On the singularities of differential equations satisfied by E-functions

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Abstract

Let ξ be a value, at an algebraic point, of a Siegel E-function. As a special case of a very general interpolation result, we prove that there exists an E-function f such that $f(1) = \xi$, and such that 1 is not a singularity of the minimal differential equation satisfied by f. We prove that the same property does not hold at the point 0, when ξ is the value at a non-zero algebraic number of the Bessel function. This answers an analogue of a question asked by Yves André for G-functions.

1 Introduction

We first recall the definition of E-functions, due to Siegel [12]. We embed the field of algebraic numbers $\overline{\mathbb{Q}}$ in $\mathbb{C}$. A power series $f(x) = \sum_{n=0}^{\infty} a_n x^n/n! \in \overline{\mathbb{Q}}[[x]]$ is said to be an E-function if

(i) $f(x)$ is solution of a non-zero linear differential equation with coefficients in $\overline{\mathbb{Q}}(x)$.

(ii) For all $\varepsilon > 0$, there exists an integer $N(\varepsilon)$ such that for all $n \geq N(\varepsilon)$, all Galois conjugates of a_n have modulus less than $n!^{\varepsilon}$.

(iii) There exists a sequence of positive integers d_n such that $d_n a_m$ are algebraic integers for all $m \leq n$ and such that for all $\varepsilon > 0$, there exists an integer $N(\varepsilon)$ such that for all $n \geq N(\varepsilon)$, $d_n \leq n!^{\varepsilon}$.

An E-function is either a polynomial or a transcendental function. In *(i)*, it does not matter if the differential equation is inhomogeneous or homogeneous, because from an inhomogeneous equation of order μ for F we readily obtain a homogeneous one of order $\mu + 1$. Unless otherwise stated, all the differential operators considered in this text will be implicitly assumed to be non-zero and have coprime coefficients in $\overline{\mathbb{Q}}[x]$. An *a priori* smaller class of E-functions, called *strict* E-functions, has also been considered in the literature: $n!^{\varepsilon}$ for all $\varepsilon > 0$ is replaced with c^n for some $c > 1$ (see for instance [1]); it is conjectured that both classes coincide. E-functions below will be understood in Siegel's sense. The statements of our results and their proofs hold *mutatis mutandis* for E-functions in the strict sense.

When $f(x)=\sum_{n=0}^{\infty} a_n x^n/n!$ is an E-function in the strict sense, the function $\sum_{n=0}^{\infty} a_n x^n$ is called a G-function. We are interested in the values taken at algebraic points by these functions. Let $\mathbf{E} := \{f(\alpha) : f \text{ is an } E\text{-function and } \alpha \in \overline{\mathbb{Q}}\}$, and define $\mathbf{G}$ in the same way where f is an analytic continuation of a G-function. These sets, which contain $\overline{\mathbb{Q}}$, were defined and proved to be rings in [5, 6] (when E-functions are considered in the strict sense; we extend here the definition to E-functions in Siegel's sense and it is still a ring).

The ring $\mathbf{G}$ of values of G-functions was studied more deeply in [5]. It is related to the ring $\mathcal{P}$ of periods (in the sense of [9], say): conjecturally, $\mathbf{G} = \mathcal{P}[1/\pi]$ (see [5, end of §2.2]). The following result was proved in [5]:

Theorem 1. *For any $\xi \in \mathbf{G}$ there exists a G-function $f(x)=\sum_{n=0}^{\infty} a_n x^n$ with coefficients a_n in $\mathbb{Q}(i)$ such that $f(1)=\xi$; moreover the radius of convergence of such a power series can be arbitrarily large.*

The following question was asked by Y. André (for the point 0), and is open:

Question 1. *Given $\xi \in \mathbf{G}$, does there always exist a G-function $f(x)$ such that $f(1)=\xi$ and f is solution of a homogeneous differential equation of which the point* 0 *(resp. the point* 1*) is not a singularity?*

In the present paper, we study analogous statements for E-functions. The ring $\mathbf{E}$ of values of E-functions is related to, and conjecturally contained in, the ring of exponential periods (see [9, 8]). Any E-function is entire, so the assertion about the radius of convergence in Theorem 1 is meaningless for E-functions. Restricting to coefficients a_n in $\mathbb{Q}(i)$, or in a given number field $\mathbb{K}$, yields a ring of values strictly contained in $\mathbf{E}$ (see [6, Theorem 4] and [7]) so this part of Theorem 1 is completely different for E-functions. On the other hand, the analogue of Question 1 makes sense for E-functions and we shall answer it.

