+j)^4\big)_{|k=-j-x}.$$

Exchanging summations, we thus get

$$\sum_{k=0}^{\infty} \big(k + x + \frac{n}{2}\big) \frac{(k+1)_n^2 (k+2x)_n^2}{(k+x)_{n+1}^4} = \sum_{s=2}^{4} \Big(\sum_{j=0}^{n} E_{j,n,s}(x)\Big)\zeta(s,x) - \sum_{s=1}^{4} \sum_{j=0}^{n} \sum_{k=0}^{j-1} \frac{E_{j,n,s}(x)}{(k+x)^s}. \tag{5}$$

Here we must explain how we have disposed of the divergent series $\sum_{k=0}^{\infty}\frac{1}{k+x}$ in (5), i.e. why the first sum over s does not start at $s=1$. The series on the left-hand side of (5) being convergent, this forces to assign the value $-\sum_{j=0}^{n}E_{j,n,1}(x))\sum_{k=0}^{j-1}\frac{1}{k+x}$ to the divergent expression $(\sum_{j=0}^{n}E_{j,n,1}(x))\sum_{k=0}^{\infty}\frac{1}{k+x+j}$. Indeed we have $\sum_{j=0}^{n}E_{j,n,1}(x)=0$ because this is the sum over the residues at all the finite poles of $\rho(k)$, hence also equal to minus its residue at infinity, which is zero. Formally, one should introduce the Lerch series $\sum_{k=0}^{\infty}\frac{z^k}{(k+x)^s}$ with $|z|<1$ and eventually to let $z\to 1$; see [10, Sec. 3.2] for details.

We now observe that $\rho(k)=-\rho(-k-2x-n)$. Since

$$\rho(-k-2x-n)=\sum_{s=1}^{4}\sum_{j=0}^{n}\frac{E_{j,n,s}(x)}{(-k-x-n+j)^s}=\sum_{s=1}^{4}\sum_{j=0}^{n}(-1)^s\frac{E_{n-j,n,s}(x)}{(k+x+j)^s},$$

we then deduce that $E_{n-j,n,s}(x)=(-1)^{s+1}E_{j,n,s}(x)$. Therefore

$$\sum_{j=0}^{n}E_{j,n,s}(x)=(-1)^{s+1}\sum_{j=0}^{n}E_{j,n,s}(x)$$

which is thus equal to 0 for $s=2$ and $s=4$, and consequently, the first sum in (5) is for $s=3$ only:

$$\sum_{k=0}^{\infty}\big(k+x+\frac{n}{2}\big)\frac{(k+1)_n^2(k+2x)_n^2}{(k+x)_{n+1}^4}=\Big(\sum_{j=0}^{n}E_{j,n,3}(x)\Big)\zeta(3,x)-\sum_{s=1}^{4}\sum_{j=0}^{n}\sum_{k=0}^{j-1}\frac{E_{j,n,s}(x)}{(k+x)^s}. \tag{6}$$

Similarly,

$$\begin{aligned}-\sum_{k=0}^{\infty}\bigg(\frac{\partial}{\partial k}\big(k{+}x+\frac{n}{2}\big)\frac{(k+1)_n^2(k+2x)_n^2}{(k+x)_{n+1}^4}\bigg)&=-\sum_{s=1}^{4}\sum_{j=0}^{n}E_{j,n,s}(x)\sum_{k=0}^{\infty}\frac{\partial}{\partial k}\frac{1}{(k+x+j)^s}\\&=\sum_{s=1}^{4}\Big(\sum_{j=0}^{n}E_{j,n,s}(x)\Big)s\zeta(s+1,x)-\sum_{s=1}^{4}\sum_{j=0}^{n}\sum_{k=0}^{j-1}\frac{sE_{j,n,s}(x)}{(k+x)^{s+1}}\end{aligned}$$

so that

$$\begin{aligned}-\sum_{k=0}^{\infty}\bigg(\frac{\partial}{\partial k}\big(k+x+\frac{n}{2}\big)\frac{(k+1)_n^2(k+2x)_n^2}{(k+x)_{n+1}^4}\bigg)&\\=\Big(\sum_{j=0}^{n}E_{j,n,3}(x)\Big)3\zeta(4,x)-\sum_{s=1}^{4}\sum_{j=0}^{n}\sum_{k=0}^{j-1}\frac{sE_{j,n,s}(x)}{(k+x)^{s+1}}.\end{aligned} \tag{7}$$

