E-FUNCTIONS OF ORDER 2 AND UNITS OF E-VALUES

T. RIVOAL AND J. ROQUES

ABSTRACT. Siegel introduced and studied the class of E-functions in 1929. They are power series, solutions of some linear differential equations, whose Taylor coefficients satisfy certain arithmetic and growth conditions. From the work of Siegel and Shidlovskii, and its refinement by Beukers, we know that E-functions solutions of a given differential system of order 1 satisfy diophantine properties that generalize the Lindemann-Weierstrass Theorem. In this paper, we give the complete classification of the vector solutions of two dimensional differential systems of order 1 whose components are algebraically dependent E-functions over $\overline{\mathbb{Q}}(z)$. As a consequence, we obtain a result that goes in the direction of a positive answer to the following question: is the group of units of the ring of values of E-functions at algebraic points equal to $\overline{\mathbb{Q}}^\times \exp(\overline{\mathbb{Q}})$? Our approach relies on the recent theory of E-operators developped by André, and on Beukers' refinement of the Siegel-Shidlovskii Theorem.

1. Introduction

An E-function is a power series

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n \in \overline{\mathbb{Q}}[[z]]$$

with coefficients in the field of algebraic numbers $\overline{\mathbb{Q}}$ such that

(1) $f(z)$ satisfies a nonzero linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$;
(2) there exists $C > 0$ such that
   (a) the maximum of the moduli of the galoisian conjuguates of a_n is bounded by C^{n+1};
   (b) there exists a sequence of positive integers d_n such that $d_n \leq C^{n+1}$ and $d_n a_m$ is an algebraic integers for all $m \leq n$.

The prototypical example is the exponential function. The class of E-functions was first introduced by Siegel ([1]) to generalize the diophantine properties of $\exp(z)$, in particular the Lindemann-Weierstrass Theorem, which we now recall because it will be used later in this paper.

Theorem 1 (Lindemann-Weierstrass Theorem). *Consider $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{Q}}$.*

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[1]His definition was slightly less restrictive, but it is now believed that both definitions define the same class of functions.

(*i*) *If* $\alpha_1, \dots, \alpha_n$ *are pairwise distinct, then the numbers* $e^{\alpha_1}, \dots, e^{\alpha_n}$ *are* $\overline{\mathbb{Q}}$*-linearly independent.*
(*ii*) *If* $\alpha_1, \dots, \alpha_n$ *are* $\overline{\mathbb{Q}}$*-linearly independent, then the numbers* $e^{\alpha_1}, \dots, e^{\alpha_n}$ *are algebraically independent over* $\overline{\mathbb{Q}}$.

The work of Siegel [11] and Shidlovskii [12] culminated with the following theorem, which can be seen as vast generalization of the Lindemann-Weierstrass Theorem.

Theorem 2 (Siegel-Shidlovskii)**.** *Let* $f_1(z), ..., f_n(z)$ *be* E*-functions such that*

$$\begin{pmatrix} f_1'(z) \\ \vdots \\ f_n'(z) \end{pmatrix} = A(z) \begin{pmatrix} f_1(z) \\ \vdots \\ f_n(z) \end{pmatrix}$$

for some $A(z) \in M_n(\overline{\mathbb{Q}}(z))$. *Denote the common denominator of the entries of* $A(z)$ *by* $T(z)$. *Then, for any* $\xi \in \overline{\mathbb{Q}}$ *such that* $\xi T(\xi) \neq 0$, *we have*

$$\deg\, \mathrm{tr}_{\overline{\mathbb{Q}}}\, \overline{\mathbb{Q}}\big(f_1(\xi), ..., f_n(\xi)\big) = \deg\, \mathrm{tr}_{\overline{\mathbb{Q}}(z)}\, \overline{\mathbb{Q}}(z)\big(f_1(z), ..., f_n(z)\big).$$

An alternative proof of the Siegel-Shidlovskii Theorem was given by Bertrand in [2] using Laurent's determinants.

In the seminal paper [1], André elucidated the structure of "E-operators" by means of their relations with G-operators. Any E-function is in the kernel of an E-operator and we recall in Section 2 some of their properties which are important for the proof of our results. Using these results, André obtained a completely new proof of the Siegel-Shidlovskii Theorem. Beukers [4] was even able to deduce from the work of André the following important refinement of a theorem of Nesterenko and Shidlovskii [8], which is itself a refinement of the above-mentioned Siegel-Shidlovskii Theorem.

Theorem 3 (Beukers)**.** *With the notations and hypothesis of Theorem 2, let us consider* $\xi \in \overline{\mathbb{Q}}$ *such that* $\xi T(\xi) \neq 0$. *For any polynomial relation*

$$P\big(f_1(\xi), ..., f_n(\xi)\big) = 0$$

with $P \in \overline{\mathbb{Q}}[X_1, ..., X_n]$, *there exists* $Q \in \overline{\mathbb{Q}}[z][X_1, ..., X_n]$ *such that*

$$Q\big(f_1(z), ..., f_n(z)\big) = 0 \quad \text{and} \quad P(X_1, ..., X_n) = Q(X_1, ..., X_n)_{|z=\xi}.$$

In order to apply the above transcendence results, the first naive question is: *when are* $f_1(z), ..., f_n(z)$ *algebraically dependent over* $\overline{\mathbb{Q}}(z)$*?* The first main result of this paper gives a complete answer to this question when $n = 2$. In what follows, for any $\gamma \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$, we denote by ${}_1F_1(1; \gamma; z)$ the hypergeometric function (which is an E-function) defined by:

$${}_1F_1(1; \gamma; z) = \sum_{n=0}^{\infty} \frac{z^n}{\gamma(\gamma+1)\cdots(\gamma+n-1)}.$$

Note that ${}_1F_1(1; 1; z) = \exp(z)$.

Theorem 4. *Let $f(z), g(z) \in \overline{\mathbb{Q}}[[z]]$ be E-functions such that*

$$\begin{pmatrix} f'(z) \\ g'(z) \end{pmatrix} = E(z) \begin{pmatrix} f(z) \\ g(z) \end{pmatrix}$$

for some $E(z) \in M_2(\overline{\mathbb{Q}}(z))$. If $f(z)$ and $g(z)$ are algebraically dependent over $\overline{\mathbb{Q}}(z)$, then one of the following cases occurs:

(*i*) *There exist $a(z), b(z), c(z), d(z) \in \overline{\mathbb{Q}}[z, z^{-1}]$ and $\alpha, \beta \in \overline{\mathbb{Q}}$ such that*

$$f(z) = a(z)e^{\alpha z} + b(z)e^{\beta z},$$
$$g(z) = c(z)e^{\alpha z} + d(z)e^{\beta z}.$$

(*ii*) *There exist $a(z), b(z), c(z), d(z) \in \overline{\mathbb{Q}}[z, z^{-1}]$, $\gamma \in \mathbb{Q} \setminus \mathbb{Z}$ and $\alpha \in \overline{\mathbb{Q}}$ such that*

$$f(z) = a(z) {}_1F_1(1; \gamma; \alpha z) + b(z),$$
$$g(z) = c(z) {}_1F_1(1; \gamma; \alpha z) + d(z).$$

Remark. Let $f(z)$ be an E-function such that

$$f'(z) = u(z)f(z) + v(z)$$

for some $u(z) \in \overline{\mathbb{Q}}(z)^{\times}$ and $v(z) \in \overline{\mathbb{Q}}(z)$. In particular, $f(z)$ and $f'(z)$ are algebraically dependent over $\overline{\mathbb{Q}}(z)$. Using Theorem 4, it is easily seen that

$$f(z) = a(z) {}_1F_1(1; \gamma; \alpha z) + b(z)$$

for some $a(z), b(z) \in \overline{\mathbb{Q}}[z, z^{-1}]$, $\gamma \in \{1\} \cup \mathbb{Q} \setminus \mathbb{Z}$ and $\alpha \in \overline{\mathbb{Q}}$. This provides a complete proof of a result suggested by André in [1, Section 4.5], which answers a question asked by Shidlovskii.

