A class of arithmetic difference operators

T. Rivoal

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Abstract

We define and study a new class of difference operators in $\overline{\mathbb{Q}}[n,\sigma]$, called Σ-operators. By analogy with the definition of E-operators by André, Σ-operators are images of G-operators in $\overline{\mathbb{Q}}[z,\delta]$ by the morphism of $\mathbb{C}$-algebras defined by $z \to \sigma$ and $\delta \mapsto n$. Here $\delta = z\frac{d}{dz}$ is Euler's operator and σ is the backward shift operator. We prove various results about these difference operators among which: determination of their slopes and generalized exponents, purity and asymptotic behavior of their solutions. These properties have a strong arithmetic coloration, in relation with the theory of G-functions. We illustrate our results by revisiting various classical and more recent examples.

1 Introduction

In this paper, we introduce an analogue for difference operators, called Σ-operators, of the notion of E-operators, a class of differential operators defined and studied by André [3]. Both classes are defined in term of G-operators.

• ***Differential and difference operators.*** We first review standard definitions and properties concerning differential and difference operators. Useful references are [3, 8, 24] for instance. We define $\partial = \frac{d}{dz}$ the usual derivation, $\delta = z\frac{d}{dz}$ the Euler derivation and σ the backward shift operator that acts on a sequence $(u_n)_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ by $\sigma u_n = u_{n-1}$ or on a meromorphic function u on $\mathbb{C}$ by $\sigma u(n) = u(n-1)$. ([1]) We define the $\mathbb{C}$-algebra of differential operators $\mathbb{C}[z,\partial]$ and $\mathbb{C}[z,\delta]$ and the $\mathbb{C}$-algebra of finite difference operators $\mathbb{C}[n,\sigma]$, with the commutation rules $z\partial - \partial z = -1$, $z\delta - \delta z = -z$ and $n\sigma - \sigma n = \sigma$ respectively. The *degree* and *order* of an operator in $\mathbb{C}[n,\sigma]$, respectively in $\mathbb{C}[z,\delta]$ or $\mathbb{C}[z,\partial]$, are the degrees in n and in σ, respectively in z and δ/∂. We define a morphism of $\mathbb{C}$-algebras $\mathcal{M} : \mathbb{C}[n,\sigma] \to \mathbb{C}[z,\delta]$ by $n \mapsto \delta$ and $\sigma \mapsto z$; it is an isomorphism, its inverse

[1] It is also standard to use the forward shift operator τ and the forward difference Δ such that $\tau u_n = u_{n+1}$ and $\Delta u_n = a_{n+1} - u_n$. We prefer to use here σ because of Eq. (1.1), and we note in passing that it is also used in the recurrences (3) and (3') in the classical paper [23]. Moreover, except in the final section 5.4, we will not consider meromorphic solutions of the difference operators delt with in this paper, because the connection with G-functions is less clear.

$\widehat{\mathcal{M}}$ being defined by $z \mapsto \sigma$ and $\delta \mapsto n$. In particular, $\mathcal{M}$ and $\widehat{\mathcal{M}}$ exchange the order and degree of the operators. Above and below, we can replace everywhere $\mathbb{C}$ by any of its subfields, for instance the field of algebraic numbers $\overline{\mathbb{Q}}$ (embedded into $\mathbb{C}$).

We recall that for all $n \in \mathbb{Z}_{\geq 0}$, we have $z^n\partial^n = \delta(\delta-1)\cdots(\delta-n+1) = \sum_{j=0}^{n} s(n,j)\delta^j$ and $\delta^n = \sum_{j=0}^{n} S(n,j)z^j\partial^j$ where the positive integers $s(n,j)$ and $S(n,j)$ are Stirling's numbers of the first and second kind respectively, with $s(n,n) = S(n,n) = 1$; see [19, Chapter 6]). We have the inclusion $\mathbb{C}[z,\delta] \subset \mathbb{C}[z,\partial]$ in the sense that given $L = \sum_{j=0}^{\eta} p_j(z)\delta^j \in \mathbb{C}[z,\delta]$, we also have $L = \sum_{k=0}^{\eta}(\sum_{j=k}^{\eta} S(j,k)z^k p_j(z))\partial^k \in \mathbb{C}[z,\partial]$, and we call this the representation of L in $\mathbb{C}[z,\partial]$. Conversely, given $L = \sum_{j=0}^{\eta} q_j(z)\partial^j \in \mathbb{C}[z,\partial]$, we have $L = \sum_{k=0}^{\eta}(\sum_{j=k}^{\eta} s(j,k)z^{-j}q_j(z))\delta^k \in \overline{\mathbb{Q}}[z,z^{-1},\delta]$. We define $\eta_1 \geq 0$ as the minimal integer such that $z^{\eta_1}\sum_{k=0}^{\eta}(\sum_{j=k}^{\eta} s(j,k)z^{-j}q_j(z))\delta^k \in \mathbb{C}[z,\delta]$ and we call $z^{\eta_1}L$ the representation of L in $\mathbb{C}[z,\delta]$. The integer η_1 is clearly at most equal to η but need not always be that large; see §3.1 when L is Fuchsian, the case of interest in the sequel.

From the differential equation point of view, the resolution of an equation of the form $L = 0$ with $L \in \mathbb{C}[z,\delta]$ is equivalent to the resolution of an equation $M = 0$ for some $M \in \mathbb{C}[z,\partial]$ with same order in δ and ∂, but not necessarily of the same degree in z. A finite difference operator can be written $R := \sum_{j=0}^{\mu} p_j(n)\sigma^j \in \mathbb{C}[n,\sigma]$. The roots of the trailing polynomial p_0 are called the singularities of R. A given sequence $(u_n)_{n\geq 0} \in \mathbb{C}^{\mathbb{Z}_{\geq 0}}$ is said to be a *solution of* R when $Ru_n := \sum_{j=0}^{\mu} p_j(n)u_{n-j} = 0$ for all $n \geq m$ for some integer $m \geq \mu$. Note that with this definition, it is irrelevant whether certain of the values $u_0, u_1, \ldots, u_{m-\mu-1}$ satisfy the recurrence relation or not; if necessary, we shall write $(u_n)_{n\geq m-\mu}$ to emphasize that $Ru_n = 0$ for all $n \geq m$. A sequence $(u_n)_{n\leq 0} \in \mathbb{C}^{\mathbb{Z}_{\leq 0}}$ is said to be a *backward solution of* R when $Ru_n = \sum_{j=0}^{\mu} p_j(n)u_{n-j} = 0$ for all $n \leq m$ for some integer $m \leq 0$, and again it is irrelevant for us whether certain of the values $u_0, u_{-1}, \ldots, u_{-m+1}$ satisfy the recurrence relation or not. However, initial values are crucial to define basis of solutions of R; see the discussion in §2.

Let $L \in \mathbb{C}[z,\delta]$ and consider a formal power series $f(z) := \sum_{n\geq 0} u_n z^n$ such that $Lf(z) = 0$. Then $(u_n)_{n\geq 0}$ is such that $\widehat{\mathcal{M}}(L)u_n = 0$ for all $n \geq \mu$, where μ is the order of $\widehat{\mathcal{M}}(L)$. Conversely, if a sequence $(u_n)_{n\geq 0}$ is such that $Ru_n = 0$ for all $n \geq m$ for some integer $m \geq \mu$ the order of R, then there exists $q(z) \in \mathbb{C}[z]$ of degree at most $m-1$ such that $\mathcal{M}(R)f(z) = q(z)$. These assertions follow from the identity

$$[z^n]\big(z^j\delta^k f(z)\big) = \sigma^j(n^k u_n), \quad \text{for } n \geq j, \tag{1.1}$$

where $[z^n]g(z)$ denotes the n-th Taylor coefficient of a power series $g \in \mathbb{C}[[z]]$. Indeed, we have that:

– If f is such that $0 = Lf(z) := \sum_{k=0}^{\eta}\sum_{j=0}^{\mu} c_{i,j}z^j\delta^k f(z)$, then for all $n \geq \mu$, $0 = [z^n](Lf(z)) = \sum_{k=0}^{\eta}\sum_{j=0}^{\mu} c_{i,j}\sigma^j(n^k u_n) = \widehat{\mathcal{M}}(L)u_n$.

– If $Ru_n = 0$ for all $n \geq m$ for some $m \geq \mu$, then the same formula shows that $0 = [z^n](\mathcal{M}(R)f(z))$ for all $n \geq m$, and this means that $\mathcal{M}(R)f(z)$ is a polynomial of degree at most $m-1$.

• ***Σ-operators.*** Of particular importance in this paper are *G-functions* and *G-operators*. A G-function is a power series $f(z) = \sum_{n\geq 0} u_n z^n$ satisfying the following three properties:

– f is solution of a differential equation $Lf(z) = 0$ for some $L \in \overline{\mathbb{Q}}[z, \partial]$; equivalently, the sequence $(u_n)_{n\geq 0}$ is solution of a difference equation $Ru_n = 0$ for some $R \in \overline{\mathbb{Q}}[n, \sigma]$.