To begin with, we give a negative answer to the analogue of Question 1 at the point 0.

Theorem 2. *Denote by $J_0(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!^2}(x/2)^{2n}$ the Bessel function. Let $\alpha \in \overline{\mathbb{Q}}^*$, and f be any E-function such that $f(1)=J_0(\alpha)$. Then* 0 *is a non-apparent singularity of any non-zero (in)homogeneous differential equation satisfied by f.*

In this theorem, and throughout the paper, in writing *(in)homogeneous* we mean that the statement holds with both *homogeneous* and *inhomogeneous* instead.

Recall that an inhomogeneous differential equation is an equation of the form $L_0 y = P_0$ where $L_0 = \sum_{i=0}^{\mu_0} a_i(x)(\frac{d}{dx})^i \in \overline{\mathbb{Q}}[x, \frac{d}{dx}]\setminus\{0\}$ and $P_0 \in \overline{\mathbb{Q}}[x]$; it is homogeneous if $P_0 = 0$. We may assume that $a_{\mu_0} \neq 0$ and $\gcd(a_0,\ldots,a_{\mu_0},P_0)=1$; then a singularity of the equation $L_0 y = P_0$ is, by definition, a zero of a_{μ_0}.

We recall also that a singularity is *apparent* if the corresponding differential equation has a basis of solutions holomorphic at that point.

At the point 1, a positive answer to the analogue of Question 1 follows from André's results [1] on E-operators (generalized to E-functions in Siegel's sense in [10]). Indeed, such an operator has no non-zero finite singularity, and any E-function is a solution of an E-operator. However, the minimal homogeneous differential equation of an E-function may have a singularity at 1 (which is then apparent since it is a right-factor of an E-operator); this happens for instance with $f(x) = (x-1)e^x$. We may strengthen Question 1 by asking the minimal (in)homogeneous differential equation of f not to have a singularity at the point 1. We shall prove the the following result, which gives a positive answer to this stronger question:

Theorem 3. *For any $\xi \in \mathbf{E}$, there is an E-function f such that $f(1) = \xi$ and 1 is not a singularity of the minimal (in)homogeneous differential equation satisfied by f.*

This result holds trivially if ξ is algebraic. Otherwise it is a special case (with $N = 1$, $\alpha_1 = 1$, $T = 1$, $\xi_{1,0} = \xi$) of the following much more general interpolation theorem.

Theorem 4. *Let $N, T \geq 1$, and $\alpha_1, \ldots, \alpha_N$ be pairwise distinct non-zero algebraic numbers. Let $\xi_{n,t} \in \mathbf{E}$ for $1 \leq n \leq N$ and $0 \leq t \leq T-1$. Then there exists an E-function f such that:*

- *For any $1 \leq n \leq N$ and any $0 \leq t \leq T-1$, we have $f^{(t)}(\alpha_n) = \xi_{n,t}$.*
- *Let $Ly = 0$, with $L \in \overline{\mathbb{Q}}[x, \frac{d}{dx}]$, be the minimal homogeneous differential equation satisfied by f; denote by μ its order. Then $\mu \geq T+1$ and for any n, we have the following equivalence: α_n is a singularity of L if, and only if, the values $\xi_{n,t}$ (for $0 \leq t \leq T-1$) are linearly dependent over $\overline{\mathbb{Q}}$.*
- *Let $L_0 y = P_0$, with $L_0 \in \overline{\mathbb{Q}}[x, \frac{d}{dx}]$ and $P_0 \in \overline{\mathbb{Q}}[x]$, be the minimal inhomogeneous differential equation satisfied by f. Then L_0 has order $\mu - 1$ and for any n, we have the following equivalence: α_n is a singularity of L_0 if, and only if, the values 1 and $\xi_{n,t}$ (for $0 \leq t \leq T-1$) are linearly dependent over $\overline{\mathbb{Q}}$.*