We now come to the second main result of this paper. In [5], Fischler and the first author took another point of view at E-functions. They defined the set $\mathbf{E}$ of all the values taken by E-functions at algebraic points. Since E-functions are entire and form a ring, it is immediate that $\mathbf{E}$ is a ring. It is very unlikely that $\mathbf{E}$ is a field and a natural problem is then to determine $\mathbf{E}^{\times}$, the group of units of $\mathbf{E}$. It is trivial that $\overline{\mathbb{Q}}^{\times} \exp(\overline{\mathbb{Q}})$ is a subgroup of $\mathbf{E}^{\times}$. ([2]) It is an open problem to prove or disprove that $\mathbf{E}^{\times} = \overline{\mathbb{Q}}^{\times} \exp(\overline{\mathbb{Q}})$. We prove a result that goes in direction of the equality $\mathbf{E}^{\times} = \overline{\mathbb{Q}}^{\times} \exp(\overline{\mathbb{Q}})$.

Theorem 5. *Let $F(z), G(z) \in \overline{\mathbb{Q}}[[z]]$ be E-functions such that*

$$\begin{pmatrix} F'(z) \\ G'(z) \end{pmatrix} = E(z) \begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

for some $E(z) \in M_2(\overline{\mathbb{Q}}(z))$. Let us assume that $\xi \in \overline{\mathbb{Q}}^{\times}$ is such that

$$F(\xi)G(\xi) = 1.$$

Then $F(\xi)$ and $G(\xi)$ are both in $\overline{\mathbb{Q}}^{\times} \exp(\overline{\mathbb{Q}})$.

[2] André proved in [1] that the units of the ring of E-functions are of the form $\beta e^{\alpha z}$, where $\alpha \in \overline{\mathbb{Q}}$ and $\beta \in \overline{\mathbb{Q}}^{\times}$.

The proof of Theorem 5 uses, as a starting point, Beukers' refinement of the Siegel-Shidlovskii Theorem stated above, which enables us to use Theorem 4.

It would be very interesting to extend Theorem 4 to higher order differential systems. There are immediate difficulties if one tries to generalize our method. For instance, the polynomial relation (3.3) below reduces one case in the proof of Theorem 4 to a linear differential equation of order 1, which is then easily solved; it is not clear how this could be generalized even with only three functions. In [9, 10], Salikhov studied *hypergeometric* E-functions solutions of a linear differential equation of order $n \geq 1$ with coefficients in $\overline{\mathbb{Q}}(z)$ and algebraically dependent of their first $n-1$ derivatives. See [3] for further results in this direction. Theorem 4 shows that hypergeometric functions enable one to describe any E-function solution of a linear differential equation of order at most 2 with coefficients in $\overline{\mathbb{Q}}(z)$ which is algebraically dependent of its first derivative over $\overline{\mathbb{Q}}(z)$. However, hypergeometric functions are not enough for linear differential equations of order 3 already, as shown by the E-function

$$\sum_{n=0}^{\infty} \frac{1 + \frac{1}{2} + \cdots + \frac{1}{n}}{n!} z^n$$

which is solution of the inhomogeneous differential equation

$$zy''(z) + (1-2z)y'(z) + (z-1)y(z) = 1$$

but is not hypergeometric.

2. First steps of the proof of Theorem 4

According to the cyclic vector lemma, there exist a linear differential operator $\mathscr{L}$ of order 2 with coefficients in $\overline{\mathbb{Q}}(z)$, a series $h(z) \in \overline{\mathbb{Q}}[[z]]$ such that $\mathscr{L}h(z) = 0$, and a matrix

$$\begin{pmatrix} p_1(z) & p_2(z) \\ p_3(z) & p_4(z) \end{pmatrix} \in \mathrm{GL}_2(\overline{\mathbb{Q}}(z))$$

such that

$$\begin{aligned} f(z) &= p_1(z)h(z) + p_2(z)h'(z) \\ g(z) &= p_3(z)h(z) + p_4(z)h'(z). \end{aligned}$$

Hence,

$$\begin{aligned} h(z) &= q_1(z)f(z) + q_2(z)g(z) \\ h'(z) &= q_3(z)f(z) + q_4(z)g(z) \end{aligned}$$

where

$$\begin{pmatrix} q_1(z) & q_2(z) \\ q_3(z) & q_4(z) \end{pmatrix} = \begin{pmatrix} p_1(z) & p_2(z) \\ p_3(z) & p_4(z) \end{pmatrix}^{-1} \in \mathrm{GL}_2(\overline{\mathbb{Q}}(z)).$$

Let $\Delta(z) \in \overline{\mathbb{Q}}[z]$ be a common denominator of the $q_i(z)$. Then,

$$k(z) := \Delta(z)h(z) = \Delta(z)q_1(z)f(z) + \Delta(z)q_2(z)g(z)$$

is an E-function (we remind the reader that the set of E-functions is a sub-$\overline{\mathbb{Q}}[z]$-algebra of $\mathbb{Q}[[z]]$), and is a solution of a linear differential operator with coefficients in $\overline{\mathbb{Q}}(z)$ of order 2, namely $\mathscr{L}\Delta^{-1}$.

By André's [1, Theorem 4.3], $k(z)$ is solution of a monic linear differential operator $\mathscr{M}$ with coefficients in $\overline{\mathbb{Q}}(z)$ of order $\nu = 1$ or 2 which is a right factor of an E-operator.

Let us first assume that $\nu = 1$. Then, it is well-known that $k(z) = q(z)e^{\alpha z}$ for some $q(z) \in \overline{\mathbb{Q}}[z]$ and $\alpha \in \overline{\mathbb{Q}}$. Therefore, there exist $a(z), c(z) \in \overline{\mathbb{Q}}(z)$ such that

$$f(z) = a(z)e^{\alpha z},$$
$$g(z) = c(z)e^{\alpha z}.$$

Since $f(z)$ and $g(z)$ are entire functions, we have $a(z), c(z) \in \overline{\mathbb{Q}}[z]$, whence the desired result.

We shall now assume that $\nu = 2$. By André's [1, Theorem 4.3], the differential operator $\mathscr{M}$ has the following properties, that will be freely used in the rest of this paper:

(1) $\mathscr{M}$ has only apparent singularities on $\mathbb{C}^{\times}$ *i.e.* it has a basis of analytic solutions near any $\xi \in \mathbb{C}^{\times}$;
(2) $\mathscr{M}$ is regular singular at 0, and its exponents at 0 are rational;
(3) $\mathscr{M}$ admits a basis of formal solutions at ∞ of the form

$$\left(\widehat{a}_1(z)e^{\alpha_1 z}, \widehat{a}_2(z)e^{\alpha_2 z}\right) = \left(\mathfrak{f}_1(z), \mathfrak{f}_2(z)\right) z^{\Gamma_\infty} e^{\Delta z}$$

where the $\mathfrak{f}_i(z) \in \overline{\mathbb{Q}}[[\frac{1}{z}]]$ are Gevrey-1 series of arithmetic type, $\Gamma_\infty \in M_2(\mathbb{Q})$ is upper-triangular and

$$\Delta = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \in M_2(\overline{\mathbb{Q}}).$$

Let us recall that a Gevrey-1 series of arithmetic type is a power series

$$\mathfrak{f}(z) = \sum_{n=0}^{\infty} n! a_n z^n \in \overline{\mathbb{Q}}[[z]]$$

with coefficients in $\overline{\mathbb{Q}}$ such that

(1) $\mathfrak{f}(z)$ satisfies a nonzero linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$;
(2) there exists $C > 0$ such that
   (a) the maximum of the moduli of the galoisian conjuguates of a_n is bounded by C^{n+1};
   (b) there exists a sequence of positive integers d_n such that $d_n \leq C^{n+1}$ and $d_n a_m$ is an algebraic integers for all $m \leq n$.