– There exists $C > 0$ such that for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and any $n \geq 0$, $|\sigma(u_n)| \leq C^{n+1}$.

– There exists $D > 0$ and a sequence of integers D_n, with $1 \leq D_n \leq D^{n+1}$, such that $D_n u_m$ are algebraic integers for all $m \leq n$.

Algebraic functions in $\overline{\mathbb{Q}}[[z]]$, polylogarithms $\sum_{n\geq 1}^{\infty} z^n/n^s$ ($s \in \mathbb{Z}$), Gauß hypergeometric function $\sum_{n\geq}(a)_n(b)_n/(n!(c)_n)z^n$ ($a, b, c \in \mathbb{Q}$, $c \notin \mathbb{Z}_{\leq 0}$) are examples of G-functions. Under a conjecture of Bombieri, D_n in the third condition can be taken of the form $c^{n+1}d_{an+b}^s$ where $d_n := \mathrm{lcm}\{1, 2, \ldots, n\}$ and $a, b, c, s \in \mathbb{Z}_{\setminus\geq 0}$; see [17].

G-operators are Fuchsian differential operators in $\overline{\mathbb{Q}}[z, \partial]$ with rational exponents, and naturally attached to G-functions. We refer to §3.2 for their definition and properties. Since any operator $\sum_{j=0}^{\eta} p_j(z)\delta^j \in \overline{\mathbb{Q}}[z, \delta]$ can be represented in a unique way as $\sum_{k=0}^{\eta}\big(\sum_{j=k}^{\eta} S(j,k)z^k p_j(z)\big)\partial^k \in \overline{\mathbb{Q}}[z, \partial]$, we can and shall say without ambiguity that an operator in $\overline{\mathbb{Q}}[z, \delta]$ is a G-operator when its representation in $\overline{\mathbb{Q}}[z, \partial]$ is a G-operator. (Conversely, for any G-operator L in $\overline{\mathbb{Q}}[z, \partial]$, their exists an integer $\eta_1 \geq 0$ such that the G-operator $z^{\eta_1}L \in \overline{\mathbb{Q}}[z, \delta]$; but we do not always have $\eta_1 = 0$. See §3.1.)

The following definition is natural because it is similar to André's definition of an E-operator in [3, §4] as a differential operator in $\overline{\mathbb{Q}}[z, \partial]$ whose Fourier-Laplace transform is a G-operator. (2)

Definition 1. *We say that $R \in \overline{\mathbb{Q}}[n, \sigma]$ is a Σ-operator if $\mathcal{M}(R) \in \overline{\mathbb{Q}}[z, \delta]$ is a G-operator.*

Note that given a G-operator $L \in \overline{\mathbb{Q}}[z, \delta]$, $R := \widehat{\mathcal{M}}(L) \in \overline{\mathbb{Q}}[n, \sigma]$ is a Σ-operator because $\mathcal{M}(R) = \mathcal{M}\widehat{\mathcal{M}}(L) = L$. We could have thus alternatively defined a Σ-operator as an operator $R \in \overline{\mathbb{Q}}[n, \sigma]$ of the form $\widehat{\mathcal{M}}(L)$ for some G-operator $L \in \overline{\mathbb{Q}}[z, \delta]$.

Any $R \in \overline{\mathbb{Q}}[\sigma]$ is a Σ-operator because $\mathcal{M}(R) \in \overline{\mathbb{Q}}[z]$ is trivially a G-operator. For any rational numbers $a, b, c \in \mathbb{Q}$, the hypergeometric operator $\mathcal{H} := \delta(\delta+c-1) - z(\delta+a)(\delta+b)$ is a G-operator, so that $\widehat{\mathcal{M}}(\mathcal{H}) = n(n+c-1) - (n+a-1)(n+b-1)\sigma$ is a Σ-operator. Another example of a Σ-operator, related to $\ln(2)$ is $\mathcal{R}_1 := (n-1)\sigma^2 - 3(2n-1)\sigma + n$. It will be studied in §5 together with Σ-operators related to $\zeta(2)$, $\zeta(3)$ and $L(2, \chi_{-3})$.

We can now state the main result of this paper. We recall that $\mathbf{G}$ is the countable subring of $\mathbb{C}$ of values taken at algebraic numbers by analytic continuations of G-functions; $\mathbf{G}$ was defined and studied in [15]. The definitions of the slopes and the generalized exponents of a difference operator are given in §4.2 and §2 respectively.

Theorem 1. *Let $R \in \overline{\mathbb{Q}}[n, \sigma]$ be a Σ-operator, of order $\mu \geq 1$. Let $L := \mathcal{M}(R) \in \overline{\mathbb{Q}}[z, \delta]$ be its associated G-operator, of order η.*

(i) 0 is the only slope of the Newton σ-polygon of R. The generalized exponents of R are of the form $\frac{1}{\xi}(1 - \frac{s}{n})$ where $\xi \in \overline{\mathbb{Q}}^*$ is a non-zero singularity of L and $s \in \mathbb{Q}$ is equal mod $\mathbb{Z}$ to a local exponent of L at ξ.

[2] We recall that the Fourier-Laplace transform is the automorphism of order 4 of $\mathbb{C}[z, \partial]$ such that $z \mapsto -\partial$ and $\partial \mapsto z$. Except the minus sign, it is similar to the isomorphism $\mathcal{M} : \mathbb{C}[n, \sigma] \to \mathbb{C}[z, \delta]$.

(ii) If $(u_n)_{n\geq 0} \in \overline{\mathbb{Q}}^{\mathbb{Z}_{\geq 0}}$ *is a solution of* R*, then* $\sum_{n\geq 0} u_n z^n$ *is a* G*-function. If* $(u_n)_{n\leq 0} \in \overline{\mathbb{Q}}^{\mathbb{Z}_{\leq 0}}$ *is a backward solution of* R*, then* $\sum_{n\geq 0} u_{-n} z^n$ *is a* G*-function.*

(iii) Let $(u_n)_{n\geq 0} \in \overline{\mathbb{Q}}^{\mathbb{Z}_{\geq 0}}$ *be a solution of* R*, and let* υ *be the degree of the polynomial* $L(\sum_{n\geq 0} u_n z^n)$*. There exist two integers* $N \geq 1$ *and* $\Omega \geq 1$ *such that for all integer* $n \geq N$ *and all integer* $\omega \geq \Omega$*,* u_n *is a finite linear combination with constant coefficients in* $\mathbf{G}$ *of absolutely convergent generalized factorial series of the form*

$$\xi^{-n} \sum_{m=0}^{\infty} \phi_m \frac{\partial^q}{\partial \varepsilon^q} \left(\frac{\Gamma(n/\omega)\Gamma(m+s+\varepsilon+1)}{\Gamma(m+s+\varepsilon+n/\omega+1)} \right)_{|\varepsilon=0} \tag{1.2}$$

where $q \in [\![0, \eta+\upsilon+1]\!]$*,* $\xi \in \overline{\mathbb{Q}}^*$ *is a singularity of* L*,* s *is an integer or is equal (mod* $\mathbb{Z}$*) to an exponent of* L *at* ξ*, and* $\sum_{m\geq 0} \phi_m z^m$ *is a* G*-function. In this linear combination, the parameters* ξ, q, s *are independent of* n *and the coefficients* $\phi_m \in \overline{\mathbb{Q}}$ *depend on* ξ, q, s, ω *but not on* n*.*

In (iii), if $L(\sum_{n\geq 0} u_n z^n) = 0$, then its degree is considered to be -1 in the result, *i.e.* $q \in [\![0, \eta]\!]$. We have in fact a more precise result in this case concerning the parameter s: it is equal (mod $\mathbb{Z}$) to an exponent of L at ξ. The parameter ω is accessory, but necessary for the (absolute) convergence of the series.

The paper is organized as follows. In §2, we make various comments on Theorem 1. In §3, we review classical properties of Fuchsian and G-operators and deduce from them a few useful properties of Σ-operators (Proposition 1). We then give the proof of Theorem 1 in §4 and conclude the paper with examples in §5.

2 Comments on Theorem 1

The generalized exponents considered in (i) determine the leading term of all possible asymptotic behaviors as $n \to +\infty$ of solutions of a difference operator. In the general case, they are of the form

$$\zeta \cdot n^{\upsilon} \cdot \left(1 + \frac{a_1}{n^{1/r}} + \frac{a_2}{n^{2/r}} + \cdots + \frac{a_r}{n^{r/r}}\right) \tag{2.1}$$

where $r \in \mathbb{Z}_{\geq 1}$, $\upsilon \in \mathbb{Q}$ and $\zeta \in \mathbb{C}^*$ are quantities associated to the difference operator (see §4.2 for details). They mean that a solution behave like $n!^{\upsilon} \zeta^n e^{\sum_{j=0}^{r-1} b_j n^{j/r} + b_r \log(n)}$ (up to a factor $\log(n)^k$ for some $k \in \mathbb{Z}_{\geq 0}$) as $n \to +\infty$. These generalized exponents play a role similar to the usual local exponents in the differential case, supplemented by other quantities also called generalized exponents when a singularity is not regular (see [9, §2.3.4]). They can be computed algorithmically (see [11]) but here we shall compute them by a different method, by studying the asymptotic behavior of series similar to those in (1.2). It is interesting to observe that they are "arithmetic" in essence for a Σ-operator. Note that (i) holds true more generally for any $R \in \mathbb{C}[n, \sigma]$ such that $\mathcal{M}(R)$ is Fuchsian: we always have $r = 1$ in (2.1) and the same meaning for ξ and s; the only difference is that ξ and s may not necessarily be in $\overline{\mathbb{Q}}$ and $\mathbb{Q}$, respectively.

