SYMMETRY PHENOMENOMS IN LINEAR FORMS IN MULTIPLE ZETA VALUES

T. RIVOAL

The following text, based on joint work with J. Cresson and S. Fischler [6, 7]*, corresponds to the talk I gave at Turun Yliopisto in may 2007 during the ANT conference. I warmly thank the organisers of this conference for the invitation, especially Tapani Matala-Aho.*

A generalisation of the Riemann zeta function $\zeta(s)$ is given by the multiple zeta value (abreviated as MZV ; note that in french, the word *polyzêtas* is now often used for these series) defined for all integers $p \geq 1$ and all p-tuples $\underline{s} = (s_1, s_2, \ldots, s_p)$ of integers ≥ 1, with $s_1 \geq 2$, by

$$\zeta(s_1, s_2, \ldots, s_p) = \sum_{k_1 > k_2 > \ldots > k_p \geq 1} \frac{1}{k_1^{s_1} k_2^{s_2} \ldots k_p^{s_p}}.$$

The integers p and $s_1+s_2+\ldots+s_p$ are respectively the depth and the weight of $\zeta(s_1, s_2, \ldots, s_p)$. MZVs naturally appear when, for example, one considers products of values of the zeta function, e.g $\zeta(n)\zeta(m) = \zeta(n+m) + \zeta(n,m) + \zeta(m,n)$. In a certain sense, this enables us to "linearise" these products. Except a few identities such as $\zeta(2,1) = \zeta(3)$ (due to Euler), the arithmetical nature of MZVs is no better understood than that of $\zeta(s)$. However, the set of MZVs has a very rich structure which is well understood, at least conjecturally. (See [16]). For example, let us consider the $\mathbb{Q}$-vector spaces $\mathcal{Z}_p$ of $\mathbb{R}$ which are spanned by the 2^{p-2} MZVs of weight $p \geq 2$: $\mathcal{Z}_2 = \mathbb{Q}\zeta(2)$, $\mathcal{Z}_3 = \mathbb{Q}\zeta(3) + \mathbb{Q}\zeta(2,1)$, $\mathcal{Z}_4 = \mathbb{Q}\zeta(4) + \mathbb{Q}\zeta(3,1) + \mathbb{Q}\zeta(2,2) + \mathbb{Q}\zeta(2,1,1)$, etc. Set $v_p = \dim_{\mathbb{Q}}(\mathcal{Z}_p)$. We have the following conjecture, whose (i) is due to Zagier and (ii) to Goncharov.

Conjecture 1. (i) *For any integer $p \geq 2$, we have $v_p = c_p$, where c_p is defined by the linear recursion $c_{p+3} = c_{p+1} + c_p$, where $c_0 = 1$, $c_1 = 0$ and $c_2 = 1$.*
(ii) *The $\mathbb{Q}$-vector spaces $\mathbb{Q}$ and $\mathcal{Z}_p$ $(p \geq 2)$ are in direct sum.*

Hence, the sequence $(v_p)_{p\geq 2}$ should grow like α^p (where $\alpha \approx 1,3247$ is a root of the polynomial $X^3 - X - 1$), which is much less than 2^{p-2}. Thus, conjecturally, there exist many linear relations between MZVs of the same weight and none between those of different

Date: August 31, 2007.
1991 *Mathematics Subject Classification.* 33C70 (Primary); 11M41, 11J72 (Secondary).

weight: in this direction, the theorem of Goncharov [9] and Terasoma [14] claims that $v_p \leq c_p$ for all integers $p \geq 2$. It remains to prove the opposite inequality to show (i), but no non-trivial lower bound for v_p is yet known: even if classical relations give $v_2 = v_3 = v_4 = 1$, we do not know how to prove that $v_5 = 2$, which is equivalent to the irrationality of $\zeta(5)/(\zeta(3)\zeta(2))$. Conjecture 1 is also interesting because it implies the following one.

Conjecture 2. *The numbers* $\pi, \zeta(3), \zeta(5), \zeta(7), \zeta(9)$, *etc, are algebraically independent over* $\mathbb{Q}$.