Remark. Let $\mathbb{K}$ be a number field that contains $\alpha_1, \ldots, \alpha_N$, and such that for any n and any t there exists an E-function $f_{n,t}$ with coefficients in $\mathbb{K}$ such that $f_{n,t}(1) = \xi_{n,t}$. Then the E-function f we construct has coefficients in $\mathbb{K}$.

The structure of this paper is as follows. In §2 we prove Theorem 2, and then in §3 we study non-zero singularities of minimal equations of E-functions, starting in §3.1 with general results of independent interest and proving Theorem 4 in §3.3.

2 Proof of Theorem 2

The proof of Theorem 2 is based of the following result of André [1] for E-functions in the strict sense with rational coefficients, generalized by Beukers [3] to all E-functions in the strict sense and by Lepetit [10, p. 143, Proposition 18] to all E-functions in Siegel's sense.

Theorem 5. *Let f be an E-function such that $f(1) = 0$. Then $f(x)/(x-1)$ is an E-function.*

We shall use also the fact that the Bessel function J_0 is a solution of $Ly = 0$, where

$$L := x\Big(\frac{d}{dx}\Big)^2 + \frac{d}{dx} + x. \tag{2.1}$$

Of course 0 is a singularity of L; a basis of local solutions at 0 is given by $(J_0(x), J_0(x)\log x + \tilde{J}_0(x))$ where $\tilde{J}_0$ is an E-function.

Let f be an E-function such that $f(1) = J_0(\alpha)$. Theorem 5 applied to the function $f(x) - J_0(\alpha x)$ provides an E-function g such that

$$f(x) = J_0(\alpha x) + (x-1)g(x). \tag{2.2}$$

Let $\mathcal{D} \in \overline{\mathbb{Q}}[x, \frac{d}{dx}]$ be a non-zero differential operator that annihilates $f(x)$, $J_0(\alpha x)$ and $g(x)$. We denote by V a Picard-Vessiot extension of $\overline{\mathbb{Q}}(x)$ that contains a basis of solutions of $\mathcal{D}$, and by G the corresponding differential Galois group. For simplicity we write $J(x) = J_0(\alpha x)$.

Let $\sigma \in G$. Then σ can be seen as a field automorphism of V that commutes with derivation and leaves any element of $\overline{\mathbb{Q}}(x)$ invariant. Applying σ to Eq. (2.2) yields

$$(\sigma f)(x) = (\sigma J)(x) + (x-1)(\sigma g)(x). \tag{2.3}$$

The function σJ is also a solution of the differential operator obtained from (2.1) by changing x to αx, so there exist scalars δ, β such that $\sigma J = \delta J + \beta(J\log + \tilde{J})$ where $\tilde{J}$ is an E-function. On the other hand, there exists an E-operator $\mathcal{D}_g$ of which g is a solution, and σg is also a solution of $\mathcal{D}_g$ so that σg is a Nilsson-Gevrey arithmetic series of order -1 (see [1]). Accordingly, it can be written as a finite sum

$$\sum_{\nu,j} \lambda_{\nu,j} h_{\nu,j}(x) x^{\nu} (\log x)^j$$

where $\nu \in \mathbb{Q}$, $j \in \mathbb{N}$, $\lambda_{\nu,j} \in \mathbb{C}$, and $h_{\nu,j}$ is an E-function. At last, assume that f is solution of an (in)homogeneous differential operator of which 0 is not a singularity (or is an apparent singularity). Then σf is a solution of the same operator, so that σf is holomorphic at 0.