(ii) can be viewed as an analogue of the purity theorems for G-functions (André-Chudnovsky-Katz [3, p. 719]) and for E-functions (André [3, p. 722, Theorem 4.3(iii)]), and also recently obtained for Mahler functions (Faverjon-Roques [14]). We do not formulate (ii) in terms of basis of solutions of R because the determination of the dimension of the $\mathbb{C}$-vector space of solutions of R leads to a well-known subtility. More precisely, given $R = \sum_{j=0}^{\mu} p_j(n)\sigma^j \in \overline{\mathbb{Q}}[n, \sigma]$, let $\zeta_R := \{n \geq \mu : p_0(n) = 0\} \cap \mathbb{Z}$. Then the $\mathbb{C}$-vector space of solutions $(u_n)_{n\geq 0}$ of the recurrence $Ru_n = 0$ for all $n \geq \mu$ has dimension $\widetilde{\mu} \in [\![\mu, \mu + \#\zeta_R]\!]$, and then by (i) it has a basis $(u_{j,n})_{n\geq 0}$, $j = 1, \ldots, \widetilde{\mu}$, such that each series $\sum_{n\geq 0} u_{j,n}z^n$ is a G-function. On the other hand, letting n_0 be the largest element of ζ_R when it is non-empty, for any given integer $m \geq \max(n_0, \mu)$, the $\mathbb{C}$-vector space of solutions $(u_n)_{n\geq m-\mu}$ of the recurrence $Ru_n = 0$ for all $n \geq m$ has dimension μ and again it has a basis $(u_{j,n})_{n\geq m-\mu}$, $j = 1, \ldots, \mu$, such that each series $\sum_{n\geq m-\mu} u_{j,n}z^n$ is a G-function. See [8, Chapter 15, §5.4] and [22, Chapitre 1, p. 3] for more details on this matter.

For instance, the $\mathbb{C}$-vector space of solutions of the Σ-operator $R_0 = (n-1)\sigma-(n-2)$ has dimension 2 if we seek solutions $(u_n)_{n\geq 0}$ (u_0 and u_2 arbitrary, $u_1 = 0$ and $u_n = (n-1)u_2$ for $n \geq 3$), but it has dimension 1 if we seek solutions $(u_n)_{n\geq 2}$ (u_2 arbitrary and $u_n = (n-1)u_2$ for $n \geq 3$); R_0 is a G-operator because $\mathcal{M}(R_0) = (z-1)\delta - 2$ is of minimal order for the order 0 arithmetic Nilsson-Gevrey series $(z-1)^2/z^2$.

The property conveyed by (iii) is that we stay in the G-realm, *i.e.*, that the coefficients of the combination are in $\mathbf{G}$, that $\sum_{m\geq 0} \phi_m z^m$ is a G-function and that $s \in \mathbb{Q}$. This is a result very similar to [16, §4.2, Theorem 5] about the asymptotic expansion at ∞ of E-functions. When $q = 0$ and $\omega = 1$, the series in (1.2) are Gevrey *séries de factorielles* of order 0 of arithmetic type in the sense of [3, p. 738]. In fact, Theorem 7.3 in [3, p. 738] is similar in spirit to (ii) in Theorem 1, but from a different point of view focused on Gevrey factorial series of arithmetic type. In the special case where $\sum_{n\geq 0} u_n z^n$ is algebraic over $\overline{\mathbb{Q}}(z)$, it is proved in [26, Theorem 3] that $q = 0$ always, that the coefficients of the linear combination are in $\overline{\mathbb{Q}}$ and that the series $\sum_{m\geq 0} \phi_m z^m$ are algebraic functions.

If we simply assume that $\sum_{n\geq 0} u_n z^n$ is solution of a Fuchsian differential equation with coefficients in $\mathbb{C}(z)$, then a classical result similar to (iii) holds (see Norlünd's book [22, Chapitre 3]) but nothing better can be said in general than the coefficients of the linear combination are in $\mathbb{C}$, that $\sum_{m=0}^{\infty} \phi_m x^m \in \mathbb{C}[[z]]$ is also solution of a Fuchsian equation with coefficients in $\mathbb{C}(z)$ and that the parameters s are in $\mathbb{C}$. In fact, Norlund's results were used (and sketched) in [18, §7.1] already when $\sum_{n\geq 0} u_n z^n$ is a G-function, but no attention was paid on the arithmetic nature of the various quantities and functions involved, as we do in (iii).

A (standard) consequence of (iii) is that u_n can be interpolated as a meromorphic function of n, because all the series on the right hand side of (1.2) define (by analytic continuation) meromorphic functions on the whole complex plane; see [22, Chapitre 3, §35]. Another consequence of (iii) is the possibility to determine the generalized asymptotic expansion of u_n, *i.e.* its expression as a $\mathbf{G}$-linear combination of terms of the form $\xi^{-n} \log(n)^q \sum_{m=0}^{\infty} \varphi_m/n^{s+m+1}$, where the series are no longer convergent but asymptotic in Poincaré sense, and ω is no longer present. The coefficients φ_m are now in the $\mathbf{G}[\gamma]$-module

$\mathbf{S}$ generated by all the values of the Gamma function at rational points, where γ is Euler's constant; see [16, §2] for the properties of this module. It is likely that $\mathbf{S} \neq \mathbf{G}$.

3 Fuchsian operators, G-operators and Σ-operators

In this section we gather various results that in particular provide useful properties satisfied by Σ-operators and not mentioned in the Introduction.

3.1 Fuchsian operators

We first present details on the representation in $\mathbb{C}[z,\delta]$ of Fuchsian operators in $\mathbb{C}[z,\partial]$. This applies in particular to G-operators, and will be used in the proof of Proposition 1 below. We recall that an operator in $\mathbb{C}[z,\partial]$ is Fuchsian when it can be written

$$L = q(z)\sum_{j=0}^{\eta}\frac{p_j(z)}{p(z)^j}\partial^{\eta-j}$$

where p, q and the p_j's are in $\mathbb{C}[z]$, p has simple roots, $p_0 = 1$ and $\deg(p_j) \leq \deg(p^j) - j$ for all j; see [9, §2.1]. For instance, the classical Gauss hypergeometric operator $\mathcal{H}(a,b;c) := z(1-z)\partial^2 + (c-(a+b+1)z)\partial - ab$ is Fuchsian because it can be written

$$\mathcal{H}(a,b;c) = z(1-z)\Big(\partial^2 + \frac{c-(a+b+1)z}{z(1-z)}\partial - \frac{abz(1-z)}{(z(1-z))^2}\Big).$$

Let $L = \sum_{j=0}^{\eta} q_j(z)\partial^j \in \mathbb{C}[z,\partial]$ be Fuchsian, with $q_\eta \neq 0$. It is known (see [18, §4.1, Lemma 1]) that its representation $z^{\eta_1}L \in \mathbb{C}[z,\delta]$ holds with $\eta_1 := \eta - \mathrm{ord}_{z=0}(q_\eta)$ and that we can write it as $z^{\eta_1}L = \sum_{j=0}^{\mu} z^j Q_j(\delta)$, with $\mu := \max\deg(q_j) - \mathrm{ord}_{z=0}(q_\eta)$, $\deg(Q_j) \leq \eta$ for all $1 \leq j \leq \mu$ and $\deg(Q_0) = \deg(Q_\mu) = \eta$. Moreover $Q_0(X)$ and $Q_\mu(-X)$ are the indicial polynomials of L at ∞ and 0 respectively. Note that the multiset of the roots of the leading polynomial in the representation $z^{\eta_1}L \in \mathbb{C}[z,\delta]$ is the multiset of the non-zero roots of q_η. Coming back to the example of Gauss hypergeometric operator, we have

$$\begin{aligned} z\mathcal{H}(a,b;c) &= (1-z)\delta^2 + (c-1+(a+b)z)\delta - abz \\ &= \delta(\delta+c-1) - z(\delta+a)(\delta+b), \end{aligned}$$

so that $\eta_1 = 1$ as expected.

The factor z^{η_1} makes no real difference for the solutions of the recurrences associated to $L \in \mathbb{C}[z,\partial]$ and $z^{\eta_1}L \in \mathbb{C}[z,\delta]$: it amounts to deal with $(u_{n+\eta_1})_{n\geq 0}$ instead of $(u_n)_{n\geq 0}$. Indeed, we can define σ^j for $j \leq 0$ by the same action on sequences: $\sigma^j u_n = u_{n-j}$, *i.e.*, $\sigma^{-1} = \tau$ the forward shift. Set $R_{\eta_1} := \widehat{\mathcal{M}}(z^{\eta_1}L) \in \mathbb{C}[n,\sigma]$. Then $R_0 := \sigma^{-\eta_1}R_{\eta_1} \in \mathbb{C}[n,\sigma,\sigma^{-1}]$ can be viewed as the definition of "$\widehat{\mathcal{M}}(L)$". Then R_0 has the same solutions (in the sense adopted in this paper) as R_{η_1} for n large enough because $R_{\eta_1}u_n = 0$ if and only if $R_0 u_{n+\eta_1} = 0$.