This conjecture seems completely out of reach. A number of diophantine results have been proved in weight 1, i.e, in the case of the Riemann zeta function (see [8]) :

(i) The number $\zeta(3)$ is irrational (Apéry [1]);

(ii) The dimension of the vector space spanned over $\mathbb{Q}$ by 1, $\zeta(3)$, $\zeta(5), \ldots, \zeta(A)$ (with A odd) grows at least as fast as $\log(A)$ ([2, 12]);

(iii) At least one of the four numbers $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational (Zudilin [19]).

These results can be proved by the study of certain series of the form

$$\sum_{k=1}^{\infty} \frac{P(k)}{(k)_{n+1}^A} \tag{0.1}$$

where $P(X) \in \mathbb{Q}[X]$, $n \geq 0$, $A \geq 1$. Here, we use the Pochhammer symbol, defined by $(k)_\alpha = k(k+1)\ldots(k+\alpha-1)$. The above series can be written as a linear combination over $\mathbb{Q}$ of 1 and the values of zeta at integers. The crucial point is we can find special polynomials P such that in these combinations only certain value of zeta occur: $\zeta(3)$ in case (i), values $\zeta(s)$ with s odd in cases (ii) and (iii). This comes from (in the last two cases, and also in certain proofs of (i)) a symmetry property linked to the very-well-poised aspect of the series (0.1) (see [2] ou [12]):

Theorem 1. *Let* $P \in \mathbb{Q}[X]$ *of degree at most* $A(n+1)-2$, *such that*

$$P(-n-X) = (-1)^{A(n+1)+1} P(X).$$

Then, the series (0.1) *is a linear combination, with rational coefficients, of* 1 *and* $\zeta(s)$ *with* s *an odd integer between* 3 *and* A.

Our aim is to present two generalisations, in arbitrary depth, of this symmetry phenomenon, and whose proofs are given in [7]. We hope that such generalisations will make new diophantine results (irrationality or linear independence) for the underlying MZVs possible.

Our first result deals with "uncoupled" series, i.e, series over all p-tuples $(k_1, \dots, k_p) \in \mathbb{N}^{*p}$:

Theorem 2. *Consider integers $p \geq 1$, $n \geq 0$ and $A \geq 1$. Let $P \in \mathbb{Q}[X_1, \dots, X_p]$ be a polynomial of degree $\leq A(n+1) - 2$ with respect to each of the variables, such that*

$$P(X_1, \dots, X_{j-1}, -X_j - n, X_{j+1}, \dots, X_p) \\ = (-1)^{A(n+1)+1} P(X_1, \dots, X_{j-1}, X_j, X_{j+1}, \dots, X_p)$$

for any $j \in \{1, \dots, p\}$. Then, the multiple series

$$\sum_{k_1, \dots, k_p \geq 1} \frac{P(k_1, \dots, k_p)}{(k_1)_{n+1}^A \dots (k_p)_{n+1}^A} \tag{0.2}$$

is a polynomial with rational coefficients, of degree at most p, in the $\zeta(s)$, for s an odd integer between 3 and A.

For example, when $A = 3$ or $A = 4$, this series is a polynomial in $\zeta(3)$. When $p = 1$, we exactly obtain Theorem 1 (for all A).

From the point of view of diophantine applications, the main drawback of Theorem 2 is that the summation of k_1, ..., k_p is uncoupled. We now describe three disadvantages of uncoupled series.

First of all, uncoupled series always give polynomials in values of ζ at integers, even if we omit the symmetry condition in Theorem 2. This remark shows that MZVs cannot really appear in this setup.

Secondly, let us consider Ball's series

$$S_n = n!^2 \sum_{k=1}^{\infty} (k + \frac{n}{2}) \frac{(k-n)_n (k+n+1)_n}{(k)_{n+1}^4}.$$

For all integer n, S_n is a linear form in 1 and $\zeta(3)$; this follows from Theorem 1. (The series S_n exactly coincids with the linear forms used by Apéry to prove the irrationality of $\zeta(3)$; without going into details, let us mention that this coincidence is not all trivial and is the first application of the *denominators conjecture* proved in [11].) For all integers $p \geq 1$, the series S_n^p is obviously an uncoupled series of the the form considered in Theorem 2 with

$$P(X_1, \dots, X_p) \\ = n!^{2p} (X_1 + \frac{n}{2}) \dots (X_p + \frac{n}{2})(X_1 - n)_n \dots (X_p - n)_n (X_1 + n + 1)_n \dots (X_p + n + 1)_n$$

and $A = 4$. Therefore, S_n^p is a polynomial in $\zeta(3)$ of degree (at most) p, from which we could hope to deduce the transcendence of $\zeta(3)$. However, S_n^p does not contain anymore diophantine information than S_n and it can only gives the irrationality of $\zeta(3)$.