2.2 The coefficients are polynomials of degree $\le 4n$

We set $Q_{0,n}(x) := \sum_{j=0}^{n} E_{j,n,3}(x)$ and

$$Q_{1,n}(x) := -\sum_{s=1}^{4}\sum_{j=0}^{n}\sum_{k=0}^{j-1}\frac{E_{j,n,s}(x)}{(k+x)^s}, \quad Q_{2,n}(x) := -\sum_{s=1}^{4}\sum_{j=0}^{n}\sum_{k=0}^{j-1}\frac{sE_{j,n,s}(x)}{(k+x)^{s+1}}$$

so that the right-hand sides of (6) and (7) are respectively equal to

$$Q_{0,n}(x)\zeta(3,x)+Q_{1,n}(x) \quad \text{and} \quad 3Q_{0,n}(x)\zeta(4,x)+Q_{2,n}(x).$$

Let us prove that for $s\in\{0,1,2\}$, the $Q_{s,n}(x)$ are in $\mathbb{Q}[x]$ and of degree $\le 4n$. We have

$$\begin{aligned} Q_{0,n}(x) &= \sum_{j=0}^{n}\frac{\partial}{\partial k}\big(\rho(k)\,(k+x+j)^4\big)_{|k=-j-x} = \sum_{j=0}^{n}\frac{\partial}{\partial \ell}\big(\ell^4\rho(\ell-j-x)\big)_{|\ell=0} \\ &= \sum_{j=0}^{n}\frac{\partial}{\partial \ell}\bigg(\Big(\frac{n}{2}-j+\ell\Big)\frac{(x+j-n-\ell)_n^2(x-j+\ell)_n^2}{(1-\ell)_j^4(1+\ell)_{n-j}^4}\bigg)_{|\ell=0}. \end{aligned} \tag{8}$$

Eq. (8) shows that $Q_{0,n}(x)\in\mathbb{Q}[x]$ and $\deg(Q_{0,n})\le 4n$. Furthermore, by Leibniz' formula

$$\begin{aligned} Q_{1,n}(x) = -\sum_{s=1}^{4}\sum_{j=0}^{n}\bigg[&\frac{1}{(4-s)!}\Big(\frac{\partial}{\partial \ell}\Big)^{4-s}\big(\ell^4\rho(\ell-j-x)\big) \\ &\times\frac{1}{(s-1)!}\Big(\frac{\partial}{\partial \ell}\Big)^{s-1}\bigg(\sum_{k=0}^{j-1}\frac{1}{k-\ell+x}\bigg)\bigg]_{|\ell=0} \\ = -\frac{1}{6}\sum_{j=0}^{n}&\Big(\frac{\partial}{\partial \ell}\Big)^3\bigg(\ell^4\rho(\ell-j-x)\sum_{k=0}^{j-1}\frac{1}{k-\ell+x}\bigg)_{|\ell=0} \\ = -\frac{1}{6}\sum_{j=0}^{n}&\Big(\frac{\partial}{\partial \ell}\Big)^3\bigg(\Big(\frac{n}{2}-j+\ell\Big)\frac{(x+j-n-\ell)_n^2(x-j+\ell)_n^2}{(1-\ell)_j^4(\ell+1)_{n-j}^4}\sum_{k=0}^{j-1}\frac{1}{k-\ell+x}\bigg)_{|\ell=0} \end{aligned}$$

and similarly,

$$\begin{aligned} Q_{2,n}(x) = -\sum_{s=1}^{4}\sum_{j=0}^{n}\bigg[&\frac{(-1)^{4-s}}{(4-s)!}\Big(\frac{\partial}{\partial \ell}\Big)^{4-s}\big(\ell^4R(\ell-j-x)\big) \\ &\times\frac{1}{(s-1)!}\Big(\frac{\partial}{\partial \ell}\Big)^{s-1}\bigg(\sum_{k=0}^{j-1}\frac{1}{(x+k-\ell)^2}\bigg)\bigg]_{|\ell=0} \\ = -\frac{1}{6}\sum_{j=0}^{n}&\Big(\frac{\partial}{\partial \ell}\Big)^3\bigg(\ell^4R(\ell-j-x)\sum_{k=0}^{j-1}\frac{1}{(k-\ell+x)^2}\bigg)_{|\ell=0} \\ = -\frac{1}{6}\sum_{j=0}^{n}&\Big(\frac{\partial}{\partial \ell}\Big)^3\bigg(\Big(\frac{n}{2}-j+\ell\Big)\frac{(x+j-n-\ell)_n^2(x-j+\ell)_n^2}{(1-\ell)_j^4(\ell+1)_{n-j}^4}\sum_{k=0}^{j-1}\frac{1}{(x+k-\ell)^2}\bigg)_{|\ell=0}. \end{aligned}$$