3.2 G-operators

We recall here some properties of G-operators and their relation with G-functions.

(i) A G-operator $L = \sum_{j=0}^{\eta} q_j(z)\partial^j \in \overline{\mathbb{Q}}[z,\partial]$ is defined as follows. The differential equation $L = 0$ gives rise to the companion differential system $\partial Y = AY$ with $A \in M_\eta(\overline{\mathbb{Q}}(z))$. We have $\partial^k Y = A_k Y$ for all $k \geq 0$, where the matrices A_k are recursively defined by $A_{k+1} = \partial A_k + A_k A$ and $A_0 = A$. Let $T \in \overline{\mathbb{Q}}[z]$ be a common denominator of the entries of A (note that T divides q_η): it is easy to see that T^k is a common denominator of the entries of A_k. We say that L is a G-operator when the following property, named the Galochkin condition, holds: there exists a sequence of positive integers $(D_k)_{k\geq 0}$ of exponential growth such that for all $k \geq 0$, we have $\frac{D_k}{m!} T^m A_m \in M_\eta(\mathcal{O}_{\overline{\mathbb{Q}}}[z])$ for all $m = 0, \ldots, k$.

(ii) If L and M are G-operators, LM is also a G-operator; in particular since $p(z) \in \overline{\mathbb{Q}}[z]$ and ∂ are G-operators, $p(z)L$, $\partial^k L$ and $\delta^k L$ are G-operators for any integer $k \geq 0$. If L is a G-operator, the operators in $\overline{\mathbb{Q}}[z,\partial]$ obtained from L by changing z to $z - \alpha$ or $1/z$ are G-operators when $\alpha \in \overline{\mathbb{Q}}$, and after multiplication by a positive power of z in the second case. Given a G-operator $L := \sum_{j=0}^{\eta} q_j(z)\partial^j$, its adjoint $L^* := \sum_{j=0}^{\eta}(-1)^j\partial^j q_j(z)$ is a G-operator.

(iii) The following properties form the *ACK theorem* and are recalled in [3, p. 719]. If L is a G-operator of order η, then L is Fuchsian, with rational exponents at any point of $\mathbb{C} \cup \{\infty\}$. Moreover, at each $\alpha \in \overline{\mathbb{Q}} \cup \{\infty\}$, the equation $L = 0$ has a basis of solutions of the form $(f_1(z-\alpha), \ldots, f_\eta(z-\alpha)) \cdot (z-\alpha)^{\Delta_\alpha}$, where each $f_j(z)$ is a G-function, the square matrix $\Delta_\alpha \in M_\eta(\mathbb{Q})$ is an upper triangular matrix, and $z - \alpha$ is understood as $1/z$ if $z = \infty$. We call such a basis is called an *ACK basis*.

(iv) If f is a G-function, then the minimal non-zero operator $L \in \overline{\mathbb{Q}}[z,\partial]$ such that $Lf(z) = 0$ is a G-operator (Chudnovsky [12]). More generally, this is also true if f is an arithmetic Nilsson-Gevrey series of order 0 (André [3, p. 720]).

(v) A G-function f can be analytically continued to a suitable cut plane, with cuts originating from the (non-zero) singularities of its associated G-operator L. Moreover, the connection coefficients of f written on a local ACK basis of $L = 0$ at some $\alpha \in \overline{\mathbb{Q}} \cup \{\infty\}$ are in the ring of G-values $\mathbf{G}$, by [15, Theorem 2].

(vi) Finally, following André [3, pp. 717–718], we can also formulate the Galochkin condition directly at the level of differential operators. To $L = \sum_{j=0}^{\eta} q_j(z)\partial^j \in \overline{\mathbb{Q}}[z,\partial]$, we associate a unique sequence $(L_n)_{n\geq 1}$ of elements of $\overline{\mathbb{Q}}[z,\partial]$ right-divisible by L and of the form $\frac{1}{n!}q_\eta(z)^n\partial^{n+\eta-1} + \sum_{j=0}^{\eta-1} q_{n,j}(z)\partial^j$ (we have $L_1 = L$); then L satisfies the Galochkin condition when there exists a constant $C > 0$ such that for all $n \geq 1$, the least common denominator of the coefficients of all the $q_{m,j}(z)$, $1 \leq m \leq n$ and $0 \leq j \leq \eta - 1$, are less than C^n. Galochkin condition could be expressed using δ instead of ∂ but it does not seem to be more "compact".

3.3 Σ-operators

In this section, we record simple properties of Σ-operators. They are mostly traductions of the corresponding properties of G-operators (the proof of which can be difficult though).

Proposition 1. (i) *The order of a Σ-operator R is equal to the number of finite non-zero singularities (counted with multiplicities) of the G-operator $\mathcal{M}(R)$.*

(ii) *Let $(u_n)_{n\geq 0} \in \overline{\mathbb{Q}}^{\mathbb{N}}$ be such that $f(z) = \sum_{n\geq 0} u_n z^n$ is a G-function. Then there exists a Σ-operator $R \in \mathbb{Q}[n,\sigma]$ of order μ (say) such that $Ru_n = 0$ for all $n \geq \mu$.*

(iii) *Let $R = \sum_{j=0}^{\mu} p_j(n)\sigma^j$ be a Σ-operator and let η be the order of the G-operator $\mathcal{M}(R)$. Then p_0 and p_μ are of degree η, and the other p_j are all of degree $\leq \eta$.*

(iv) *The roots of the polynomials p_0 and p_μ are rational numbers.*

(v) *The product of two Σ-operators is a Σ-operator. Any right divisor in $\overline{\mathbb{Q}}[n,\sigma]$ of a Σ-operator is a Σ-operator.*

(vi) *(Ore property) Let R_1, R_2 be Σ-operators. Then a left common multiple in $\overline{\mathbb{Q}}[n,\sigma]$ of both R_1 and R_2 of minimal degree in n is a Σ-operator.*

(vii) *If $\sum_{j=0}^{\mu} p_j(n)\sigma^j$ is a Σ-operator, then for any $t \in \mathbb{Q}$, $\sum_{j=0}^{\mu} p_j(n+t)\sigma^j$ is also a Σ-operator.*

Proof of Proposition 1. (i) This is a general property due to the fact that $\mathcal{M}(R)$ is a Fuchsian differential operator, see §3.1.

(ii) Let $L \in \overline{\mathbb{Q}}[z,\delta]$ be a G-operator such that $Lf(z) = 0$. The operator $R := \widehat{\mathcal{M}}(L)$ is a Σ-operator and then we use the fact that $Ru_n = 0$ for all $n \geq \mu$, where μ is the order of $\widehat{\mathcal{M}}(L)$.

(iii) This is again a consequence of the fact that $\mathcal{M}(R)$ is a Fuchsian differential operator, see §3.1.

(iv) This is a traduction of the fact that the local exponents at 0 and ∞ of a G-operator are rational numbers, see §3.1.

(v) Let R_1, R_2 be two Σ-operators: there exist two G-operators L_1 and L_2 such that $R_1 = \mathcal{M}(L_1)$ and $R_2 = \mathcal{M}(L_2)$. Now $R_1R_2 = \mathcal{M}(L_1)\mathcal{M}(L_2) = \mathcal{M}(L_1L_2)$ and the conclusion follows because L_1L_2 is a G-operator.

Let R be Σ-operator such that $R = R_1R_2$ with $R_1, R_2 \in \overline{\mathbb{Q}}[n,\sigma]$. Then the G-operator $\mathcal{M}(R)$ factorizes as $\mathcal{M}(R_1)\mathcal{M}(R_2)$ in $\overline{\mathbb{Q}}[z,\delta] \subset \overline{\mathbb{Q}}[z,\partial]$. Since any right divisor in $\overline{\mathbb{Q}}[z,\partial]$ of a G-operator is a G-operator, $\mathcal{M}(R_2)$ is a G-operator and R_2 is a Σ-operator.

(vi) Let R be a left common multiple of R_1, R_2 of minimal degree in n. Then $\mathcal{M}(R)$ is a left common multiple of the G-operators $\mathcal{M}(R_1)$ and $\mathcal{M}(R_2)$ of minimal order in δ. It is known that $\mathcal{M}(R)$ is then a G-operator (Ore property), and thus R is a Σ-operator.