Expanding both sides of Eq. (2.3) as polynomials in $\log x$, and taking the coefficient of $\log x$, yields

$$0 = \beta J(x) + (x-1)\sum_{\nu} \lambda_{\nu,1} h_{\nu,1}(x) x^{\nu}. \tag{2.4}$$

Now all E-functions $h_{\nu,1}$ are holomorphic at 1, so that taking $x = 1$ in Eq. (2.4) yields $\beta = 0$ (since $J(1) = J_0(\alpha) \neq 0$ because it is a transcendental number for all $\alpha \in \overline{\mathbb{Q}}^*$ by Siegel's theorem [12]).

We have proved that for any $\sigma \in G$ there exists a scalar δ such that $\sigma J = \delta J$. This implies

$$\sigma\Big(\frac{J'}{J}\Big) = \frac{\sigma(J)'}{\sigma(J)} = \frac{\delta J'}{\delta J} = \frac{J'}{J}$$

so that $J'/J \in \overline{\mathbb{Q}}(x)$ (see for instance [4, Proposition 1.9 and Theorem 1.6]). This contradicts another result of Siegel [12] that J_0 and J_0' are algebraically independent over $\overline{\mathbb{Q}}(x)$.

3 Non-zero singularities: proof of Theorem 4

3.1 Non-zero singularities and linear dependence of values

The following version of the Siegel-Shidlovskii theorem is due to Beukers [3, Corollary 1.4] in the strict sense, and to André [2] in Siegel's sense.

Theorem 6. *Let* $f_1, \ldots, f_M$ *be E-functions, linearly independent over* $\overline{\mathbb{Q}}(x)$*, such that the vector* ${}^t(f_1, \ldots, f_M)$ *is solution of a first-order linear differential system with entries in* $\overline{\mathbb{Q}}(x)$*.*

Let $\alpha \in \overline{\mathbb{Q}}^*$*. If* α *is not a singularity of this system, then* $f_1(\alpha), \ldots, f_M(\alpha)$ *are linearly independent over* $\overline{\mathbb{Q}}$*.*

(By *singularity*, we mean a pole of one of the entries of the matrix of the system.) In particular, let f be an E-function. Denote by μ the minimal order of a non-zero homogeneous linear differential equation satisfied by f. Then Theorem 6 applies to the functions $f_j := f^{(j-1)}$ with $M = \mu - 1$, so that:

If α *is not a singularity of the non-zero minimal linear differential equation of* f*, then* $f(\alpha), f'(\alpha), \ldots, f^{(\mu-1)}(\alpha)$ *are linearly independent over* $\overline{\mathbb{Q}}$*.*

The following observation is a converse statement; it is not difficult to prove but we did not find explicitly it in the literature. It is very convenient to ensure that a point is not a singularity of the minimal differential equation of an E-function; it will be used in the proof of Theorem 4.

Proposition 1. *Let* $L = \sum_{i=0}^{\mu} a_i(x)(\frac{d}{dx})^i \in \overline{\mathbb{Q}}[x, \frac{d}{dx}] \setminus \{0\}$ *be a differential operator such that* $\gcd(a_0, \ldots, a_\mu) = 1$*.*

Let $\alpha \in \overline{\mathbb{Q}}$ *and* f *be a solution of the differential equation* $Ly = 0$*, defined around* α*, such that* $f(\alpha), f'(\alpha), \ldots, f^{(\mu-1)}(\alpha)$ *are linearly independent over* $\overline{\mathbb{Q}}$*. Then* $a_\mu(\alpha) \neq 0$ *so that* α *is not a singularity of this differential equation* $Ly = 0$*.*

Proof. On the contrary, assume that $a_\mu(\alpha) = 0$; then we have $\sum_{i=0}^{\mu-1} a_i(\alpha) f^{(i)}(\alpha) = 0$. This is a $\overline{\mathbb{Q}}$-linear relation between $f(\alpha), f'(\alpha), \ldots, f^{(\mu-1)}(\alpha)$. Since by assumption $f(\alpha), f'(\alpha), \ldots, f^{(\mu-1)}(\alpha)$ are $\overline{\mathbb{Q}}$-linearly independent, this relation has to be trivial, so that $a_i(\alpha) = 0$ for any i. This contradicts the assumption that $\gcd(a_0, \ldots, a_\mu) = 1$. □

Combining Theorem 6 and Proposition 1 we obtain the following corollary.