It follows that $Q_{1,n}(x)$ and $Q_{2,n}(x)$ are in $\mathbb{Q}[x]$ and of degree $\leq 4n$, because for all $j \in \{0,\dots,n\}$, $k \in \{0,\dots,j-1\}$ and any ℓ, we have that

$$\frac{(x+j-n-\ell)_n}{x+k-\ell} \in \mathbb{Q}[x].$$

2.3 Proof that $S_n(x) = \mathcal{O}\big(\frac{1}{x^{2n+3}}\big)$

We shall prove that $S_n(x) = \mathcal{O}\big(\frac{1}{x^{2n+3}}\big)$ as $x \to \infty$ in any open angular sector that does not contain the negative real axis. The methods of [10, Sec. 3.1] can not be used here because they lead to divergent series.

We first observe that it is enough to consider the case $x \to +\infty$ on the real axis. Indeed, $S_n(x) = 3Q_{0,n}(x)\zeta(4,x) + Q_{2,n}(x)$ so that we know a priori that $S_n(x)$ has an asymptotic expansion in any angular sector that does not contain the negative real axis. Thus the leading power of this expansion can be determined by letting $x \to +\infty$ on the real positive axis.

Let $N \geq 0$ be an integer. We assume that $x \geq 1$. Consider the positively oriented square C_N with sides $[-\frac{1}{2} - iN, N + \frac{1}{2} - iN]$, $[N + \frac{1}{2} - iN, N + \frac{1}{2} + iN]$, $[N + \frac{1}{2} + iN, -\frac{1}{2} + iN]$, $[-\frac{1}{2} + iN, -\frac{1}{2} - iN]$. Then by the residues theorem,

$$\frac{1}{2i\pi}\int_{C_N} \frac{\pi^2}{\sin(\pi t)^2}\rho(t)dt = \sum_{k=0}^{N} \frac{\partial}{\partial k}\left(\big(k+x+\frac{n}{2}\big)\frac{(k+1)_n^2(k+2x)_n^2}{(k+x)_{n+1}^4}\right).$$

(The only poles of the integrand inside C_N are those of $\frac{\pi^2}{\sin(\pi t)^2}$ because $x \geq 1$.)

For any fixed real number u, $\frac{\pi^2}{\sin(\pi(u+iv))^2} = \mathcal{O}(e^{-2\pi|v|})$ as the real number $v \to \pm\infty$, and $\rho(t) = \mathcal{O}(1/t^3)$ as $t \to \infty$. Letting $N \to +\infty$, it follows that

$$\frac{1}{2i\pi}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\pi^2}{\sin(\pi t)^2}\rho(t)dt = -\sum_{k=0}^{\infty} \frac{\partial}{\partial k}\left(\big(k+x+\frac{n}{2}\big)\frac{(k+1)_n^2(k+2x)_n^2}{(k+x)_{n+1}^4}\right) = S_n(x).$$

Then for any $t \in [-\frac{1}{2} - i\infty, -\frac{1}{2} + i\infty]$ and any $x \geq 1$, we have $|x^{2n+3}\rho(t)| \leq c_n|(t+1)_n^2|$ for some constant $c_n > 0$ independent of x and t. Since $\int_{-1/2+i\mathbb{R}} |\frac{\pi^2}{\sin(\pi t)^2}(t+1)_n^2||dt| < \infty$ for any n, and $\lim_{x\to+\infty} x^{2n+3}\rho(t) = 4^n(t+1)_n^2$, it follows by the dominated convergence theorem that

$$\lim_{x\to+\infty} x^{2n+3}S_n(x) = \frac{4^n}{2i\pi}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\pi^2}{\sin(\pi t)^2}(t+1)_n^2 dt.$$

Similarly, we can prove that

$$\frac{1}{2i\pi}\int_{-\frac{1}{2}+i\infty}^{-\frac{1}{2}-i\infty} \pi\cot(\pi t)\rho(t)dt = \sum_{k=0}^{\infty} \big(k+x+\frac{n}{2}\big)\frac{(k+1)_n^2(k+2x)_n^2}{(k+x)_{n+1}^4} = R_n(x).$$

However, we can not deduce from this representation that $\lim_{x\to+\infty} x^{2n+3}R_n(x)$ is finite by the method above, because $\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \pi\cot(\pi t)(t+1)_n^2 dt$ is divergent. In fact, it turns out that $\lim_{x\to+\infty} x^{2n+3}R_n(x)$ is not finite when $n \geq 1$, because it can be proved that $\lim_{x\to+\infty} x^2R_n(x)$ is finite and non-zero.