The prototypical example is Euler's series

$$\sum_{n=0}^{\infty} n! z^n.$$

To complete the proof of Theorem 4, we shall now consider several cases in the next sections.

3. Proof of Theorem 4 in the case $\alpha_1 \neq \alpha_2$

In what follows, we will freely use several terminologies and results from the resummation theory of linear differential equations; see [6] and the references therein for instance. The set of singular directions of $\mathscr{M}$ is

$$\Sigma := \{\pm d + 2\pi\mathbb{Z}\}$$

where $d = \arg(\alpha_2 - \alpha_1)$; we set $\Sigma = \{d_i, i \in \mathbb{Z}\}$ with $d_i < d_{i+1}$. Moreover, $\mathscr{M}$ is of level 1. Therefore, the $\mathfrak{f}_i(z)$ are 1-sommable in any direction not in Σ. We denote by $\widetilde{\mathbb{C}^\times}$ the Riemann surface of the logarithm, and by $\widetilde{M}_\infty$ the field whose elements are the functions defined and meromorphic on some domain of the form $\{z \in \widetilde{\mathbb{C}^\times} : |z| > R\}$. For any real numbers $\theta_1 < \theta_2$, we consider the angular sector

$$S(\theta_1, \theta_2) = \{z \in \widetilde{\mathbb{C}^\times} : \theta_1 < \arg(1/z) < \theta_2\}.$$

For any $\theta \in (d_i - \frac{\pi}{2}, d_{i+1} + \frac{\pi}{2})$, there exists a unique $\mathfrak{f}_{\mathrm{i},\theta}(z) \in \widetilde{M}_\infty$ which is 1-Gevrey asymptotic to $\mathfrak{f}_{\mathrm{i}}(z)$ at ∞ in any closed sector included in $S(d_i - \frac{\pi}{2}, d_{i+1} + \frac{\pi}{2})$ and which is such that

$$\big(a_{1,\theta}(z)e^{\alpha_1 z}, a_{2,\theta}(z)e^{\alpha_2 z}\big) := \big(\mathfrak{f}_{1,\theta}(z), \mathfrak{f}_{2,\theta}(z)\big) z^{\Gamma_\infty} e^{\Delta z}$$

is a basis of solutions of $\mathscr{M}$.

For any $\theta \in \mathbb{R} \setminus \Sigma$, we let $c_{1,\theta}, c_{2,\theta} \in \mathbb{C}$ be such that ([3])

$$k(z) = c_{1,\theta}a_{1,\theta}(z)e^{\alpha_1 z} + c_{2,\theta}a_{2,\theta}(z)e^{\alpha_2 z}.$$

We consider several cases.

3.1. There exists $\theta \in \mathbb{R} \setminus \Sigma$ such that $c_{1,\theta} = 0$ or $c_{2,\theta} = 0$.

Assume for instance that $c_{2,\theta} = 0$. Then, we have $k(z) = c_{1,\theta}a_{1,\theta}(z)e^{\alpha_1 z}$ and, hence, $k(z)e^{-\alpha_1 z} = c_{1,\theta}a_{1,\theta}(z)$ has at most polynomial growth at infinity on any closed sector included in $S(d_i - \frac{\pi}{2}, d_{i+1} + \frac{\pi}{2})$, where d_i is such that $\theta \in (d_i - \frac{\pi}{2}, d_{i+1} + \frac{\pi}{2})$. Since $k(z)$, and hence $k(z)e^{-\alpha_1 z}$, is an entire function with at most exponential growth of order 1 at infinity, it follows from the Phragmén-Lindelöf theorem that $k(z)e^{-\alpha_1 z}$ has at most polynomial growth at infinity. Therefore, $k(z)e^{-\alpha_1 z} \in \mathbb{C}[z]$. Hence, $k(z)e^{-\alpha z} \in \mathbb{C}[z] \cap \overline{\mathbb{Q}}[[z]] = \overline{\mathbb{Q}}[z]$. It follows that there exist $a(z), c(z) \in \overline{\mathbb{Q}}(z)$ such that

$$f(z) = a(z)e^{\alpha_1 z},$$
$$g(z) = c(z)e^{\alpha_1 z}.$$

Since $f(z)$ and $g(z)$ are entire functions, we have $a(z), c(z) \in \overline{\mathbb{Q}}[z]$, whence the desired result.

3.2. For all $\theta \in \mathbb{R} \setminus \Sigma$, $c_{1,\theta} \neq 0$ and $c_{2,\theta} \neq 0$.

We choose some $\theta \in \mathbb{R} \setminus \Sigma$, for instance in $(d_0 - \frac{\pi}{2}, d_1 + \frac{\pi}{2})$.

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[3]The results of [5] show that $c_{1,\theta}$, $c_{2,\theta}$ (and similar connection constants in other sections) are in a certain "arithmetical" subset $\mathbf{S}$ of $\mathbb{C}$ defined in terms of G-values and Γ-values, but this precision will not used here.

3.2.1. $\alpha_1 \neq 0$ *and* $\alpha_2 \neq 0$. Since $f(z)$ and $g(z)$ are algebraically dependent over $\overline{\mathbb{Q}}(z)$, the functions $h(z)$ and $h'(z)$ are algebraically dependent over $\overline{\mathbb{Q}}(z)$ and, hence, $k(z)$ and $k'(z)$ are algebraically dependent over $\overline{\mathbb{Q}}(z)$. Let $Q(X,Y) \in \overline{\mathbb{Q}}(z)[X,Y] \setminus \{0\}$ be such that

$$Q\big(k(z), k'(z)\big) = 0.$$

Let K be the field extension of $\mathbb{C}(z)$ generated by $a_{1,\theta}(z)e^{\alpha_1 z}$, $a_{2,\theta}(z)e^{\alpha_2 z}$ and their derivatives. Then, K is a Picard-Vessiot extension for $\mathscr{M}$ over $\mathbb{C}(z)$. The differential Galois group G of $\mathscr{M}$ over $\mathbb{C}(z)$ is

$$G = \{\sigma \in \mathrm{Aut}(K/\mathbb{C}(z)) \mid \forall x \in K, \sigma(x)' = \sigma(x')\},$$

i.e. its elements are the field automorphisms of the extension $\mathbb{C}(z) \subset K$ commuting with the action of the usual derivative. For any $\sigma \in G$, we have

$$Q\big(\sigma(k(z)), \sigma(k(z))'\big) = \sigma\Big(Q\big(k(z), k'(z)\big)\Big) = 0. \tag{3.1}$$

A special subgroup of G is given by Ramis' exponential torus, whose elements can be described as follows (see [7]). For any $\chi \in \mathrm{Hom}(\mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2, \mathbb{C}^\times)$ (here, Hom refers to the group homomorphisms), there exists a unique element σ_χ of G whose action on K is determined by

$$\sigma_\chi\big(a_{i,\theta}(z)e^{\alpha_i z}\big) = a_{i,\theta}(z)e^{\alpha_i z}\chi(\alpha_i).$$