Corollary 1. *Let* f *be an E-function, and* $Ly = 0$ *denote its minimal homogeneous linear differential equation. Let* $\alpha \in \overline{\mathbb{Q}}^*$*. Then* α *is a singularity of* L *if, and only if,* $f(\alpha), f'(\alpha), \ldots, f^{(\mu-1)}(\alpha)$ *are linearly dependent over* $\overline{\mathbb{Q}}$*, where* μ *is the order of* L*.*

We can also adapt this proof to the inhomogeneous setting.

Proposition 2. *Let f be an E-function, and $L_0 y = P_0$ denote its minimal non-zero inhomogeneous linear differential equation. Let $\alpha \in \overline{\mathbb{Q}}^*$. Then α is a singularity of this equation $L_0 y = P_0$ if, and only if, 1, $f(\alpha)$, $f'(\alpha)$, ..., $f^{(\mu_0 - 1)}(\alpha)$ are linearly dependent over $\overline{\mathbb{Q}}$, where μ_0 is the order of L_0.*

Proof. We write $L_0 = \sum_{i=0}^{\mu_0} a_i(x)(\frac{d}{dx})^i \in \overline{\mathbb{Q}}[x, \frac{d}{dx}]$ with $a_{\mu_0} \neq 0$, and we may assume that $\gcd(a_0, \ldots, a_{\mu_0}, P_0) = 1$. Then a singularity of the equation $L_0 y = P_0$ is, by definition, a zero of a_{μ_0}.

$\Rightarrow$ We have $\sum_{i=0}^{\mu_0-1} a_i(\alpha) f^{(i)}(\alpha) = P_0(\alpha)$ since $a_{\mu_0}(\alpha) = 0$. This is a $\overline{\mathbb{Q}}$-linear relation between 1 and $f(\alpha)$, $f'(\alpha)$, ..., $f^{(\mu_0-1)}(\alpha)$, and it is non-trivial because a_0, ..., a_{μ_0}, P_0 have no common zero.

$\Leftarrow$ If α is not a singularity, we apply Theorem 6 to the functions 1, f, f', ..., $f^{(\mu_0-1)}$: these functions are linearly independent over $\overline{\mathbb{Q}}(x)$ by minimality of the inhomogeneous differential equation $L_0 y = P_0$. We deduce that the values at α of these functions are linearly independent over $\overline{\mathbb{Q}}$. □

3.2 Beukers' desingularization lemma

We shall use the following desingularization lemma. It is due to Beukers [3, Theorem 1.5] for E-functions in the strict sense and was extended by Lepetit [10, §6.5, Théorème 13] to E-functions in Siegel's sense.

Lemma 1. *Let g_1, ..., g_M be E-functions, linearly independent over $\overline{\mathbb{Q}}(x)$, such that the vector ${}^t(g_1, \ldots, g_M)$ is solution of a first-order linear differential system with entries in $\overline{\mathbb{Q}}(x)$. Then there exist E-functions h_1, ..., h_M such that:*

- *The vector ${}^t(h_1, \ldots, h_M)$ is solution of a first-order linear differential system with coefficients in $\overline{\mathbb{Q}}[x, 1/x]$ (that is, without non-zero finite singularity).*
- *Each function g_ℓ can be written as a linear combination of $h_1, \ldots, h_M$ with coefficients in $\overline{\mathbb{Q}}[x]$.*

3.3 Completion of the proof of Theorem 4

To begin with, we notice that if we prove Theorem 4 with an additional point $\alpha_0 \in \overline{\mathbb{Q}}^*$, then we can eventually forget about this point and the conclusion of Theorem 4 will hold with the points α_n, $1 \leq n \leq N$. To this additional point (distinct from α_1, ..., α_N) we attach the values $\xi_{0,t} = e^{t+1}$ for $0 \leq t \leq T-1$.