3 Connections with other works

3.1 Cohen's continued fraction for $\zeta(4,x)$

In [6], Cohen presented certain continued fractions for values of the Riemann zeta function and the Gamma function. In particular he stated the following one (in his notations):

$$\zeta(4,x+1) \approx \frac{(2x+1)/3|}{|1P_x(1)} + \frac{1^8 2x(2x+2)|}{|3P_x(2)} + \frac{2^8(2x-1)(2x+3)|}{|5P_x(3)} + \cdots \tag{9}$$

where

$$P_x(\ell) = 2x^4 + 4x^3 + (2\ell^2 - 2\ell + 4)x^2 + (2\ell^2 - 2\ell + 2)x - \ell(\ell-1)(\ell^2 - \ell + 1).$$

He wrote that $\approx$ means "asymptotic expansion as the integer $x \to \infty$", and that it is not an equality.

Maple implementation of Zeilberger's algorithm shows that our sequences $(S_n(x+1))_{n\geq 0}$, $(Q_{0,n}(x+1))_{n\geq 0}$ and $(Q_{2,n}(x+1))_{n\geq 0}$ are solutions of the linear recurrence

$$n^5U_n + (2n-1)P_x(n)U_{n-1} + (n+1)^3(n+2x)(n-2-2x)U_{n-2} = 0.$$

It is then not difficult to prove that $\frac{Q_{2,n}(x+1)}{3Q_{0,n}(x+1)}$ are the convergents of Cohen's continued fraction (9). See also Lange's paper [7] for many continued fractions related to Hurwitz zeta function, though (9) does not seem to be listed.

Cohen then mentioned that Apéry's "continued fraction acceleration" method shows

$$\zeta(4) = \frac{13|}{|C(1)} + \frac{2\cdot 3\cdot 4\cdot 1^7|}{|C(2)} + \frac{5\cdot 6\cdot 7\cdot 2^7|}{|C(3)} + \cdots \tag{10}$$

where

$$C(n) = 3(2n-1)(45n^4 - 90n^3 + 72n^2 - 27n + 4). \tag{11}$$

He also wrote that the convergents $\frac{a_n}{b_n}$ of the continued fraction (10) are such that

$$\zeta(4) - \frac{a_n}{b_n} \approx \frac{c(-1)^n}{(2+\sqrt{3})^{6n}}$$

for some constant $c \neq 0$, which is not enough to prove the irationality of $\zeta(4)$. The continued fraction (10) had been announced before in [4], with details given in [5].

3.2 Zudilin's approximations to $\zeta(4)$

In [12, Section 2], Zudilin showed that for any integer $n \geq 0$

$$Z_n := -\sum_{k=0}^{\infty} \frac{\partial}{\partial k}\left(\left(k+\frac{n}{2}\right)\frac{(k-n)_n^2(k+n+1)_n^2}{(k)_{n+1}^4}\right) = u_n 3\zeta(4) + v_n$$

where u_n and v_n are rational numbers. In particular,

$$u_n = \sum_{j=0}^{n} \frac{\partial}{\partial j}\left(\left(\frac{n}{2}-j\right)\binom{n}{j}^4\binom{n+j}{n}^2\binom{2n-j}{n}^2\right)$$

and the expression for v_n is more complicated. He also proved that $(Z_n)_n$, $(u_n)_n$ and $(v_n)_n$ are solutions of the linear recurrence

$$n^5 U_n + C(n)U_{n-1} - 3(3n-2)(3n-4)(n-1)^3 U_{n-2} = 0$$

where $Q(n)$ is Cohen's polynomial (11). It can be verified that $\frac{u_n}{v_n}$ coincide with the convergents $\frac{a_n}{b_n}$ of (10); see [12, Section 2, Theorem 2].

We observe that $Z_n = S_n(n+1)$ and $u_n = Q_{0,n}(n+1)$. Since $\zeta(4, n+1) = \zeta(4) - \sum_{j=1}^{n} \frac{1}{j^4}$, the specialization of Theorem 1 at $x = n+1$:

$$S_n(n+1) = 3Q_{0,n}(n+1)\zeta(4, n+1) + Q_{2,n}(n+1)$$

becomes

$$Z_n = u_n 3\zeta(4) + Q_{2,n}(n+1) - 3u_n \sum_{j=1}^{n} \frac{1}{j^4}$$

and thus we recover Zudilin's sequence $(v_n)_n$ by the identity

$$v_n = Q_{2,n}(n+1) - 3u_n \sum_{j=1}^{n} \frac{1}{j^4}.$$

3.3 Prévost's remainder Padé approximants for $\zeta(s, x)$

In [8], Prévost showed a very original method to prove the irrationality of $\zeta(2)$ and $\zeta(3)$. We present the slightly modified approach he recently presented in [9]. For any integer $x \geq 1$, we have