Let us now describe $\mathrm{Hom}(\mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2, \mathbb{C}^\times)$. Consider relatively prime relative integers p, q such that $\alpha_1/\alpha_2 = p/q$. Note that $p, q \neq 0$ because $\alpha_1, \alpha_2 \neq 0$ and $p \neq q$ because $\alpha_1 \neq \alpha_2$. Up to renumbering, one can assume that $p > q$. Then, the elements of $\mathrm{Hom}(\mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2, \mathbb{C}^\times)$ are the maps

$$\begin{aligned} \chi_{s,t} : \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 &\to \mathbb{C}^\times \\ m\alpha_1 + n\alpha_2 &\mapsto s^m t^n \end{aligned}$$

for $s, t \in \mathbb{C}^\times$ such that $s^q = t^p$. For all $t \in \mathbb{C}^\times$, the algebraic relation (3.1) in the special case $\sigma = \sigma_{\chi(t^p, t^q)}$ is:

$$Q\big(c_{1,\theta}a_{1,\theta}(z)e^{\alpha_1 z}t^p + c_{2,\theta}a_{2,\theta}(z)e^{\alpha_2 z}t^q, c_{1,\theta}(a_{1,\theta}(z)e^{\alpha_1 z})'t^p + c_{2,\theta}(a_{2,\theta}(z)e^{\alpha_2 z})'t^q\big) = 0. \tag{3.2}$$

We write

$$Q(X,Y) = \sum_{i,j} q_{i,j}(z)X^iY^j,$$

and denote by δ the total degree of $Q(X,Y)$. The term of higher degree in t of (3.2) is equal to

$$c_{1,\theta}^{p\delta} \sum_{i+j=\delta} q_{i,j}(z)\big(a_{1,\theta}(z)e^{\alpha_1 z}\big)^i \big((a_{1,\theta}(z)e^{\alpha_1 z})'\big)^j$$

and, hence,

$$\sum_{i+j=\delta} q_{i,j}(z)\big(a_{1,\theta}(z)e^{\alpha_1 z}\big)^i \big((a_{1,\theta}(z)e^{\alpha_1 z})'\big)^j = 0. \tag{3.3}$$

We now have a non trivial homogeneous algebraic relation with coefficients in $\overline{\mathbb{Q}}(z)$ between $a_{1,\theta}(z)e^{\alpha_1 z}$ and $\big(a_{1,\theta}(z)e^{\alpha_1 z}\big)'$ and, hence, their quotient

$$\frac{\big(a_{1,\theta}(z)e^{\alpha_1 z}\big)'}{a_{1,\theta}(z)e^{\alpha_1 z}} = \frac{a'_{1,\theta}(z)}{a_{1,\theta}(z)} + \alpha_1$$

is algebraic over $\overline{\mathbb{Q}}(z)$. It follows that the logarithmic derivative

$$u(z) := \frac{a'_{1,\theta}(z)}{a_{1,\theta}(z)}$$

is algebraic over $\overline{\mathbb{Q}}(z)$. We have $a_{1,\theta}(z) = \exp(\int u(z))$. Consider the Puiseux expansion of $u(z)$ at ∞:

$$u(z) = \sum_{k=-N}^{\infty} u_k z^{-k/\ell}$$

for some integer $\ell \geq 1$. We have

$$\int u(z) = \sum_{k=\ell+1}^{\infty} \frac{u_k}{1-k/\ell} z^{1-k/\ell} + u_\ell \log(z) + \sum_{k=-N}^{\ell-1} \frac{u_k}{1-k/\ell} z^{1-k/\ell},$$

up to some additive constant. Since $a_{1,\theta}(z)$ has at most polynomial growth at infinity in any closed sector included in $S(d_0 - \frac{\pi}{2}, d_1 + \frac{\pi}{2})$, we see that $u_k = 0$ for $k \leq \ell - 1$ and, hence,

$$a_{1,\theta}(z) = C z^{u_\ell} \exp\Big(\sum_{k=\ell+1}^{\infty} \frac{u_k}{1-k/\ell} z^{1-k/\ell}\Big), \tag{3.4}$$

for some $C \in \mathbb{C}^\times$. Since $a_{1,\theta}(z)e^{\alpha_1 z}$ is a solution of $\mathscr{M}$, it is analytic on $\widetilde{\mathbb{C}^\times}$ and has at most polynomial growth at 0 in any sector with finite aperture (recall that $\mathscr{M}$ is regular on $\mathbb{C}^\times$ and regular singular at 0). Therefore, $a_{1,\theta}(z)$ is analytic on $\widetilde{\mathbb{C}^\times}$ and has at most polynomial growth at 0 in any sector with finite aperture. Moreover, the equation (3.4) shows that $a_{1,\theta}(z^\ell)/z^{\ell u_\ell}$ (has trivial monodromy at ∞ and) is analytic near ∞. Therefore, $a_{1,\theta}(z^\ell)/z^{\ell u_\ell}$ is analytic on $\mathbb{C}^\times \cup \{\infty\}$ and has at most polynomial growth at 0. It follows that $a_{1,\theta}(z^\ell)/z^{\ell u_\ell}$ is of the form $p(z^{-1})$ for some $p(X) \in \mathbb{C}[X]$. Hence, $a_{1,\theta}(z)e^{\alpha_1 z} = z^c q(z^{1/\ell})e^{\alpha_1 z}$ for some $c \in \mathbb{C}$ and $q(X) \in \mathbb{C}[X]$. By analytic continuation, $z^c q(e^{2\pi i/\ell} z^{1/\ell})e^{\alpha_1 z}$ is also a solution of $\mathscr{M}$. This solution is a linear combination with coefficients in $\mathbb{C}$ of the $a_{i,\theta}(z)e^{\alpha_i z}$. Since $\alpha_1 \neq \alpha_2$, we conclude that there exists $\lambda \in \mathbb{C}$ such that $z^c q(e^{2\pi i/\ell} z^{1/\ell})e^{\alpha_1 z} = \lambda z^c q(z^{1/\ell})e^{\alpha_1 z}$. This implies that $q(z^{1/\ell}) = z^{m/\ell} r_1(z)$ for some relative integer m and some $r_1(z) \in \mathbb{C}[z]$. Therefore,

$$a_{1,\theta}(z)e^{\alpha_1 z} = z^{c_1} r_1(z) e^{\alpha_1 z}$$

for some $c_1 \in \mathbb{C}$. Since the exponents of $\mathscr{M}$ at 0 are rational, we must have $c_1 \in \mathbb{Q}$.

Using a similar argument for $a_{2,\theta}(z)$, we see that

$$a_{2,\theta}(z)e^{\alpha_2 z} = z^{c_2} r_2(z) e^{\alpha_2 z}$$

for some $c_2 \in \mathbb{Q}$ and $r_2(z) \in \mathbb{C}[z]$.