For each n and each t, we denote by $f_{n,t}$ an E-function such that $f_{n,t}(\alpha_t) = \xi_{n,t}$; it exists since $h(ax)$ is an E-function whenever h is an E-function and $a \in \overline{\mathbb{Q}}$.

We denote by A the $\overline{\mathbb{Q}}[x]$-module generated by the constant function 1 and the $f_{n,t}$, *i.e.* the set of elements of the form

$$P + \sum_{n=0}^{N} \sum_{t=0}^{T-1} P_{n,t} f_{n,t},$$

where P and the $P_{n,t}$ are polynomials with algebraic coefficients. Then A is a finitely generated torsion-free $\overline{\mathbb{Q}}[x]$-module, and $\overline{\mathbb{Q}}[x]$ is a principal ring, so that A is free: there exist $\overline{\mathbb{Q}}[x]$-linearly independent elements $g_1, \ldots, g_m \in A$ such that each $f_{n,t}$ can be written as

$$f_{n,t}(x) = \sum_{\ell=1}^{m} Q_{n,t,\ell}(x) g_\ell(x)$$

with polynomials $Q_{n,t,\ell} \in \overline{\mathbb{Q}}[x]$, and the function 1 can be written in this way too. Each g_ℓ is an E-function; in particular it is solution of a homogeneous linear differential equation of order μ_ℓ, say. Denote by B the $\overline{\mathbb{Q}}(x)$-vector space generated by the functions $g_\ell^{(j)}$ with $1 \leq \ell \leq m$ and $0 \leq j \leq \mu_\ell - 1$; then B is invariant under derivation. Since $g_1, \ldots, g_m$ are $\overline{\mathbb{Q}}(x)$-linearly independent elements, there exist $M \geq m$ and E-functions $g_{m+1}, \ldots, g_M$ such that $(g_1, \ldots, g_M)$ is a basis of B. In particular,

- The E-functions $g_1, \ldots, g_M$ are linearly independent over $\overline{\mathbb{Q}}(x)$.
- The vector ${}^t(g_1, \ldots, g_M)$ is solution of a first-order linear differential system with entries in $\overline{\mathbb{Q}}(x)$.

Applying Lemma 1 to $g_1, \ldots, g_M$, we obtain E-functions $h_1, \ldots, h_M$ such that the vector ${}^t(h_1, \ldots, h_M)$ is solution of a first-order linear differential system with coefficients in $\overline{\mathbb{Q}}[x, 1/x]$. They are $\overline{\mathbb{Q}}(x)$-linearly independent since the vector space they span over $\overline{\mathbb{Q}}(x)$ has dimension M (it contains the linearly independent E-functions $g_1, \ldots, g_M$). Moreover each $f_{n,t}$ (and also the constant function 1) is a linear combination of $h_1, \ldots, h_M$ with coefficients in $\overline{\mathbb{Q}}[x]$. It is very important here to have polynomial coefficients (and not only coefficients in $\overline{\mathbb{Q}}(x)$) because it enables us to evaluate at $x = \alpha_n$ and obtain for any $0 \leq n \leq N$ and any $0 \leq t \leq T - 1$:

$$\xi_{n,t} \in V_n, \text{ where } V_n := \operatorname{Span}_{\overline{\mathbb{Q}}}(h_1(\alpha_n), \ldots, h_M(\alpha_n)).$$

We have also $1 \in V_n$.

Let $0 \leq n \leq N$. Since $h_1, \ldots, h_M$ are $\overline{\mathbb{Q}}(x)$-linearly independent E-functions that satisfy a first-order linear differential system of which α_n is not a singularity, the André-Beukers theorem (Theorem 6 above) shows that $h_1(\alpha_n), \ldots, h_M(\alpha_n)$ are linearly independent over $\overline{\mathbb{Q}}$. In other words, $\dim_{\overline{\mathbb{Q}}} V_n = M$.