(vii) Let $t \in \mathbb{Q}$ and $R_t := \sum_{j=0}^{\mu} p_j(n+t)\sigma^j$. The solutions of $\mathcal{M}(R_t) = \sum_{j=0}^{\mu} p_j(\delta+t)z^j$ are of the form $z^{-t}g(z)$ where g is any solution of the G-operator $\mathcal{M}(R_0)$: by [20, p. 151, Lemme 8], this implies that $\mathcal{M}(R_t)$ is itself a G-operator. Hence, R_t is a Σ-operator. $\square$

We conclude this section with the following remarks. Given a G-operator $L_0 := \sum_{j=0}^{\eta} q_j(z)\partial^j \in \overline{\mathbb{Q}}[z,\partial]$ with adjoint $L_0^* = \sum_{j=0}^{\eta}(-1)^j\partial^j q_j(z) \in \overline{\mathbb{Q}}[z,\partial]$, let η_0 and η_0^* be the minimal non-negative integers such that $L := z^{\eta_0} L_0$ and $L_* := z^{\eta_0^*} L_0^*$ are their representations in $\overline{\mathbb{Q}}[z,\delta]$; they are still G-operators. We have $L = \sum_{j=0}^{\eta} q_j(z) z^{\eta_0 - j}(\delta - j + 1)_j$ and $L_* = \sum_{j=0}^{\eta} (-1)^j z^{\eta_0^* - j}(\delta - j + 1)_j q_j(z)$. Hence,

$$R := \widehat{\mathcal{M}}(L) = \sum_{j=0}^{\eta} \sigma^{\eta_0 - j} q_j(\sigma)(n-j+1)_j \text{ and } R_* := \widehat{\mathcal{M}}(L_*) = \sum_{j=0}^{\eta} (-1)^j (n-\eta_0^*+1)_j \sigma^{\eta_0^* - j} q_j(\sigma)$$

are both Σ-operators. Note that the solutions of R and R_* depend on η_0 and η_0^* only by the first value of n from which each recurrence is solved, see §3.1. The four operators considered in §§5.1-5.3 are such that $\mathcal{R}_1 = \mathcal{R}_{1,*}$, $\mathcal{R}_2 = \mathcal{R}_{2,*}$, $\mathcal{R}_3 = \mathcal{R}_{3,*}$ and $\mathcal{R}_4 = \mathcal{R}_{4,*}$ because they come from four self-adjoint G-operators. The adjoint of the hypergeometric operator $\mathcal{H}(a,b;c)$ is $\mathcal{H}(1-a, 1-b; 2-c)$.

If $(u_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$ in $\overline{\mathbb{Q}}^{\mathbb{N}}$ are solutions of Σ-operators, then $(u_n + v_n)_{n\geq 0}$, $(u_n \cdot v_n)_{n\geq 0}$ and $(\sum_{k=0}^{n} u_k v_{n-k})_{n\geq 0}$ are also solutions of Σ-operators. In particular $(r(n)u_n)_{n\geq 0}$ is a solution of a Σ-operator, where $r(X) \in \overline{\mathbb{Q}}(X)$ and the roots of the denominator of $r(X)$ are in $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$. Indeed, let R_1 and R_2 be Σ-operators such that $R_1 u_n = 0$ and $R_2 v_n = 0$. The functions $f(z) = \sum_{n\geq 0} u_n z^n$ and $g(z) = \sum_{n\geq 0} v_n z^n$ are G-functions by Theorem 1(ii). Then the sum $f(z) + g(z)$, the Hadamard product $f \odot g$ and ordinary product fg are also G-functions. They are all solutions of G-operators by Chudnovsky theorem [12], the images of which by $\widehat{\mathcal{M}}$ are the requested Σ-operators.

4 Proof of Theorem 1

In this section, we first prove a lemma and then give the proof of Theorem 1.

4.1 Composition of a G-function and of an algebraic function

We shall need the following useful result.

Lemma 1. *Let F be a G-function and $A \in \overline{\mathbb{Q}}[[z]]$ be an algebraic function over $\overline{\mathbb{Q}}(z)$ such that $A(0) = 0$. Then $F \circ A$ is a G-function.*

This lemma is proved in [3, p. 717, Footnote 9] when A is further assumed to be a rational function. Note that it is not true in general that the composition $F_1 \circ F_2$ of two G-functions (with $F_2(0) = 0$) is a G-function. In fact, it is already not true in general that $\alpha \circ F$ is a G-function when F is a G-function such that $F(0) = 0$ and α is algebraic function over $\overline{\mathbb{Q}}(z)$ holomorphic at $z = 0$. For instance, $\sqrt{1 - 2\log(1-z)} \in \mathbb{Q}[[z]]$ is not a G-function because it has a singularity at $z = 1 - \sqrt{e} \notin \overline{\mathbb{Q}}$, while the finite singularities of any G-function are in $\overline{\mathbb{Q}}$.

Proof. We shall check that $F \circ A$ satisfies the three conditions (in the same order) stated in the introduction to be a G-function.

That $F \circ A$ satisfies a linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$ follows from the following general statement, due to Stanley [28, p. 180, Theorem 2.7]. Let $\mathbb{K}$ be a subfield of $\mathbb{C}$ and $F \in \mathbb{K}[[z]]$ be a solution of a non-zero linear differential equation with coefficients in $\mathbb{K}(z)$. Then, for any algebraic function A over $\mathbb{K}(z)$, holomorphic at $z = 0$ such that $A(0) = 0$, the function $F \circ A$ is solution of a non-zero linear differential equation with coefficients in $\mathbb{K}(z)$.

Writing $F(A(z)) = \sum_{n=0}^{\infty} a_n z^n$, the second condition on the archimedean growth obviously holds if $\sigma = id$ because both F and A have positive radii of convergence, hence this is also the case of $F \circ A$. The general case can be reduced to the case $\sigma = id$. Indeed, let $\mathbb{K}$ be a Galois number field containing the Taylor coefficients of $F(A(z))$ and those of $F(z) = \sum_{n=0}^{\infty} b_n z^n$ and $A(z) = \sum_{n=1}^{\infty} c_n z^n$. Then, for any $\sigma \in \mathrm{Gal}(\mathbb{K}/\mathbb{Q})$, we have

$$\sum_{n=0}^{\infty} \sigma(a_n) z^n = \sum_{n=0}^{\infty} \sigma(b_n) \Big(\sum_{m=1}^{\infty} \sigma(c_m) z^m \Big)^n,$$

where $\sum_{n=0}^{\infty} \sigma(b_n) z^n$ is a G-function and $\sum_{m=1}^{\infty} \sigma(c_m) z^m$ is algebraic over $\overline{\mathbb{Q}}(z)$.

It remains to check the third condition on the growth of the denominators. For any integer $n \geq 0$, we set $A(z)^n = \sum_{m=0}^{\infty} c_{m,n} z^m \in \overline{\mathbb{Q}}[[z]]$, with $c_{m,n} = 0$ for $0 \leq m \leq n-1$. The series

$$\sum_{m,n \geq 0} c_{m,n} z^m x^n = \sum_{n=0}^{\infty} A(z)^n x^n = \frac{1}{1 - xA(z)}$$

is a bivariate algebraic function. We now use Safonov's theorem [27, p. 273], a multivariate generalization of Eisenstein's theorem, to conclude that there exists an integer $C \geq 1$ such that $C^{m+n+1} c_{m,n}$ is an algebraic integer for all $m, n \geq 0$. Now, we have

$$F(A(z)) = \sum_{n=0}^{\infty} b_n \sum_{m=n}^{\infty} c_{m,n} z^m = \sum_{n=0}^{\infty} b_n \sum_{m=0}^{\infty} c_{m+n,n} z^{m+n} = \sum_{k=0}^{\infty} \Big(\sum_{n=0}^{k} b_n c_{k,n} \Big) z^k.$$

Since $\sum_{n=0}^{\infty} b_n z^n$ is a G-function, there exists a sequence of positive integers $(B_k)_{k \geq 0}$ such that $B_k b_n$ is an algebraic integer for all $n \leq k$ and $B_k \leq B^{k+1}$ for some $B \geq 1$. We also assume without loss of generality that B_k is the least possible positive denominator for each $k \geq 0$, so that B_k divides B_{k+1} for all $k \geq 0$. Therefore, $B_k C^{2k+1} \sum_{n=0}^{k} b_n c_{k,n}$ is an algebraic integer for all $k \geq 0$, and the third condition holds with $d_k := B_k C^{2k+1}$ and $D := BC^2$. This completes the proof that $F \circ \alpha$ is a G-function.