Hence, we have

$$k(z) = z^{c_1} r_1(z) e^{\alpha_1 z} + z^{c_2} r_2(z) e^{\alpha_2 z} \tag{3.5}$$

for some $c_i \in \mathbb{Q}$ and $r_i(z) \in \mathbb{C}(z)^\times$. By analytic continuation along a simple loop around 0, we get

$$k(z) = e^{2\pi i c_1} z^{c_1} r_1(z) e^{\alpha_1 z} + e^{2\pi i c_2} z^{c_2} r_2(z) e^{\alpha_2 z}. \tag{3.6}$$

Equating the right-hand side of (3.5) with the the right-hand side of (3.6), we get $e^{2\pi i c_1} = e^{2\pi i c_2} = 1$, *i.e.* $c_1, c_2 \in \mathbb{Z}$. It follows that

$$f(z) = s_1(z) e^{\alpha_1 z} + s_2(z) e^{\alpha_2 z} \tag{3.7}$$

for some $s_i(z) \in \mathbb{C}(z)$. A simple linear algebra argument shows that $s_i(z) \in \overline{\mathbb{Q}}(z)$. We claim that we actually have $s_1(z), s_2(z) \in \overline{\mathbb{Q}}[z, z^{-1}]$. Assume at the contrary that $s_1(z)$ or $s_2(z)$ has a pole $\xi \in \overline{\mathbb{Q}}^\times$. Let us denote by n the maximum between the order of ξ as a pole of $s_1(z)$ and of $s_2(z)$. Then, multiplying equation (3.7) by $(z-\xi)^n$ and letting $z = \xi$, we obtain a non trivial linear relation with coefficients in $\overline{\mathbb{Q}}$ between $e^{\alpha_1 \xi}$ and $e^{\alpha_2 \xi}$, which contradicts Lindemann's Theorem.

The same kind of arguments proves a similar result for $g(z)$.

3.2.2. $\alpha_1 = 0$ *or* $\alpha_2 = 0$. Assume for instance that $\alpha_2 = 0$ (so that $\alpha_1 \neq 0$). Arguing as in Section 3.2.1, we see that

$$a_{1,\theta}(z) = z^\gamma p(z)$$

for some $\gamma \in \mathbb{Q}$ and some $p(z) \in \mathbb{C}[z]$. In particular, $a_{1,\theta}(z) e^{\alpha_1 z}$ is solution of some differential equation $y'(z) = a(z) y(z)$ with $a(z) \in \mathbb{C}(z)$. By euclidean division, it follows that

$$\mathscr{M} = \left(\frac{d}{dz} - b(z)\right)\left(\frac{d}{dz} - a(z)\right)$$

for some $b(z) \in \mathbb{C}(z)$. Since $k(z)$ is a solution of $\mathscr{M}$, the function

$$v(z) := k'(z) - a(z) k(z)$$

satisfies $v'(z) = b(z) v(z)$ and, hence, $v(z) = e^{B(z)}$ where $B(z)$ is some primitive of the rational function $b(z)$ (note that $v(z) \neq 0$ because the order of $\mathscr{M}$ is equal to 2). Using the fact that $v(z)$ is a meromorphic function on $\mathbb{C}$, we see that

$$v(z) = r(z) e^{q(z)}$$

for some $r(z) \in \mathbb{C}(z)$ and $q(z) \in z\mathbb{C}[z]$. Since $a_{2,\theta}(z)$ is a solution of $\mathscr{M}$, the function

$$w(z) := a_{2,\theta}(z)' - a(z) a_{2,\theta}(z)$$

satisfies $w'(z) = b(z) w(z)$. Moreover, $w(z)$ is non zero (otherwise, $a_{1,\theta}(z) e^{\alpha_1 z}$ and $a_{2,\theta}(z)$ would be solutions of $y'(z) = a(z) y(z)$ and, hence, would be linearly dependent over $\mathbb{C}$). It follows that

$$w(z) = \kappa v(z)$$

for some $\kappa \in \mathbb{C}^\times$. But $w(z)$ has at most polynomial growth at ∞ in any closed sector included in $S(d_0 - \frac{\pi}{2}, d_1 + \frac{\pi}{2})$, so that $v(z)$ has the same property and, hence, belongs to $\mathbb{C}(z)$. Using the variation of constants method, we get that there exists $C \in \mathbb{C}^\times$ such that

$$k(z) = z^\gamma p(z) e^{\alpha_1 z}\left(\int_{z_0}^{z} x^{-\gamma} p(x)^{-1} v(x) e^{-\alpha_1 x} dx + C\right), \tag{3.8}$$

where $z_0 \in \mathbb{C}^\times$ is not a pole of $p(x)^{-1}v(x) \in \mathbb{C}(z)$. We shall now express (3.8) by means of hypergeometric series. For this, we need some lemmas.

Lemma 1. *For all $\gamma \notin \mathbb{Z}$, for all $Q(z) \in \mathbb{C}(z)$, for any $z_0 \in \mathbb{C}^\times$ which is not a pole of $Q(z)$, there exists $R(z) \in \mathbb{C}(z)$ with at most simple poles on $\mathbb{C}^\times$ and whose set of poles in $\mathbb{C}^\times$ is included in the set of poles in $\mathbb{C}^\times$ of $Q(z)$, and there exist $\lambda(z), \mu(z) \in \mathbb{C}(z)$ and $\nu \in \mathbb{C}$ such that*

$$\int_{z_0}^{z} x^{-\gamma} Q(x) e^{-\alpha_1 x} dx = \lambda(z) z^{-\gamma} e^{-\alpha_1 z} + \mu(z) \int_{z_0}^{z} x^{-\gamma} R(x) e^{-\alpha_1 x} dx + \nu.$$

Proof. Using the decomposition in partial fractions of $Q(z)$, we see that it is sufficient to prove the lemma for $Q(z) = \frac{1}{(z-\xi)^n}$ with $\xi \in \mathbb{C}^\times$ and $n \in \mathbb{N}^*$. We proceed by induction on n. The result is obvious for $n = 1$. Assume that the result is true for some $n \in \mathbb{N}^*$. Set

$$Q(z) = \frac{1}{(z-\xi)^{n+1}}.$$

An integration by parts shows that

$$\begin{aligned}
&\int_{z_0}^{z} x^{-\gamma} Q(x) e^{-\alpha_1 x} dx \\
&\quad = \left[x^{-\gamma} e^{-\alpha_1 x} \frac{-1}{n} \frac{1}{(x-\xi)^n}\right]_{x=z_0}^{x=z} + \int_{z_0}^{z} \frac{-1}{n} \frac{1}{(x-\xi)^n} (\alpha_1 + \gamma x^{-1}) x^{-\gamma} e^{-\alpha_1 x} dx \\
&\quad = z^{-\gamma} e^{-\alpha_1 z} \frac{-1}{n} \frac{1}{(z-\xi)^n} - z_0^{-\gamma} e^{-\alpha_1 z_0} \frac{-1}{n} \frac{1}{(z_0-\xi)^n} \\
&\qquad - \frac{\alpha_1}{n} \int_{z_0}^{z} \frac{1}{(x-\xi)^n} x^{-\gamma} e^{-\alpha_1 x} dx - \frac{\gamma}{n} \int_{z_0}^{z} \frac{1}{(x-\xi)^n} x^{-\gamma-1} e^{-\alpha_1 x} dx.
\end{aligned}$$

The induction hypothesis leads to the desired result. □

Lemma 2. *For all $Q(z) \in \mathbb{C}(z)$, for all $z_0 \in \mathbb{C}^\times$ which is not a pole of $Q(z)$, there exists $R(z) \in \mathbb{C}(z)$ with at most simple poles on $\mathbb{C}$ and whose set of poles in $\mathbb{C}^\times$ is included in the set of poles in $\mathbb{C}^\times$ of $Q(z)$, and there exist $\lambda(z), \mu(z) \in \mathbb{C}(z)$ and $\nu \in \mathbb{C}$ such that*

$$\int_{z_0}^{z} Q(x) e^{-\alpha_1 x} dx = \lambda(z) e^{-\alpha_1 z} + \mu(z) \int_{z_0}^{z} R(x) e^{-\alpha_1 x} dx + \nu.$$

Proof. Similar to the proof of Lemma 1. □

In what follows, for any $\gamma \in \mathbb{C} \setminus \mathbb{Z}$ and $\alpha \in \mathbb{C}$, we set:

$$\mathcal{E}_{\gamma,\alpha}(z) = z^{\gamma} \int_0^z x^{-\gamma} e^{-\alpha x} dx = \sum_{n=0}^{\infty} \frac{(-\alpha)^n z^{n+1}}{(n-\gamma+1)n!}$$

and, for $\gamma \in \mathbb{Z}$, we set

$$\mathcal{E}_{\gamma,\alpha}(z) = e^{-\alpha z}.$$

If $\gamma \in \mathbb{Q}$ and $\alpha \in \overline{\mathbb{Q}}$, then $\mathcal{E}_{\gamma,\alpha}(z)$ is an E-function.

