A stability property of G-functions

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October 26, 2019

Siegel [4, p. 239] introduced in 1929 the notion of G-function as a generalization of the series $1/(1-z) = \sum_{n=0}^{\infty} z^n$ and $-\log(1-z) = \sum_{n=1}^{\infty} z^n/n$. We fix an embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}$.

Definition 1. *A power series $F(z) = \sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[[z]]$ is a G-function if*

(i) *$F(z)$ is solution of a non-zero linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$.*

(ii) *There exists $C > 0$ such that for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and any $n \geq 0$, $|\sigma(a_n)| \leq C^{n+1}$.*

(iii) *There exists $D > 0$ and a sequence of integers d_n, with $1 \leq d_n \leq D^{n+1}$, such that $d_n a_m$ are algebraic integers for all $m \leq n$.*

Note that (i) implies that the a_n's all lie in a certain number field $\mathbb{K}$, so that in (ii) there are only finitely many Galois conjugates $\sigma(a_n)$ of a_n to consider, with $\sigma \in \mathrm{Gal}(\mathbb{K}/\mathbb{Q})$ (assuming for simplicity that $\mathbb{K}$ is a Galois extension of $\mathbb{Q}$). G-functions form a ring stable under $\frac{d}{dz}$ and $\int_0^z$; they are not entire in general but they can be analytically continued in suitably cut planes. Any algebraic function over $\overline{\mathbb{Q}}(z)$ and regular (ie holomorphic) at $z = 0$ is a G-function.

The following stability property satisfied by G-functions is often quoted but no proof seems to have been given in the literature. I give a proof in this note.

Proposition 1. *Let $F(z)$ be a G-function and $\alpha(z)$ an algebraic function over $\overline{\mathbb{Q}}(z)$, holomorphic at $z = 0$ such that $\alpha(0) = 0$.*

Then $F(\alpha(z))$ is a G-function.

Property (i) follows from the following general statement, due to Stanley [3, p. 180, Theorem 2.7]. Let $\mathbb{L}$ be a subfield of $\mathbb{C}$ and $F(z) \in \mathbb{L}[[z]]$ be a solution of a non-zero linear differential equation with coefficients in $\mathbb{L}(z)$. Then, for any algebraic function $\alpha(z)$ over $\mathbb{L}(z)$, holomorphic at $z = 0$ such that $\alpha(0) = 0$, the function $F(\alpha(z))$ is solution of a non-zero linear differential equation with coefficients in $\mathbb{L}(z)$. See [1, Theorem 3] for a quantitative version of Stanley's result.

Writing $F(\alpha(z)) = \sum_{n=0}^{\infty} a_n z^n$, Property (ii) obviously holds if $\sigma = id$ because both F and α have positive radii of convergence, hence this is also the case of $F \circ \alpha$. The general case can be reduced to the case $\sigma = id$. Indeed, let $\mathbb{K}$ be a Galoisian number field containing the Taylor coefficients of $F(\alpha(z))$ and those of $F(z) = \sum_{n=0}^{\infty} b_n z^n$ and $\alpha(z) = \sum_{n=1}^{\infty} c_n z^n$. Then, for any $\sigma \in \mathrm{Gal}(\mathbb{K}/\mathbb{Q})$, we have

$$\sum_{n=0}^{\infty} \sigma(a_n) z^n = \sum_{n=0}^{\infty} \sigma(b_n) \Big(\sum_{m=1}^{\infty} \sigma(c_m) z^m \Big)^n,$$

where $\sum_{n=0}^{\infty} \sigma(b_n) z^n$ is a G-function and $\sum_{m=1}^{\infty} \sigma(c_m) z^m$ is algebraic over $\overline{\mathbb{Q}}(z)$.