We recall from the beginning of the proof that $\xi_{0,i} = e^{i+1}$ for any $0 \leq i \leq T - 1$. Therefore $1, \xi_{0,0}, \xi_{0,1}, \ldots, \xi_{0,T-1}$ are $\overline{\mathbb{Q}}$-linearly independent elements of V_0, so that $T+1 \leq \dim_{\overline{\mathbb{Q}}} V_0 = M$. We claim that for any $0 \leq n \leq N$ it is possible to choose values $\xi_{n,t} \in V_n$, for $T \leq t \leq M - 1$, with the following properties:

- if $\xi_{n,0}, \ldots, \xi_{n,T-1}$ are linearly independent over $\overline{\mathbb{Q}}$, then so are $\xi_{n,0}, \ldots, \xi_{n,M-1}$ (and then this is a basis of V_n);

- if 1, $\xi_{n,0}$, …, $\xi_{n,T-1}$ are linearly independent over $\overline{\mathbb{Q}}$, then so are 1, $\xi_{n,0}$, …, $\xi_{n,M-2}$ (and then this is a basis of V_n too).

If 1, $\xi_{n,0}$, …, $\xi_{n,T-1}$ are linearly independent over $\overline{\mathbb{Q}}$, we use the Steinitz exchange lemma to construct $\xi_{n,t} \in V_n$, for $T \leq t \leq M-2$, in such a way that 1, $\xi_{n,0}$, …, $\xi_{n,M-2}$ is a basis of V_n; then we let $\xi_{n,M-1} = 1$.

Otherwise, if $\xi_{n,0}$, …, $\xi_{n,T-1}$ are linearly independent over $\overline{\mathbb{Q}}$ and 1 belongs to the subspace spanned by these numbers, we construct $\xi_{n,t} \in V_n$, for $T \leq t \leq M-1$, in such a way that $\xi_{n,0}$, …, $\xi_{n,M-1}$ is a basis of V_n.

At last, if $\xi_{n,0}, \ldots, \xi_{n,T-1}$ are linearly dependent, we simply choose $\xi_{n,t} \in V_n$ arbitrarily for $T \leq t \leq M-1$. This concludes the proof of our claim.

Now we shall construct our E-function f as

$$f(x) = \sum_{i=1}^{M} S_i(x) h_i(x) \text{ with } S_1, \ldots, S_M \in \overline{\mathbb{Q}}[x]. \tag{3.1}$$

To be able to choose the polynomials S_i in a suitable way, we compute the t-th derivative of f for any $t \geq 0$:

$$f^{(t)}(x) = \sum_{i=1}^{M} S_i^{(t)}(x) h_i(x) + \sum_{i=1}^{M} \sum_{k=0}^{t-1} \binom{t}{k} S_i^{(k)}(x) h_i^{(t-k)}(x). \tag{3.2}$$

We shall construct now algebraic numbers $\xi_{n,t,i}$ such that if $S_i^{(t)}(\alpha_n) = \xi_{n,t,i}$ for any $0 \leq n \leq N$, $1 \leq i \leq M$ and $0 \leq t \leq M-1$, then $f^{(t)}(\alpha_n) = \xi_{n,t}$ for any n, t.

For $t = 0$, since $\xi_{n,0} \in V_n$ for any n, we may write $\xi_{n,0} = \sum_{i=1}^{M} \xi_{n,0,i} h_i(\alpha_n)$ for some $\xi_{n,0,i} \in \overline{\mathbb{Q}}$. Then the desired property $f(\alpha_n) = \xi_{n,0}$ follows at once from Eq. (3.1) provided that $S_i(\alpha_n) = \xi_{n,0,i}$.

If the values $S_i^{(t')}(\alpha_n) \in \overline{\mathbb{Q}}$ have been chosen already for any $0 \leq t' \leq t-1$, with $t \geq 1$, we write for any n:

$$\xi_{n,t} - \sum_{i=1}^{M} \sum_{k=0}^{t-1} \binom{t}{k} S_i^{(k)}(\alpha_n) h_i^{(t-k)}(\alpha_n) = \sum_{i=1}^{M} \xi_{n,t,i} h_i(\alpha_n) \text{ for some } \xi_{n,t,i} \in \overline{\mathbb{Q}}. \tag{3.3}$$