Lemma 3. *Consider $\gamma \in \mathbb{C}$, $R(z) \in \mathbb{C}(z)$, $z_0 \in \mathbb{C}^{\times}$ which is not a pole of $R(z)$, $C \in \mathbb{C}$, and $\varphi(z) := z^{\gamma}(\int_{z_0}^{z} x^{-\gamma} R(x) e^{-\alpha_1 x} dx + C)$. Assume that $\varphi(z)$ is meromorphic over $\mathbb{C}$. Then, there exists $\lambda(z), \mu(z) \in \mathbb{C}(z)$ such that*

$$\varphi(z) = \lambda(z) e^{-\alpha_1 z} + \mu(z) \mathcal{E}_{\gamma,\alpha_1}(z) \text{ if } \gamma \notin \mathbb{Z}$$

and

$$\varphi(z) = \lambda(z) e^{-\alpha_1 z} + \mu(z) \text{ if } \gamma \in \mathbb{Z}.$$

Proof. Let us first assume that $\gamma \notin \mathbb{Z}$. By Lemma 1, there exists $R(z) \in \mathbb{C}(z)$ with at most simple poles on $\mathbb{C}^{\times}$ and whose set of poles in $\mathbb{C}^{\times}$ is included in the set of poles in $\mathbb{C}^{\times}$ of $Q(z)$, and there exist $\lambda(z), \mu(z) \in \mathbb{C}(z)$ and $C' \in \mathbb{C}$ such that

$$\varphi(z) = \lambda(z) e^{-\alpha_1 z} + \mu(z) z^{\gamma} \int_{z_0}^{z} x^{-\gamma} R(x) e^{-\alpha_1 x} dx + C' z^{\gamma}.$$

If $\mu(z) = 0$, then we must have $C' = 0$ because $\varphi(z) - \lambda(z) e^{-\alpha_1 z}$ is meromorphic over $\mathbb{C}$ and $\gamma \notin \mathbb{Z}$, and, hence, the result is proved. We now assume that $\mu(z) \neq 0$. If $\xi \in \mathbb{C}^{\times}$ is a (simple) pole of $R(z)$, then ξ is a logarithmic singularity of $\int_{z_0}^{z} x^{-\gamma} R(x) e^{-\alpha_1 x} dx$ (because $x^{-\gamma} R(x) e^{-\alpha_1 x}$ itself has a simple pole at ξ), and this contradicts the fact that $\varphi(z)$ is meromorphic over $\mathbb{C}$. Therefore, $R(z) = z^{-n} S(z)$ for some integer $n \geq 0$ and some $S(z) \in \mathbb{C}[z]$. Hence,

$$\varphi(z) = \lambda(z) e^{-\alpha_1 z} + \mu(z) z^{\gamma} \int_{z_0}^{z} x^{-\gamma-n} S(x) e^{-\alpha_1 x} dx + C' z^{\gamma}.$$

But, $\int_{z_0}^{z} x^{-\gamma-n} S(x) e^{-\alpha_1 x} dx$ is a linear combination with coefficients in $\mathbb{C}$ of functions of the form $\int_{z_0}^{z} x^{-\gamma-n+k} e^{-\alpha_1 x} dx$ for $k \in \mathbb{N}$. Using integrations by parts, we conclude that $\int_{z_0}^{z} x^{-\gamma-n} S(x) e^{-\alpha_1 x} dx$ is, up to an additive constant in $\mathbb{C}$, a linear combination with coefficients in $\mathbb{C}(z)$ of $z^{-\gamma} e^{-\alpha_1 z}$ and $\int_{z_0}^{z} x^{-\gamma} e^{-\alpha_1 x} dx$. Hence, there exist $\widetilde{\lambda}(z), \widetilde{\mu}(z), \widetilde{\nu}(z) \in \mathbb{C}(z)$ such that

$$\varphi(z) = \widetilde{\lambda}(z) e^{-\alpha_1 z} + \widetilde{\mu}(z) \mathcal{E}_{\gamma,\alpha_1}(z) + \widetilde{\nu}(z) z^{\gamma}.$$

We must have $\widetilde{\nu}(z) = 0$ because $\varphi(z) - \widetilde{\lambda}(z) e^{-\alpha_1 z} - \widetilde{\mu}(z) \mathcal{E}_{\gamma,\alpha_1}(z)$ is meromorphic over $\mathbb{C}$. This yields the desired result.

Let us assume that $\gamma \in \mathbb{Z}$. By Lemma 2, there exists $R(z) \in \mathbb{C}(z)$ with at most simple poles on $\mathbb{C}$ and whose set of poles in $\mathbb{C}^{\times}$ is included in the set of poles in $\mathbb{C}^{\times}$ of $Q(z)$, and there exist $\lambda(z), \mu(z) \in \mathbb{C}(z)$ and $C' \in \mathbb{C}$ such that

$$\varphi(z) = \lambda(z)e^{-\alpha_1 z} + \mu(z)\int_{z_0}^{z} R(x)e^{-\alpha_1 x}dx + C'z^{\gamma}.$$

If $\mu(z) = 0$, the result is proved. Therefore, we now assume that $\mu(z) \neq 0$. If $\xi \in \mathbb{C}$ is a (simple) pole of $R(z)$, then ξ is a logarithmic singularity of $\int_{z_0}^{z} R(x)e^{-\alpha_1 x}dx$ (because $R(x)e^{-\alpha_1 x}$ itself has a simple pole at ξ), and this contradicts the fact that $\varphi(z)$ is meromorphic over $\mathbb{C}$. Hence, $R(z) \in \mathbb{C}[z]$. Using integration by parts, we see that $\int_{z_0}^{z} R(x)e^{-\alpha_1 x}dx$ is, up to an additive constant in $\mathbb{C}$, of the form $\eta(z)e^{-\alpha_1 z}$ for some $\eta(z) \in \mathbb{C}[z]$, which gives the desired result. $\square$

Let us first assume that $\gamma \notin \mathbb{Z}$. Using (3.8) and Lemma 3, we see that there exist $\lambda(z), \mu(z) \in \mathbb{C}(z)$ such that

$$k(z) = \lambda(z) + \mu(z)\mathcal{E}_{\gamma,\alpha_1}(z)e^{\alpha_1 z}. \tag{3.9}$$

A simple linear algebra argument shows that $\lambda(z)$ and $\mu(z)$ belong to $\overline{\mathbb{Q}}(z)$.