It remains to check Property (iii). For any integer $n \geq 0$, we set

$$\alpha(z)^n = \sum_{m=0}^{\infty} c_{m,n} z^m \in \overline{\mathbb{Q}}[[z]],$$

with $c_{m,n} = 0$ for $0 \leq m \leq n-1$. The series

$$\sum_{m,n \geq 0} c_{m,n} z^m x^n = \sum_{n=0}^{\infty} \alpha(z)^n x^n = \frac{1}{1 - x\alpha(z)}$$

is a bivariate algebraic function. We now use Safonov's Theorem [2, p. 273], a multivariate generalization of Eisenstein's Theorem, to conclude that there exists an integer $C \geq 1$ such that $C^{m+n+1} c_{m,n}$ is an algebraic integer for all $m, n \geq 0$. Now, we have

$$\begin{aligned} F(\alpha(z)) &= \sum_{n=0}^{\infty} b_n \sum_{m=n}^{\infty} c_{m,n} z^m \\ &= \sum_{n=0}^{\infty} b_n \sum_{m=0}^{\infty} c_{m+n,n} z^{m+n} \\ &= \sum_{k=0}^{\infty} \Big(\sum_{n=0}^{k} b_n c_{k,n} \Big) z^k. \end{aligned}$$

Since $\sum_{n=0}^{\infty} b_n z^n$ is a G-function, there exists a sequence of integers $B_k \geq 1$ such that $B_k b_n$ is an algebraic integer for all $n \leq k$ and $B_k \leq B^{k+1}$ for some $B \geq 1$. Hence,

$$B_k C^{2k+1} \sum_{n=0}^{k} b_n c_{k,n}$$

is an an algebraic integer for all $k \geq 0$, and (iii) holds with $D := BC^2$. This completes the proof that $F \circ \alpha$ is a G-function.

Safonov's Theorem is proved under the assumption that the Taylor coefficients of the multivariate algebraic series are in $\mathbb{Q}$. The general case used above can be easily deduced. Indeed, consider an algebraic series

$$F(X_1, \ldots, X_s) := \sum_{n_1 \geq 0, \ldots, n_s \geq 0} c_{n_1, \ldots, n_s} X^{n_1} \cdots X^{n_s} \in \overline{\mathbb{Q}}[[X_1, \ldots, X_s]].$$

Obviously, the coefficients $c_{n_1,\dots,n_s}$ all lie into a certain Galoisian number field $\mathbb{Q}(\beta)$ of degree $d \geq 1$, say. Hence, there exists d multivariate sequences of *rational* numbers $(u_{j,n_1,\dots,n_s})_{n_1,\dots,n_s\geq 0}$, $j = 0, \dots, d-1$, such that

$$c_{n_1,\dots,n_s} = \sum_{j=0}^{d-1} u_{j,n_1,\dots,n_s}\beta^j.$$

Now, each series

$$\sum_{n_1\geq 0,\dots,n_s\geq 0} u_{j,n_1,\dots,n_s} X^{n_1}\cdots X^{n_s} \in \mathbb{Q}[[X_1,\dots,X_s]]$$

is an algebraic one because it is a $\overline{\mathbb{Q}}$-linear combination of the algebraic series

$$\sum_{n_1\geq 0,\dots,n_s\geq 0} \sigma(c_{n_1,\dots,n_s}) X^{n_1}\cdots X^{n_s} \in \overline{\mathbb{Q}}[[X_1,\dots,X_s]]$$

where σ runs through $\mathrm{Gal}(\mathbb{Q}(\beta)/\mathbb{Q})$. We can thus apply Safonov's Theorem to each of them separately and let the integers $C_j \geq 1$ denote their respective Eisenstein's constant, ie $C_j^{n_1+\cdots+n_s+1} u_{j,n_1,\dots,n_s} \in \mathbb{Z}$. Let also the integer $B \geq 1$ denote a denominator of β, ie $B\beta$ is an algebraic integer. Then, $D := \mathrm{lcm}(C_0, C_1 B \dots, C_{d-1}B^{d-1})$ is such that $D^{n_1+\cdots+n_s+1}c_{n_1,\dots,n_s}$ is an algebraic integer for all $n_1, \dots, n_s \geq 0$.

I thank Alin Bostan for bringing [2] to my attention.

References

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