Indeed $h_i^{(t-k)}$ is a linear combination of h_1, …, h_M with coefficients in $\overline{\mathbb{Q}}[x, 1/x]$ due to the differential system satisfied by these functions, so that $h_i^{(t-k)}(\alpha_n) \in V_n$. Moreover $\binom{t}{k} S_i^{(k)}(\alpha_n) \in \overline{\mathbb{Q}}$ and V_n is a $\overline{\mathbb{Q}}$-vector space that contains $\xi_{n,t}$, so the left hand side of Eq. (3.3) belongs to V_n and the algebraic coefficients $\xi_{n,t,i}$ exist. Using Eq. (3.2) we obtain immediately that $S_i^{(t)}(\alpha_n) = \xi_{n,t,i}$ for all i implies $f^{(t)}(\alpha_n) = \xi_{n,t}$.

This concludes the proof of our claim; of course it is possible to find polynomials S_1, …, S_M such that $S_i^{(t)}(\alpha_n)$ is equal to that algebraic number $\xi_{n,t,i}$ for any $0 \leq n \leq N$,

$1 \leq i \leq M$ and $0 \leq t \leq M-1$. Then the function f defined by Eq. (3.1) satisfies $f^{(t)}(\alpha_n) = \xi_{n,t}$ for any n, t. This proves the first assertion of Theorem 4.

Now recall that the values $f^{(t)}(\alpha_0) = \xi_{0,t}$, $0 \leq t \leq M-1$, are linearly independent over $\overline{\mathbb{Q}}$ since e, e^2, ..., e^T are. Therefore the functions f, f', ..., $f^{(M-1)}$ are linearly independent over $\overline{\mathbb{Q}}[x]$ (that is, over $\overline{\mathbb{Q}}(x)$). Denoting by μ the order of the minimal homogeneous differential equation satisfied by f, this implies $\mu \geq M$. On the other hand, the $\overline{\mathbb{Q}}(x)$-vector space spanned by h_1, ..., h_M has dimension M, is stable under taking derivatives and contains f so that $\mu \leq M$. Finally we have $\mu = M$, and we have noticed that $M \geq T+1$ so that $\mu \geq T+1$.

Corollary 1 then shows that for any n we have the following equivalence: α_n is a singularity of the minimal homogeneous differential equation satisfied by f if, and only if, the values $f^{(t)}(\alpha_n) = \xi_{n,t}$ (for $0 \leq t \leq M-1$) are linearly dependent over $\overline{\mathbb{Q}}$. By construction of the $\xi_{n,t}$ for $T \leq t \leq M-1$, this is equivalent to the linear dependence of $\xi_{n,t}$ for $0 \leq t \leq T-1$. Therefore the second assertion of Theorem 4 holds.

To conclude, let us consider the minimal inhomogeneous differential equation $L_0 y = P_0$ satisfied by f, and denote by μ_0 its order. The family $(f, f', \dots, f^{(M-1)})$ is a basis of the $\overline{\mathbb{Q}}(x)$-vector space spanned by h_1, ..., h_M, and this vector space contains the constant function 1. This provides a non-trivial linear relation between 1, f, f', ..., $f^{(M-1)}$, that is an inhomogeneous differential equation of order at most $M-1$ satisfied by f. Denoting by μ_0 the order of the minimal inhomogeneous differential equation $L_0 y = P_0$ satisfied by f, we have $\mu_0 \leq M-1$. Actually equality holds, because f then satisfies an homogeneous differential equation of order $\mu_0 + 1$ so that $\mu_0 + 1 \geq M$.

Proposition 2 shows that for any n, α_n is a singularity of this equation $L_0 y = P_0$ if, and only if, the values 1 and $f^{(t)}(\alpha_n) = \xi_{n,t}$ (for $0 \leq t \leq M-2$) are linearly dependent over $\overline{\mathbb{Q}}$. By construction of the $\xi_{n,t}$ for $T \leq t \leq M-1$, this is equivalent to the linear dependence of 1 and the $\xi_{n,t}$ for $0 \leq t \leq T-1$.

This concludes the proof of Theorem 4.

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