Therefore, there exist $a(z), b(z), c(z), d(z) \in \overline{\mathbb{Q}}(z)$ such that

$$\begin{aligned} f(z) &= a(z)\mathcal{E}_{\gamma,\alpha_1}(z)e^{\alpha_1 z} + b(z), \\ g(z) &= c(z)\mathcal{E}_{\gamma,\alpha_1}(z)e^{\alpha_1 z} + d(z). \end{aligned}$$

We have the following relation

$$\mathcal{E}_{\gamma,\alpha_1}(z)e^{\alpha_1 z} = \frac{\gamma\alpha_1^2}{z}\left({}_1F_1(1;\gamma;-\alpha_1 z) - 1 + \frac{\alpha_1 z}{\gamma}\right);$$

this is a direct consequence of the fact that they satisfy the same nonhomogenous differential of order one, namely:

$$zy'(z) + (\gamma + \alpha_1 z)y(z) = z.$$

Therefore,

$$f(z) = \widetilde{a}(z)_1F_1(1;\gamma;-\alpha_1 z) + \widetilde{b}(z), \tag{3.10}$$

$$g(z) = \widetilde{c}(z)_1F_1(1;\gamma;-\alpha_1 z) + \widetilde{d}(z) \tag{3.11}$$

for some $\widetilde{a}(z), \widetilde{b}(z), \widetilde{c}(z), \widetilde{d}(z) \in \overline{\mathbb{Q}}(z)$. Assume that $\xi (\in \overline{\mathbb{Q}})$ is a non-zero pole of $\widetilde{a}(z)$ or $\widetilde{b}(z)$. Let us denote by n the order of ξ as a pole of $\widetilde{a}(z)$. Let us denote by m the order of ξ as a pole of and of $\widetilde{b}(z)$. Let us first assume that $m > n$. Then, multiplying equation (3.10) by $(z-\xi)^m$ and letting $z = \xi$, we get

$$0 = \Big((z-\xi)^m\widetilde{b}(z)\Big)_{|z=\xi}$$

and this is a contradiction because $\Big((z-\xi)^m\widetilde{b}(z)\Big)_{|z=\xi} \neq 0$. So, we have $n \leq m$. Then, multiplying equation (3.10) by $(z-\xi)^n$ and letting $z = \xi$, we obtain that ${}_1F_1(1;\gamma;-\alpha_1\xi)$ belongs to $\overline{\mathbb{Q}}$, and this is a contradiction by [12, p. 192, Theorem 3]. Hence, $\widetilde{a}(z)$ and $\widetilde{b}(z)$

do not have poles on $\mathbb{C}^\times$ and, hence, belong to $\overline{\mathbb{Q}}[z, z^{-1}]$, whence the desired result. A similar argument shows that $\widetilde{c}(z)$ and $\widetilde{d}(z)$ belong to $\overline{\mathbb{Q}}[z, z^{-1}]$.

We shall now assume that $\gamma \in \mathbb{Z}$. Using (3.8) and Lemma 3, we see that there exists $\lambda(z), \mu(z) \in \mathbb{C}(z)$ such that

$$k(z) = \lambda(z) + \mu(z)e^{\alpha_1 z}. \tag{3.12}$$

It is easily seen that $\lambda(z), \mu(z) \in \overline{\mathbb{Q}}(z)$. Therefore, there exist $a(z), b(z), c(z), d(z) \in \overline{\mathbb{Q}}(z)$ such that

$$f(z) = a(z)e^{\alpha_1 z} + b(z),$$
$$g(z) = c(z)e^{\alpha_1 z} + d(z).$$

The proof of the fact that $a(z), b(z), c(z)$ and $d(z)$ actually belong to $\overline{\mathbb{Q}}[z, z^{-1}]$ is similar to the proof of the similar result in the case $\gamma \notin \mathbb{Z}$, using Lindemann's Theorem.

4. Proof of Theorem 4 in the case $\alpha_1 = \alpha_2$

In this case, $\mathfrak{f}_1(z)$ and $\mathfrak{f}_2(z)$ are convergent at ∞. We let $\alpha = \alpha_1 = \alpha_2$. One can decompose $k(z)$ as a linear combination with coefficients in $\mathbb{C}$ of the $\widehat{a}_i(z)e^{\alpha_i z}$ and, hence, as a linear combination with coefficients in $\mathbb{C}$ of functions of the form $z^\lambda(\ln(z))^\mu \mathfrak{f}_i(z)e^{\alpha z}$ with $\lambda \in \mathbb{Q}$ and $\mu \in \{0, 1\}$. The entire function $k(z)e^{-\alpha z}$ has at most polynomial growth at ∞ and, hence, $k(z)e^{-\alpha z} \in \mathbb{C}[z]$. So, $k(z)e^{-\alpha z} \in \mathbb{C}[z] \cap \overline{\mathbb{Q}}[[z]] = \overline{\mathbb{Q}}[z]$. Therefore, there exist $a(z), c(z) \in \overline{\mathbb{Q}}(z)$ such that

$$f(z) = a(z)e^{\alpha z},$$
$$g(z) = c(z)e^{\alpha z}.$$

Since $f(z)$ and $g(z)$ are entire functions, we have $a(z), c(z) \in \overline{\mathbb{Q}}[z]$, whence the desired result.

5. A remark on the proof of Theorem 4

An essential ingredient in the proof of Theorem 4 is the fact that some nonzero solution $u(z)$ of $\mathcal{M}$ has an algebraic logarithmic derivative $u'(z)/u(z)$ over $\overline{\mathbb{Q}}(z)$. The existence of such a solution can be easily derived using differential Galois theory. Indeed, we know that the functions $k(z)$ and $k'(z)$ are algebraically dependent over $\overline{\mathbb{Q}}(z)$. Therefore, the differential Galois group of $\mathscr{M}$ over $\overline{\mathbb{Q}}(z)$ does not contain $\mathrm{SL}_2(\overline{\mathbb{Q}})$. It follows that $\mathscr{M}$ is reducible over the algebraic closure $\overline{\overline{\mathbb{Q}}(z)}$ of $\overline{\mathbb{Q}}(z)$ *i.e.* there exist $a(z), b(z) \in \overline{\overline{\mathbb{Q}}(z)}$ such that

$$\mathscr{M} = \left(\frac{d}{dz} - b(z)\right)\left(\frac{d}{dz} - a(z)\right).$$

Therefore, any nonzero $u(z)$ such that $u'(z) = a(z)u(z)$ has the required property. It is very likely that we can use this fact a starting point for a variant of the proof of Theorem 4, but this would not led to substantial simplifications of the proof we have given.

6. Proof of Theorem 5

If $F(z)$ and $G(z)$ are $\overline{\mathbb{Q}}(z)$-linearly dependent, then both satisfy a differential equation of order 1 with coefficients in $\overline{\mathbb{Q}}(z)$. Using André's theory, we deduce that $F(z) = p(z)e^{\alpha z}$ and $G(z) = q(z)e^{\alpha z}$ for some $p(z), q(z) \in \overline{\mathbb{Q}}[z, z^{-1}]$ and $\alpha \in \overline{\mathbb{Q}}$. Hence, $F(\xi)$ and $G(\xi)$ are both in $\overline{\mathbb{Q}}^{\times} \exp(\overline{\mathbb{Q}})$ as expected. Actually, the equation $F(\xi)G(\xi) = 1$ forces that $\alpha = 0$ by Lindemann's theorem, so that $F(\xi)$ and $G(\xi)$ are in fact in $\overline{\mathbb{Q}}^{\times}$.