For completeness, let us mention that Safonov's theorem is proved in [27] under the assumption that the Taylor coefficients of the multivariate algebraic series are in $\mathbb{Q}$. The general case used above be deduced from it. Indeed, consider an algebraic series

$$F(X_1, \dots, X_s) := \sum_{n_1 \geq 0, \dots, n_s \geq 0} c_{n_1, \dots, n_s} X^{n_1} \cdots X^{n_s} \in \overline{\mathbb{Q}}[[X_1, \dots, X_s]].$$

The coefficients $c_{n_1, \dots, n_s}$ all lie into a Galois number field $\mathbb{Q}(\alpha)$ of degree $d \geq 1$. Hence, there exists d multivariate sequences of *rational* numbers $(u_{j,n_1,\dots,n_s})_{n_1,\dots,n_s \geq 0}$, $j = 0, \dots, d-1$, such that $c_{n_1,\dots,n_s} = \sum_{j=0}^{d-1} u_{j,n_1,\dots,n_s}\alpha^j$. Now, each series $\sum_{n_1\geq 0,\dots,n_s\geq 0} u_{j,n_1,\dots,n_s} X^{n_1}\cdots X^{n_s} \in \mathbb{Q}[[X_1,\dots,X_s]]$ is an algebraic function over $\mathbb{Q}(X_1,\dots,X_s)$ because it is a $\overline{\mathbb{Q}}$-linear combination of the algebraic series

$$\sum_{n_1\geq 0,\dots,n_s\geq 0} \sigma(c_{n_1,\dots,n_s}) X^{n_1}\cdots X^{n_s} \in \overline{\mathbb{Q}}[[X_1,\dots,X_s]]$$

where σ runs through $\mathrm{Gal}(\mathbb{Q}(\alpha)/\mathbb{Q})$. We can thus apply Safonov's theorem to each of them separately and let the integers $C_j \geq 1$ denote their respective Eisenstein's constant, *i.e.* $C_j^{n_1+\cdots+n_s+1} u_{j,n_1,\dots,n_s} \in \mathbb{Z}$. Let also the integer $M \geq 1$ denote a denominator of α. Then, $D := \mathrm{lcm}(C_0, C_1 M \dots, C_{d-1}M^{d-1})$ is such that $D^{n_1+\cdots+n_s+1} c_{n_1,\dots,n_s}$ is an algebraic integer for all $n_1,\dots,n_s \geq 0$. □

4.2 Proof of Theorem 1

• **Proof of** (i)**.** We use definitions in [11, §3.2] for what follows; the forward shift operator τ is used there but the notions and results can be adapted in a straightforward way for σ. We first recall that in our situation $R = \sum_{j=0}^{\mu} p_j(n)\sigma^j$ with $\deg(p_j) \leq \deg(p_0) = \deg(p_\mu) =: d$ for all j because $\mathcal{M}(R)$ is a Fuchsian operator. The Newton σ-polygon of R is the boundary of the lower convex hull in the half-strip $[0,\mu]\times\mathbb{R}^+$ of the region above the polygonal line with vertices $(j, \mathrm{val}_{t=0}(t^d p_j(1/t))$ for $j = 0,\dots,\mu$. Since $(0,0)$ and $(\mu,0)$ are amongst these vertices, this polygon is simply $\big(\{0\}\times\mathbb{R}^+\big)\cup\big([0,\mu]\times\{0\}\big)\cup\big(\{\mu\}\times\mathbb{R}^+\big)$. Hence, 0 is the only slope of R. Moreover, the Newton σ-polynomial of R for the slope 0 is $\sum_j' \lambda_j X^j$ where λ_j is the leading coefficient of p_j and the sum is restricted to the $j \in [\![0,\mu]\!]$ such that $\deg(p_j) = d$: the multiset of the roots of this polynomial is the multiset of the inverses of the non-zero singularities of $\mathcal{M}(R)$ (by multiset, we mean that an element appears as many times as its multiplicity as a root/singularity).

The generalized exponents of R are of the form

$$\zeta\cdot n^v\cdot\Big(1+\frac{a_1}{n^{1/r}}+\frac{a_2}{n^{2/r}}+\cdots+\frac{a_r}{n^{r/r}}\Big)$$

where v is a slope of the Newton σ-polygon of R, ζ is a root of the Newton σ-polynomial of R for the slope v, and r is a ramification index. Here, $v = 0$ and $\zeta = 1/\xi$ where ξ is a non-zero singularity of $\mathcal{M}(R)$. We shall now prove that $r = 1$ and determine the possible values for a_1. This will also give a different proof that $v = 0$ and $\zeta = 1/\xi$.

In [22, Chapitre 3, §29 & §34], for each non-zero singularity ξ of $L := \mathcal{M}(R)$, Norlünd associates μ independent solutions of R: they are of the form (for any large enough integer n)

$$\xi^{-n}\sum_{s\in S_\xi}\sum_{0\leq q\leq\eta}\sum_{k=0}^{\infty}\phi_{\xi,q,s,\omega,k}\frac{\partial^q}{\partial\varepsilon^q}\Big(\frac{\Gamma(k+s+\varepsilon+1)\Gamma(n/\omega)}{\Gamma(k+s+\varepsilon+n/\omega+1)}\Big)_{|\varepsilon=0}, \tag{4.1}$$

where η is the order of L, the elements of the finite S_ξ are distinct mod $\mathbb{Z}$ and each coincides mod $\mathbb{Z}$ with a local exponent of L at ξ, and the coefficients $\phi_{\xi,q,s,\omega,k} \in \mathbb{C}$ are not all zero. Any solution of R is a finite $\mathbb{C}$-linear combination of such series (for all n large enough).

We can rewrite the inner series in (4.1) as

$$\frac{\Gamma(n/\omega)}{\Gamma(n/\omega+s+1)}\sum_{k=0}^{\infty}\phi_{\xi,q,s,\omega,k}\frac{\partial^q}{\partial\varepsilon^q}\Big(\frac{\Gamma(k+s+\varepsilon+1)}{(n/\omega+s+\varepsilon+1)_k}\Big)_{|\varepsilon=0}. \tag{4.2}$$

Then the generalized asymptotic expansion of the series in (4.2) is of the form

$$\sum_{p=0}^{q}\log(n/\omega)^p\sum_{k=0}^{\infty}\frac{\widehat{\phi}_{\xi,p,q,s,\omega,k}}{(n/\omega)^k}, \tag{4.3}$$

where the $q+1$ series in (4.3) may no longer be convergent but are asymptotic series as $n\to+\infty$ in the Poincaré sense, and the coefficients $\widehat{\phi}_{\xi,p,q,s,\omega,k}$ are not all zero. Moreover, we also have the asymptotic expansion as $n\to+\infty$:

$$\frac{\Gamma(n/\omega)}{\Gamma(n/\omega+s+1)}\sim\sum_{k=0}^{\infty}\frac{(-1)^k\binom{k+s+1}{k}P_k(s+1)}{(n/\omega)^{k+s+1}}, \tag{4.4}$$

where the polynomial $P_k(x)\in\mathbb{Q}[x]$ of degree k are defined by the Taylor expansion $(\frac{t}{e^t-1})^x=\sum_{k\geq 0}P_k(x)t^k/k!$; see [5, p. 615]. Note that $P_0(x)=1$ so that $\frac{\Gamma(n/\omega)}{\Gamma(n/\omega+s+1)}\sim(n/\omega)^{-s-1}$, as confirmed by Stirling's formula.

Combining (4.3) and (4.4) to obtain the asymptotic expansion of (4.2), we then obtain the leading term in the asymptotic expansion of (4.1):

$$\xi^{-n}\sum_{s\in S_\xi}\sum_{0\leq q\leq\eta}\sum_{k=0}^{\infty}\phi_{\xi,q,s,\omega,k}\frac{\partial^q}{\partial\varepsilon^q}\Big(\frac{\Gamma(k+s+\varepsilon+1)\Gamma(n/\omega)}{\Gamma(k+s+\varepsilon+n/\omega+1)}\Big)_{|\varepsilon=0}\sim c_0\frac{\log(n)^p}{\xi^n\,n^{s+m}}=:E_n \tag{4.5}$$

where $c_0\neq 0$, $m\in\mathbb{Z}_{\geq 1}$, s is an element of S_ξ such that $s+m$ is minimal, and $p\in[\![0,\eta]\!]$ is maximal.

The generalized exponents of R are computed from the asymptotic expansion of E_{n+1}/E_n as $n\to+\infty$. We have

$$\frac{E_{n+1}}{E_n}=\frac{1}{\xi}\cdot\Big(1-\frac{s+m}{n}+\mathcal{O}\big(\frac{1}{n\log(n)}\big)\Big).$$

Hence, by definition (*cf* [11, §3.2]) the generalized exponents of R are of the form $\xi^{-1}(1-(s+m)/n)$.

• **Proof of** (*ii*)**.** Let $L\in\overline{\mathbb{Q}}[z,\delta]$ be a G-operator such that $R=\mathcal{M}(L)$ is of order μ and let $f(z)=\sum_{n\geq 0}u_nz^n$. The recurrence relation $Ru_n=0$ for all $n\geq m$ (for some integer $m\geq\mu$) means that $Lf(z)$ is a polynomial in $\overline{\mathbb{Q}}[z]$ of degree at most $m-1$, hence $\partial^mLf(z)=0$ for some integer $m\geq 0$. Now, ∂^mL is a G-operator and by the ACK theorem, f is a G-function.

Solving the recurrence $R:=\sum_{j=0}^{\mu}p_j(n)\sigma^j=0$ backward (*i.e.*, for $n\leq m\leq 0$) amounts to solving the recurrence $\widetilde{R}:=\sum_{j=0}^{\mu}p_{\mu-j}(\mu-n)\sigma^j=0$ forward (*i.e.*, for $n\geq -m$).