Finally, the multiple series which appear in irrationality proofs are generally of the form

$$\sum_{k_1 \geq \cdots \geq k_p \geq 1} \frac{P(k_1, \ldots, k_p)}{(k_1)_{n+1}^A \cdots (k_p)_{n+1}^A}, \tag{0.3}$$

i.e, the summation is over ordered indices; it is to this kind of series that one can apply the algorithm decribed in [6]. For example, when $p = 2$, $A = 2$ and

$$P(X_1, X_2) = n!(X_1 - X_2 + 1)_n (X_2 - n)_n (X_2)_{n+1},$$

Sorokin [13] shows that the sum (0.3) is exactly the linear form in 1 and $\zeta(3)$ used by Apéry. More generaly, a conjecture of Vasilyev [15] claimed that a certain multiple integral, equals to

$$n!^{p-\varepsilon} \sum_{k_1 \geq \cdots \geq k_p \geq 1} \frac{(k_1 - k_2 + 1)_n \ldots (k_{p-1} - k_p + 1)_n (k_p - n)_n}{(k_1)_{n+1}^2 \ldots (k_{p-1})_{n+1}^2 (k_p)_{n+1}^{2-\varepsilon}}, \tag{0.4}$$

is a rational linear form in zeta values at integers ≥ 2 of the same parity as $\varepsilon \in \{0, 1\}$. The integral formulation of this conjecture was proved in [20] and a refined version was proved in [11]: the method is to prove that the series (0.4) is also equal to a simple series to which Theorem 1 applies. Zlobin [18] recently obtained a completely different proof by a direct study of the series (0.4), in the spirit of the combinatorial methods developped in [6, 7]. It is then possible to prove results of essentially the same nature as those of [2, 12]: this confirms our feeling that multiple series with ordered indices are the interesting ones.

We showed in [6] that any convergent series of the form (0.3) can be written as a rational linear form in MZVs of weight at most pA and of depth at most p (this result was also obtained independently by Zlobin [17]). Furthermore, we produced an algorithm, implemented [5] in Pari, to explicitly compute such a linear combination. This enabled us to discover the symmetry property that we now describe in the special case of depth 2 for the reader's convenience.

Theorem 3. *Consider integers $n \geq 0$ et $A \geq 1$, with n even. Let $P \in \mathbb{Q}[X_1, X_2]$ be a polynomial in two variables, of degree $\leq A(n+1) - 2$ in each one, such that*

$$\begin{cases} P(X_1, X_2) = -P(X_2, X_1) \\ P(-n - X_1, X_2) = (-1)^{A(n+1)+1} P(X_1, X_2) \\ P(X_1, -n - X_2) = (-1)^{A(n+1)+1} P(X_1, X_2) \end{cases} \tag{0.5}$$

Then, the double series (0.3) *is a linear combination, with rational coefficients,*

- *of 1,*
- *of the values* $\zeta(s)$ *with* s *an odd integer such that* $3 \leq s \leq 2A$*,*
- *of the differences* $\zeta(s,s') - \zeta(s',s)$ *with* s*,* s' *odd integers such that* $3 \leq s < s' \leq A$*.*

(Let us note here that in the series (0.3), the variables $k_1, \ldots, k_p$ are linked by non-strict inequalities, as in [6], but contrary to the definition of MZVs. This does not cause any problems, since it is easy to go from statements with non-strict inequalities to statements with strict inequalities, and vice-versa.)

Of course, in (0.5), the third condition is a consequence of the first two. If $A = 4$, this theorem shows that the double series

$$\sum_{k_1 \geq k_2 \geq 1} \frac{P(k_1,k_2)}{(k_1)_{n+1}^4 (k_2)_{n+1}^4}$$

is a linear form in 1, $\zeta(3)$, $\zeta(5)$ and $\zeta(7)$ (which was far from obvious a priori since this a double series). For $A = 3$, we get a linear form in 1, $\zeta(3)$, $\zeta(5)$. Finally, for $A = 2$, we get a linear form in 1 and $\zeta(3)$.