Let us now assume that $F(z)$ and $G(z)$ are $\overline{\mathbb{Q}}(z)$-linearly independent. By Beukers' Theorem 1.5 in [4], there exist

$$M(z) = \begin{pmatrix} m_{1,1}(z) & m_{1,2}(z) \\ m_{2,1}(z) & m_{2,2}(z) \end{pmatrix} \in M_2(\overline{\mathbb{Q}}[z])$$

and two E-functions $f(z), g(z) \in \overline{\mathbb{Q}}[[z]]$ satisfying a differential system

$$\begin{pmatrix} f'(z) \\ g'(z) \end{pmatrix} = E(z) \begin{pmatrix} f(z) \\ g(z) \end{pmatrix}$$

for some matrix $E(z) \in M_2(\overline{\mathbb{Q}}[z, z^{-1}])$, such that

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix} = M(z) \begin{pmatrix} f(z) \\ g(z) \end{pmatrix}.$$

We have

$$F(\xi)G(\xi) = \big(m_{1,1}(\xi)f(\xi) + m_{1,2}(\xi)g(\xi)\big)\big(m_{2,1}(\xi)f(\xi) + m_{2,2}(\xi)g(\xi)\big) = 1.$$

In other words, we have

$$p\big(f(\xi), g(\xi)\big) = 0.$$

where

$$p(X, Y) = \big(m_{1,1}(\xi)X + m_{1,2}(\xi)Y\big)\big(m_{2,1}(\xi)X + m_{2,2}(\xi)Y\big) - 1 \in \overline{\mathbb{Q}}[X, Y] \setminus \{0\}.$$

By Beukers' Theorem 1.3 in [4], one can lift this algebraic relation to an algebraic relation between $f(z)$ and $g(z)$ over $\overline{\mathbb{Q}}(z)$ *i.e.* there exists for $P(X, Y) \in \overline{\mathbb{Q}}[z][X, Y]$ such that

$$P\big(f(z), g(z)\big) = 0 \text{ and } P(X, Y)_{|z=\xi} = p(X, Y).$$

By Theorem 4, one of the following cases occurs :

(1) There exist $a(z), b(z), c(z), d(z) \in \overline{\mathbb{Q}}[z, z^{-1}]$ and $\alpha, \beta \in \overline{\mathbb{Q}}$ such that

$$f(z) = a(z)e^{\alpha z} + b(z)e^{\beta z},$$
$$g(z) = c(z)e^{\alpha z} + d(z)e^{\beta z}.$$

(2) There exists $a(z), b(z), c(z), d(z) \in \overline{\mathbb{Q}}[z, z^{-1}]$, $\gamma \in \mathbb{Q} \setminus \mathbb{Z}$ and $\alpha \in \overline{\mathbb{Q}}$ such that

$$f(z) = a(z)\,{}_1F_1(1; \gamma; \alpha z) + b(z),$$
$$g(z) = c(z)\,{}_1F_1(1; \gamma; \alpha z) + d(z).$$

6.1. **Assume that we are in case (1).** Then, we have

$$f(\xi) = a_1 e^{\alpha\xi} + b_1 e^{\beta\xi} \text{ and } g(\xi) = c_1 e^{\alpha\xi} + d_1 e^{\beta\xi}$$

where $a_1 = a(\xi)$, $b_1 = b(\xi)$, $c_1 = c(\xi)$ and $d_1 = d(\xi)$ belong to $\overline{\mathbb{Q}}$. Note that $\alpha \neq \beta$ because $F(z)$ and $G(z)$ are linearly independent over $\overline{\mathbb{Q}}(z)$. The connection between (F, G) and (f, g) implies that

$$F(\xi) = ae^{\alpha\xi} + be^{\beta\xi}, \quad G(\xi) = ce^{\alpha\xi} + de^{\beta\xi}$$

with $a, b, c, d \in \overline{\mathbb{Q}}$ such that $\{a, b\} \neq \{0\}$ and $\{c, d\} \neq \{0\}$ (because $F(\xi), G(\xi) \neq 0$). The condition $F(\xi)G(\xi) = 1$ becomes

$$ace^{2\alpha\xi} + bde^{2\beta\xi} + (ad + bc)e^{(\alpha+\beta)\xi} - e^0 = 0. \tag{6.1}$$

We first consider the case $\alpha\beta = 0$. At most one of α, β can be equal to 0. Let us assume that $\beta = 0$ and $\alpha \neq 0$. Then, (6.1) reads

$$ace^{2\alpha\xi} + (ad + bc)e^{\alpha\xi} + bd - 1 = 0.$$

Hence, if at least one of ac and $ad + bc$ is $\neq 0$, $e^{\alpha\xi}$ is an algebraic number, which forces $\alpha\xi = 0$ (Lindemann's Theorem again): impossible. Hence, $ac = 0$, $ad + bc = 0$ and, thus, $bd = 1$. These three conditions implies that if $a = 0$, resp. $c = 0$, then $c = 0$, resp. $a = 0$, so that $F(\xi) = b$ and $G(\xi) = d$ in both cases. The case $\beta \neq 0$ and $\alpha = 0$ is similar and leads to $F(\xi) = a$ and $G(\xi) = c$.

We now consider the case $\alpha\beta \neq 0$. If $\alpha+\beta \neq 0$, the four algebraic numbers $2\alpha\xi, 2\beta\xi, (\alpha+\beta)\xi, 0$ are pairwise distinct and, by the Lindemann-Weierstrass Theorem, (6.1) is impossible. Hence, $\alpha + \beta = 0$ and the same theorem applied to (6.1) implies in this case that $ac = 0$, $bd = 0$ and $ad + bc = 1$. The first and third equations implies that exactly one of a and c is 0. Assume that $a \neq 0$ and $c = 0$: then $ad = 1$ and thus $d \neq 0$ and $b = 0$, so that $F(\xi) = ae^{\alpha\xi}$ and $G(\xi) = de^{-\alpha\xi}$. If $a = 0$ and $c \neq 0$: then $bc = 1$ and thus $b \neq 0$ and $d = 0$, so that $F(\xi) = be^{-\alpha\xi}$ and $G(\xi) = ce^{\alpha\xi}$.

In all cases, the conclusion is that $F(\xi)$ and $G(\xi)$ are in $\overline{\mathbb{Q}}^{\times} \exp(\overline{\mathbb{Q}})$.

6.2. **Assume that we are in case (2).** Then, we have

$$f(\xi) = a_1 \cdot {}_1F_1(1; \gamma; \alpha\xi) + b_1 \text{ and } g(\xi) = c_1 \cdot {}_1F_1(1; \gamma; \alpha\xi) + d_1.$$

where $a_1 = a(\xi)$, $b_1 = b(\xi)$, $c_1 = c(\xi)$ and $d_1 = d(\xi)$ belong to $\overline{\mathbb{Q}}$. Note that $\alpha \neq \beta$ because $F(z)$ and $G(z)$ are linearly independent over $\overline{\mathbb{Q}}(z)$. The connection between (F, G) and (f, g) implies that

$$F(\xi) = a \cdot {}_1F_1(1; \gamma; \alpha\xi) + b \text{ and } G(\xi) = c \cdot {}_1F_1(1; \gamma; \alpha\xi) + d$$

with $a, b, c, d \in \overline{\mathbb{Q}}$ such that $\{a, b\} \neq \{0\}$ and $\{c, d\} \neq \{0\}$ (because $F(\xi), G(\xi) \neq 0$).

If $a = c = 0$, then $F(\xi)$ and $G(\xi)$ are algebraic. If a or c is nonzero, then the equation $F(\xi)G(\xi) = 1$ implies that ${}_1F_1(1; \gamma; \alpha\xi)$ is algebraic, and hence $F(\xi)$ and $G(\xi)$ are algebraic.

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T. Rivoal, Institut Fourier, Université Grenoble 1, CNRS UMR 5582, 100 rue des Maths, BP 74, 38402 Saint-Martin d'Hères cedex, France.
Tanguy.Rivoal@ujf-grenoble.fr

J. Roques, Institut Fourier, Université Grenoble 1, CNRS UMR 5582, 100 rue des Maths, BP 74, 38402 Saint-Martin d'Hères cedex, France.
Julien.Roques@ujf-grenoble.fr