Now, there exists an integer k such that $\widetilde{L} := z^k\mathcal{M}(\widetilde{R}) \in \overline{\mathbb{Q}}[z,\delta]$ is the operator obtained from the G-operator $\mathcal{M}(R)$ by changing z to $1/z$: again, it is a G-operator. Hence, since $\widetilde{L}(\sum_{n\geq -m} u_{-n}z^n) = q(z)$ for some $q(z) \in \overline{\mathbb{Q}}[z]$, the series $\sum_{n\geq -m} u_{-n}z^n$ is a G-function.

• **Proof of** (iii)**.** Let R be a Σ-operator of order μ such that $R = \widehat{\mathcal{M}}(L)$ for some G-operator L of order η. As seen in (ii), there exists an integer $m_0 \geq \mu$ such that $(\partial^{m_0}L)(\sum_{n\geq 0} u_n z^n) = 0$. As recalled above, $L_0 := \partial^{m_0}L$ is a G-operator, of order $\eta_0 := \eta + m_0$ say, and its singularities are those of L. L_0 has μ non-zero singularities because L is Fuchsian (Proposition 1(i)). The principle of what follows is classical, and are an explicitation of the results mentioned in (i) in this situation. It can be found for instance in [22, Chapitre 3], and Norlünd's results have also been used in "the G-functions" situation in [18, §7.1]. However, no attention was paid there on the arithmetic nature of the various coefficients and functions involved. To do this, we have in fact to provide the details of the computations.

We denote by $\xi_1,\ldots,\xi_{\mu_0} \in \overline{\mathbb{Q}}$ ($\mu_0 \leq \mu$) the distinct non-zero singularities of L_0 counted without multiplicity; 0 might also be a singularity of L_0 but we discard it from the discussion. Let $\mathrm{Ind}_{M,\alpha}(X)$ denote the indicial polynomial of $M \in \mathbb{C}[z,\partial]$ at $\alpha \in \mathbb{C}$. The operator ∂^{m_0} has no finite singularity, so that $\mathrm{Ind}_{\partial^{m_0},\alpha}(X) = X(X-1)\cdots(X-m_0+1)$. Moreover, $\mathrm{Ind}_{L_0,\alpha}(X) = \mathrm{Ind}_{\partial^{m_0},\alpha}(X-\varpi)\cdot\mathrm{Ind}_{L,\alpha}(X)$ for some integer ϖ; see the proof of [25, Lemma 1]. Therefore, at ξ_j, the set S_j of the local exponents of L_0 is the union of the set of the local exponents of L and of a finite set of integers.

To simplify the exposition, we now assume that on any half-line $e^{i\alpha}[0,+\infty)$ lies at most one ξ_j. We then define the half-lines $C_j := \xi_j[1,+\infty)$ ($j = 1,\ldots,\mu_0$) originating from ξ_j and going to ∞ in the direction $\arg(\xi_j)$; these H_j do not intersect pairwise. The function f can be analytically continued to the simply connected domain $\mathcal{D} := \mathbb{C}\setminus\cup_{j=1}^{\mu_0}C_j$. The general case when two or more ξ_j lie on certain half-lines $e^{i\alpha}[0,\infty)$ can be done similarly but it requires to define different cuts and suitable determination of logarithms on these cuts. This is done in details in [18, §7.1] and the result (iii) we want to prove follows again by analysing [18, Eq. (7.5)]) in the light of the computations done below to guarantee the "G-aspect". Now, using an ACK basis for f at each $\xi_j \neq 0$, we have in a slit neighborhood of $z = 1$:

$$f(\xi_j z) = \sum_{s\in S_j}\sum_{k\in K_{j,s}} \rho_{j,k,s}(z-1)^s\log_j(z-1)^k F_{j,k,s}(z-1) \tag{4.6}$$

where $S_j \subset \mathbb{Q}$ is defined above, $K_{j,s}$ are finite subsets of $\mathbb{Z}_{\geq 0}$ (possibly empty) such that $\sum_{s\in S_j}\#K_{j,s} = \eta_0$, $F_{j,k,s}(z) \in \overline{\mathbb{Q}}[[z]]$ are G-functions, and $\rho_{j,k,s} \in \mathbf{G}$. The functions $\log(z-1)$ and $(z-1)^s$ are defined by $0 \leq \arg(z-1) < 2\pi$. We set

$$F_{j,k,s}(z) = \sum_{m=0}^{\infty}\phi_{j,k,s,m}z^m. \tag{4.7}$$

For all $n \geq 0$, we have

$$u_n = \frac{1}{2i\pi}\int_\gamma \frac{f(z)}{z^{n+1}}dz, \tag{4.8}$$

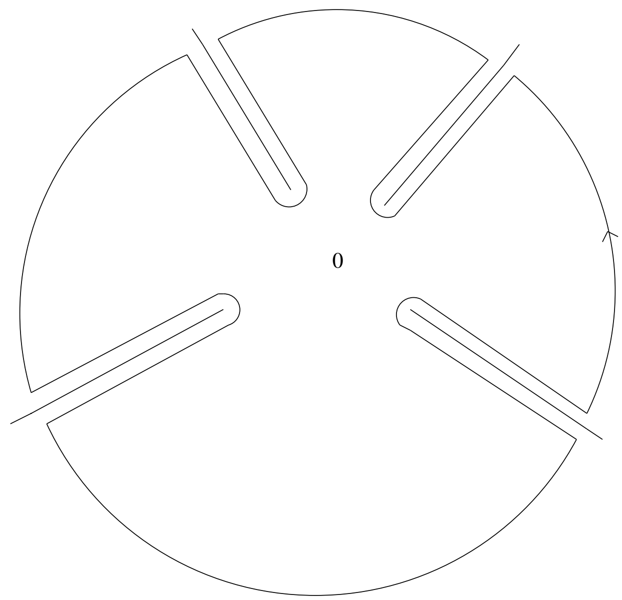

Figure 1: The cuts C_j and the contour γ.

where γ is the contour oriented in the positive sense represented in Figure 1 (with four cuts). Since there exists $N \in \mathbb{Z}_{\geq 1}$ (that depends on L_0, and thus R and $(u_n)_{n\geq 0}$) such that $|f(z)| \leq |z|^{N-1}$ as $z \to \infty$ in $\mathcal{D}$, we can let the contour γ "tends to infinity" for any $n \geq N$ in (4.8). From now on, we assume that $n \geq N$. We thus have

$$u_n = \sum_{j=1}^{\mu_0} \frac{\xi_j^{-n}}{2i\pi} \int_{\widetilde{\gamma}_j} \frac{f(\xi_j z)}{z^{n+1}} dz \tag{4.9}$$

where $\widetilde{\gamma}_j$ is now a Hankel type contour composed of two half-lines parallel to $[1, +\infty)$ and joined by a half-circle of center 1 in the negative sense, and close enough to $[1, +\infty)$ so that $f(\xi_j z)$ has no singularity along γ_j. ([3]) If ξ_j is a pole of f, the corresponding integral in (4.9) is simply equal to the residue of $f(z)/z^{n+1}$ at $z = \xi_j$, but the formalism adopted here covers this case as well (in a more complicated way than necessary).

On each integral, we would like to use in (4.9) the expansion (4.6) together with (4.7). However, we only know that the series $F_{j,k,s}(z-1)$ converge in a disk of center 1 and (unknown) positive radius but this is not enough to substitute (4.6) into (4.9) and inverse the summation and integral signs. A method described in [22, Chapitre 3] elegantly solves this problem. We set $x = 1 - z^{-\omega}$ where ω is a positive integer to be specified later; we have

$$z - 1 = (1-x)^{-1/\omega} - 1 = (x/\omega) f_\omega(x)$$

[3]Let us mention that the integrals $v_{j,n} := \frac{1}{2i\pi}\int_{\xi_j\widetilde{\gamma}_j} \frac{f(z)}{z^{n+1}}dz$ are solutions of R provided n is large enough. The proof is based on the observation that $0 = \frac{1}{2i\pi}\int_{\xi_j\widehat{\gamma}_j}\frac{Lf(z)}{z^{n+1}}dz = Rv_{j,n}$ because, by repeated integration by parts, we see that $\frac{1}{2i\pi}\int_{\xi_j\widehat{\gamma}_j}\frac{z^m\delta^\ell f(z)}{z^{n+1}}dz = \sigma^m(n^\ell v_{j,n})$ for all large enough n.

where $f_\omega(x) \in 1+x\mathbb{Q}[[x]]$ is algebraic over $\overline{\mathbb{Q}}(x)$, with radius of convergence 1 and such that $f_\omega(0)=1$ and $f_\omega(x)>0$ for all $x\in[0,1]$. Observe that $\log(f_\omega(x))$ and $f_\omega(x)^s := e^{s\log(f_\omega(x))}$ are G-functions by Lemma 1, both with radius of convergence 1. ([4]) Since the functions $F_{j,k,s}((1-x)^{-1/\omega}-1)$ are also G-functions (by Lemma 1 again) with radius of convergence 1 for all $\omega \geq \Omega'$ for some $\Omega' \geq 1$ (by [22, pp. 62–63]), we have for any $s\in\mathbb{Q}$ and $p\in\mathbb{Z}_{\geq 0}$ that

$$f_\omega(x)^s \log\big(f_\omega(x)\big)^p F_{j,k,s}\big((1-x)^{-1/\omega}-1\big) =: \Phi_{j,k,p,s,\omega}(x) \in \overline{\mathbb{Q}}[[x]]$$

where the series $\Phi_{j,k,p,s,\omega}(x) := \sum_{m=0}^\infty \phi_{j,k,p,s,\omega,m}x^m$ are G-functions all with radius of convergence 1 provided $\omega \geq \Omega'$.