To state our main result in arbitrary depth, we need the following notation. For integers $p \geq 0$ and $s_1, \ldots, s_p \geq 2$, we set

$$\zeta^{\mathrm{as}}(s_1, \ldots, s_p) = \sum_{\sigma \in \mathfrak{S}_p} \varepsilon_\sigma \zeta(s_{\sigma(1)}, \ldots, s_{\sigma(p)}),$$

where ε_σ is the signature of the permutation σ. We call such a linear combination of MZVs an *antisymmetric MZV* (even if, for $p \geq 2$, it is not an MZV in general). These are convergent series since each s_i is supposed ≥ 2. For $p = 1$, we have $\zeta^{\mathrm{as}}(s) = \zeta(s)$. The natural convention is to set $\zeta^{\mathrm{as}}(s_1, \ldots, s_p) = 1$ when $p = 0$ because there exists one unique bijection of the empty set onto itself. For $p = 2$, we have $\zeta^{\mathrm{as}}(s_1, s_2) = \zeta(s_1, s_2) - \zeta(s_2, s_1)$ and, when $p = 3$,

$$\zeta^{\mathrm{as}}(s_1, s_2, s_3) \\ = \zeta(s_1,s_2,s_3) + \zeta(s_2,s_3,s_1) + \zeta(s_3,s_1,s_2) - \zeta(s_2,s_1,s_3) - \zeta(s_1,s_3,s_2) - \zeta(s_3,s_2,s_1).$$

By definition, for all $\sigma \in \mathfrak{S}_p$, we have

$$\zeta^{\mathrm{as}}(s_{\sigma(1)}, \ldots, s_{\sigma(p)}) = \varepsilon_\sigma \zeta^{\mathrm{as}}(s_1, \ldots, s_p),$$

and $\zeta^{\mathrm{as}}(s_1, \ldots, s_p) = 0$ once two of the s_i's are equal. It seems reasonable to us that in general an antisymmetric MZV is not a polynomial in values of the Riemann zeta function. However, any "symmetric" MZV (defined as $\zeta^{\mathrm{as}}(s_1, \ldots, s_p)$ but omiting the signature ε_σ) is a polynomial in $\zeta(s)$ (by [10], Theorem 2.2).

Let $\mathscr{A}_p$ denotes the set of polynomials $P(X_1,\dots,X_p)\in\mathbb{Q}[X_1,\dots,X_p]$ such that:

$$\begin{cases}\text{For all } \sigma\in\mathfrak{S}_p, \text{ we have}\\ \qquad P(X_{\sigma(1)},X_{\sigma(2)},\dots,X_{\sigma(p)})=\varepsilon_\sigma P(X_1,X_2,\dots,X_p).\\ \\ \text{For all } j\in\{1,\dots,p\}, \text{ we have}\\ \qquad P(X_1,\dots,X_{j-1},-X_j-n,X_{j+1},\dots,X_p)\\ \qquad\qquad =(-1)^{A(n+1)+1}P(X_1,\dots,X_{j-1},X_j,X_{j+1},\dots,X_p).\end{cases}$$

There are redondances in these conditions. If the first one is satisfied, then it is enough to check the second one for one single value of j. For example, $\mathscr{A}_2$ is exactly the set of polynomials P satisfying the conditions (0.5). Moreover, if $P\in\mathscr{A}_p$ then P has the same degree in each variable $X_1,\dots,X_p$. Clearly, the definition of $\mathscr{A}_p$ also depends on the parity of $A(n+1)$. We can now state our main result.

Theorem 4. *Consider integers $n\geq 0$ and $A,p\geq 1$, with n even. Let $P\in\mathscr{A}_p$ be of degree $\leq A(n+1)-2$ in each of the variables. Then, the series*

$$\sum_{k_1\geq\ldots\geq k_p\geq 1}\frac{P(k_1,\dots,k_p)}{(k_1)_{n+1}^A\cdots(k_p)_{n+1}^A}\tag{0.6}$$

is a rational linear combination of products of the form

$$\zeta(s_1)\dots\zeta(s_q)\zeta^{\mathrm{as}}(s'_1,\dots,s'_{q'}),$$

where

$$\begin{cases} q,q'\geq 0 \text{ integers such that } 2q+q'\leq p,\\ s_1,\dots,s_q,s'_1,\dots,s'_{q'} \text{ odd integers } \geq 3,\\ s_i\leq 2A-1 \text{ for all } i\in\{1,\dots,q\},\\ s'_i\leq A \text{ for all } i\in\{1,\dots,q'\}.\end{cases}\tag{0.7}$$

When $q'=0$, the antisymmetric MZV $\zeta^{\mathrm{as}}(s'_1,\dots,s'_{q'})$ is equal to 1 and we obtain a product of values of ζ at odd integers. When $q=q'=0$, this produit is empty and we obtain 1.