$$\zeta(s) = \sum_{k=1}^{x-1} \frac{1}{k^s} + \zeta(s, x).$$

He then computed explicitly the Padé approximants $[n+1/n](x)$ at $x = \infty$ of $\zeta(2, x)$, respectively the Padé approximants $[n+2/n](x)$ at $x = \infty$ of $\zeta(3, x)$. After taking $x = n+1$, he obtained Apéry's famous sequences for $\zeta(2)$ and $\zeta(3)$.

For $s = 2$, the denominators of $[n+1/n](x)$ are

$$P_n(x) = \sum_{j=0}^{n} \binom{n+1}{j+1}\binom{n+j+2}{j+1}\binom{x-1}{j}, \quad n \geq 0,$$

and they satisfy the orthogonality relation

$$\int_{i\mathbb{R}} P_n(x)P_m(x)\frac{x^2}{\sin(\pi x)^2}dx = 0, \quad n \neq m.$$

For $s = 3$, the denominators $Q_n(x)$ of $[n+2/n](x)$ are such that

$$Q_n(x^2) = \sum_{j=0}^{n} \frac{1}{j+1}\binom{n+1}{j+1}\binom{n+j+2}{j+1}\binom{x-1}{j}\binom{x+1}{j}, \quad n \geq 0,$$

and they satisfy the orthogonality relation

$$\int_{i\mathbb{R}} Q_n(x)Q_m(x)\frac{x^5\cos(\pi x)}{\sin(\pi x)^3}dx = 0, \quad n \neq m.$$

The two families of orthogonal polynomials $(P_n)_n$ and $(Q_n)_n$ are specializations of Wilson's orthogonal polynomials [11].

Recently, Prévost [9] proved that the Padé approximants $[n+1/n](x)$ of $\zeta(s,x)$ at $x = \infty$ converge to $\zeta(s,x)$ for any fixed real number $s > 1$, but convergence is still an open problem when s is a complex number. Moreover, except for $s = 2, 3$, no expression of these approximants is known, even for $s = 4$. In this case, the problem is to find explicit expressions for polynomials $A_n(x)$ (of degree n) such that

$$\int_{i\mathbb{R}} A_n(x)A_m(x)\frac{x^8(2+\cos(2\pi x))}{\sin(\pi x)^4}dx = 0, \quad n \neq m.$$

Unfortunately, the weight function $\frac{x^8(2+\cos(2\pi x))}{\sin(\pi x)^4}$ is not of the form studied by Wilson. The sequence $(Q_{0,n}(x))_n$ is not orthogonal for this weight, but it is bi-orthogonal in the following sense: for any n and m such that $0 \leq m \leq 2n-1$, we have

$$\int_{i\mathbb{R}} x^{m+5}Q_{0,n}(x)\,\frac{\cos(\pi x)}{\sin(\pi x)^3}dx = 0 = \int_{i\mathbb{R}} x^{m+8}Q_{0,n}(x)\,\frac{2+\cos(2\pi x)}{\sin(\pi x)^4}dx.$$

3.4 Beukers and Bel's p-adic irrationality proofs

In [3], Calegari proved the irrationality of the 2-adic numbers $\zeta_2(2)$ and $\zeta_2(3)$, as well as of the 3-adic numbers $\zeta_2(3)$. His proof used overconvergent p-adic modular forms. Later, Beukers [2] obtained another proof of these facts, of a more classical flavor. In fact, he essentially used Prévost's Padé approximants for $\zeta(2,x)$ and $\zeta(3,x)$, though his formulas are written differently. The Padé type approximants constructed in [10] for $\zeta(s,x)$ contain as initial cases Beukers and Prévost approximants; Bel [1] used them to prove certain linear independence results for values of p-adic Hurwitz zeta functions. It would be interesting to know if Theorem 1 or its generalization could be used to prove the irrationality of the numbers $\zeta_p(4)$ for some p. The arithmetic and asymptotic properties of Zudilin's series Z_n are not good enough to imply the irrationality of $\zeta(4)$, but a modification of Z_n conjecturally proves that $\zeta(4) \notin \mathbb{Q}$.

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Tanguy Rivoal, Institut Fourier, CNRS et Université Grenoble Alpes, CS 40700, 38058 Grenoble cedex 9, France

tanguy.rivoal@univ-grenoble-alpes.fr