If $p=1$, Theorem 4 states that (0.6) is a linear combination of 1 and the $\zeta(s)$ with odd s such that $3\leq s\leq A$: this is just Theorem 1.

If $p=2$, we obtain exactly Theorem 3.

If $p=3$, the theorem states that the series is a linear combination of

- products of at most two values of ζ at odd integers ≥ 3,
- antisymmetric MZVs $\zeta^{\mathrm{as}}(s_1,s_2)$ with $s_1,s_2\geq 3$ odd,
- antisymmetric MZVs $\zeta^{\mathrm{as}}(s_1,s_2,s_3)$ with $s_1,s_2,s_3\geq 3$ odd.

In depth $p \geq 4$, terms such as $q \geq 1$ and $q' \geq 2$ can appear: it seems that the series is not always the sum of a polynomial in values of $\zeta(s)$ (with s odd) and of a linear combination of antisymmetric MZVs $\zeta^{\mathrm{as}}(s_1, \dots, s_q)$ with $s_1, \dots, s_q$ odd.

When $A \leq 2$, we necessarily have $q' = 0$ in all the products, which implies the following corollary.

Corollary 1. *Under the hypotheses of Theorem 4, if $A \leq 2$, then the series* (0.6) *is a polynomial in $\zeta(3)$ with rationals coefficients.*

Theorem 4 also contains, for example, the following special case.

Corollary 2. *Consider integers $n, r, t, \varepsilon \geq 0$ and $A, p \geq 1$, with n even, such that*

$$\varepsilon \equiv (A+1)(n+1) + 1 \bmod 2$$

and

$$\varepsilon + (4r+2)p + 2t \leq (A-1)(n+1) + 4r.$$

Then, the convergent series

$$\sum_{k_1 \geq \ldots \geq k_p \geq 1} \Big[\prod_{i=1}^{p} (k_i + \frac{n}{2})\Big]^{\varepsilon} \frac{\Big[\prod_{1 \leq i < j \leq p} (k_i - k_j - r)_{2r+1} (k_i + k_j + n - r)_{2r+1}\Big] \Big[\prod_{i=1}^{p} (k_i - t)_{2t+n+1}\Big]}{(k_1)_{n+1}^{A} \ldots (k_p)_{n+1}^{A}}$$

is a linear combination as described in Theorem 4.

An example of application of this corollary is the following series (in which we take $t = 0$ and the Pochhammer symbols $(k_i)_{n+1}$ at the numerator cancel out with those at the denominator):

$$\sum_{k_1 \geq k_2 \geq k_3 \geq 1} \big(k_1 + \frac{1}{2}\big)\big(k_2 + \frac{1}{2}\big)\big(k_3 + \frac{1}{2}\big) \times \frac{(k_1 - k_2)(k_2 - k_3)(k_1 - k_3)(k_1 + k_2 + 1)(k_1 + k_3 + 1)(k_2 + k_3 + 1)}{(k_1)_2^4 \, (k_2)_2^4 \, (k_3)_2^4}$$
$$= -\frac{1}{4} - \zeta(3) + \frac{1}{4}\,\zeta(5) + \zeta(3)^2 - \frac{1}{4}\,\zeta(7).$$

$$\sum_{k_1 \geq k_2 \geq 1} \big(k_1 + \frac{1}{2}\big)\big(k_2 + \frac{1}{2}\big) \frac{(k_1 - k_2 - 1)_3 (k_1 + k_2)_3 (k_1 - 1)_4 (k_2 - 1)_4}{(k_1)_2^7 \, (k_2)_2^7}$$
$$= -1156 + 891\,\zeta(3) + \frac{189}{2}\,\zeta(5) + 78\big(\zeta(5,3) - \zeta(3,5)\big).$$

Finally, let us mention that the series described in the above theorems are related to multiple hypergeometric series related to root systems: see [3, 4] for example as well as the discussion in [7].

References

[1] R. Apéry, *Irrationalité de $\zeta(2)$ et $\zeta(3)$*, Journées Arithmétiques (Luminy, 1978), Astérisque, no. **61**, 1979, p. 11–13.

[2] K. Ball and T. Rivoal, *Irrationalité d'une infinité de valeurs de la fonction zêta aux entiers impairs*, Invent. Math. **146** (2001), no. 1, p. 193–207.

[3] G. Bhatnagar and M. Schlosser, *C_n and D_n very well-poised ${}_{10}\phi_9$ transformations*, Constr. Approx. **14** (1998), p. 531–567.

[4] H. Coksun, *An Elliptic BC_n Bailey Lemma, Multiple Rogers–Ramanujan Identities and Euler's Pentagonal Number Theorems*, Trans. AMS, to appear

[5] J. Cresson, S. Fischler and T. Rivoal, Algorithm available at `http//www.math.u-psud.fr/~fischler/algo.html`.

[6] J. Cresson, S. Fischler and T. Rivoal *Séries hypergéométriques multiples et polyzêtas*, Bulletin de la Soc. Math. de France, to appear.

[7] J. Cresson, S. Fischler and T. Rivoal, *Phénomènes de symétrie dans des formes linéaires en polyzêtas*, J. reine angew. Math., to appear.

[8] S. Fischler, *Irrationalité de valeurs de zêta (d'après Apéry, Rivoal, ...)*, Sém. Bourbaki 2002/03, Astérisque **294**, 2004, exp. no. 910, p. 27–62.

[9] A. Goncharov, *Multiple polylogarithms and mixed Tate motives*, preprint available at `http//front.math.ucdavis.edu/math.AG/0103059`, 2001.

[10] M. Hoffman, *Multiple harmonic series*, Pacific J. of Math. **152** (1992), p. 275–290.

[11] C. Krattenthaler and T. Rivoal, *Hypergéométrie et fonction zêta de Riemann*, Memoirs of the AMS **186** (2007), 93 pages.

[12] T. Rivoal, *La fonction zêta de Riemann prend une infinité de valeurs irrationnelles aux entiers impairs*, C. R. Acad. Sci. Paris, Ser. I **331** (2000), no. 4, p. 267–270.

[13] V. Sorokin, *Apéry's theorem*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. [Moscow Univ. Math. Bull.] **53** (1998), no. 3, p. 48–53 [48–52].

[14] T. Terasoma, *Mixed Tate motives and multiple zeta values*, Invent. Math. **149** (2002), no. 2, p. 339–369.

[15] D. Vasilyev, *Approximations of zero by linear forms in values of the Riemann zeta-function*, Doklady Nats. Akad. Nauk Belarusi **45** (2001), no. 5, p. 36–40, in russian ; extended version in english anglais : *On small linear forms for the values of the Riemann zeta-function at odd points*, preprint no.1 (558), Nat. Acad. Sci. Belarus, Institute Math., Minsk (2001), 14 pages.

[16] M. Waldschmidt, *Valeurs zêta multiples : une introduction*, J. Théor. Nombres Bordeaux **12** (2000), no. 2, p. 581–595.

[17] S. Zlobin, *Expansion of multiple integrals in linear forms*, Mat. Zametki [Math. Notes] **77** (2005), no. 5, 683–706 [630–652].

[18] S. Zlobin, *Properties of coefficients of certain linear forms in generalized polylogarithms*, Fundamentalnaya i Prikladnaya Matematika [Fundamental and Applied Mathemetics] **11** (2005), no. 6, p. 41–58,

[19] W. Zudilin, *One of the numbers $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ is irrational*, Uspekhi Mat. Nauk [Russian Math. Surveys] **56** (2001), no. 4, p. 149–150 [774–776].

[20] W. Zudilin, *Well-poised hypergeometric service for diophantine problems of zeta values*, J. Théor. Nombres Bordeaux **15** (2003), no. 2, p. 593–626.

T. Rivoal, Institut Fourier, CNRS UMR 5582, Université Grenoble 1, 100 rue des Maths, BP 74, 38402 Saint-Martin d'Hères cedex, France.