Now, for all $n \geq N+1$, we have

$$u_n = \sum_{j=1}^{\mu_0} \frac{\xi_j^{-n}}{2i\pi\omega}\int_{\widehat{\gamma}_{j,\omega}} (1-x)^{n/\omega-1} f(\xi_j(1-x)^{-1/\omega})dx \tag{4.10}$$

where the direct contour $\widehat{\gamma}_{j,\omega}$ is now a closed path surrounding $[0,1]$, oriented in the positive sense and passing through 1. Moreover, $\widehat{\gamma}_{j,\omega}$ "tends" to $[0,1]$ when $\omega \to +\infty$. Therefore, there exists $\Omega \geq \Omega'$ such that for any $\omega \geq \Omega$, we can invert the summation and integral signs below, because $\widehat{\gamma}_{j,\omega}$ is then inside the closed unit disk: we have

$$u_n = \sum_{j=1}^{\mu_0}\sum_{s\in S_j}\sum_{k\in K_{j,s}} \rho_{j,k,s}\Big(\frac{\xi_j^{-n}}{2i\pi\omega}\int_{\widehat{\gamma}_{j,\omega}} (1-x)^{n/\omega-1}\Big(\frac{x}{\omega}f_\omega(x)\Big)^s \log\Big(\frac{x}{\omega}f_\omega(x)\Big)^k F_{j,k,s}\big((1-x)^{-1/\omega}-1\big)dx\Big). \tag{4.11}$$

Now, a standard computation shows that ([5])

$$\begin{aligned}\int_{\widehat{\gamma}_{j,\omega}} &(1-x)^{n/\omega-1}\big(xf_\omega(x)/\omega\big)^s \log\big(xf_\omega(x)/\omega\big)^k F_{j,k,s}\big((1-x)^{-1/\omega}-1\big)dx = \\ e^{2i\pi s}&\int_0^1 (1-x)^{n/\omega-1}(x/\omega)^s f_\omega(x)^s\big(\log(x/\omega)+\log(f_\omega(x))+2i\pi\big)^k F_{j,k,s}\big((1-x)^{-1/\omega}-1\big)dx \\ -&\int_0^1 (1-x)^{n/\omega-1}(x/\omega)^s f_\omega(x)^s\big(\log(x/\omega)+\log(f_\omega(x))\big)^k F_{j,k,s}\big((1-x)^{-1/\omega}-1\big)dx.\end{aligned} \tag{4.12}$$

[4] The function f_ω is holomorphic and does not vanish in the domain $|x|<1$, and $f_\omega(0)=1$. Hence $\log(f_\omega)$ is well defined and holomorphic on $|x|<1$, and the radius of convergence of its Taylor expansion at $x=0$ is 1 because f_ω is singular at $x=1$.

[5] If $s\in\mathbb{Z}$ and $k=0$, then (4.12) is simply equal to 0.

Both integrals can be delt with in a similar way. We have

$$\begin{aligned}
&\int_0^1 (1-x)^{n/\omega-1}(x/\omega)^s f_\omega(x)^s\big(\log(x/\omega)+\log(f_\omega(x))\big)^k F_{j,k,s}\big((1-x)^{-1/\omega}-1\big)dx \\
&\quad=\sum_{p=0}^{k}\binom{k}{p}\int_0^1(1-x)^{n/\omega-1}(x/\omega)^s\log(x/\omega)^{k-p}\Phi_{j,k,p,s,\omega}(x)dx \\
&\quad=\sum_{p=0}^{k}\binom{k}{p}\sum_{m=0}^{\infty}\phi_{j,k,p,s,\omega,m}\int_0^1 x^m(1-x)^{n/\omega-1}(x/\omega)^s\log(x/\omega)^{k-p}dx \\
&\quad=\sum_{p=0}^{k}\binom{k}{p}\sum_{m=0}^{\infty}\phi_{j,k,p,s,\omega,m}\int_0^1 x^m(1-x)^{n/\omega-1}\Big(\frac{\partial^{k-p}\big((x/\omega)^t\big)}{\partial t^{k-p}}\Big)_{|t=s}dx \\
&\quad=\sum_{p=0}^{k}\binom{k}{p}\sum_{m=0}^{\infty}\phi_{j,k,p,s,\omega,m}\frac{\partial^{k-p}}{\partial t^{k-p}}\Big((1/\omega)^t\int_0^1 x^{m+t}(1-x)^{n/\omega-1}dx\Big)_{|t=s} \\
&\quad=\sum_{p=0}^{k}\binom{k}{p}\sum_{m=0}^{\infty}\phi_{j,k,p,s,\omega,m}\frac{\partial^{k-p}}{\partial t^{k-p}}\Big((1/\omega)^t\frac{\Gamma(m+t+1)\Gamma(n/\omega)}{\Gamma(m+t+n/\omega+1)}\Big)_{|t=s} \\
&\quad=\sum_{p=0}^{k}\binom{k}{p}\sum_{q=0}^{k-p}\binom{k-p}{q}(1/\omega)^s\log(1/\omega)^{k-p-q} \\
&\qquad\times\sum_{m=0}^{\infty}\phi_{j,k,p,s,\omega,m}\frac{\partial^q}{\partial\varepsilon^q}\Big(\frac{\Gamma(m+s+\varepsilon+1)\Gamma(n/\omega)}{\Gamma(m+s+\varepsilon+n/\omega+1)}\Big)_{|\varepsilon=0} \qquad (4.13)
\end{aligned}$$

By a similar process, we have

$$\begin{aligned}
&\int_0^1 (1-x)^{n/\omega-1}(x/\omega)^s f_\omega(x)^s\big(\log(x/\omega)+\log(f_\omega(x))+2i\pi\big)^k F_{j,k,s}\big((1-x)^{-1/\omega}-1\big)dx \\
&\quad=\sum_{r=0}^{k}\binom{k}{r}(2i\pi)^{k-r} \\
&\qquad\times\int_0^1 (1-x)^{n/\omega-1}(x/\omega)^s f_\omega(x)^s\big(\log(x/\omega)+\log(f_\omega(x))\big)^r F_{j,k,s}\big((1-x)^{-1/\omega}-1\big)dx \\
&\quad=\sum_{r=0}^{k}\binom{k}{r}(2i\pi)^{k-r}\sum_{p=0}^{r}\binom{r}{p}\sum_{q=0}^{r-p}\binom{r-p}{q}(1/\omega)^s\log(1/\omega)^{r-p-q} \\
&\qquad\times\sum_{m=0}^{\infty}\phi_{j,k,p,s,\omega,m}\frac{\partial^q}{\partial\varepsilon^q}\Big(\frac{\Gamma(m+s+\varepsilon+1)\Gamma(n/\omega)}{\Gamma(m+s+\varepsilon+n/\omega+1)}\Big)_{|\varepsilon=0} \qquad (4.14)
\end{aligned}$$

Using Eqs. (4.13) and (4.14) in (4.11), we thus have for all $n \geq N$ and all $\omega \geq \Omega$

(independently of n):

$$u_n = \sum_{j=1}^{\mu_0}\sum_{s\in S_j}\sum_{k\in K_{j,s}}\sum_{r=0}^{k}\sum_{p=0}^{r}\sum_{q=0}^{r-p}\Bigg(\binom{k}{r}\binom{r}{p}\binom{r-p}{q}e^{2i\pi s}\rho_{j,k,s}(2i\pi)^{k-r-1}(1/\omega)^{s+1}\log(1/\omega)^{r-p-q}$$
$$\times\, \xi_j^{-n}\sum_{m=0}^{\infty}\phi_{j,k,p,s,\omega,m}\frac{\partial^q}{\partial\varepsilon^q}\Big(\frac{\Gamma(m+s+\varepsilon+1)\Gamma(n/\omega)}{\Gamma(m+s+\varepsilon+n/\omega+1)}\Big)_{|\varepsilon=0}\Bigg)$$
$$-\sum_{j=1}^{\mu_0}\sum_{s\in S_j}\sum_{k\in K_{j,s}}\sum_{p=0}^{k}\sum_{q=0}^{k-p}\Bigg(\binom{k}{p}\binom{k-p}{q}\rho_{j,k,s}(1/\omega)^{s+1}\log(1/\omega)^{k-p-q}/(2i\pi)$$
$$\times\, \xi_j^{-n}\sum_{m=0}^{\infty}\phi_{j,k,p,s,\omega,m}\frac{\partial^q}{\partial\varepsilon^q}\Big(\frac{\Gamma(m+s+\varepsilon+1)\Gamma(n/\omega)}{\Gamma(m+s+\varepsilon+n/\omega+1)}\Big)_{|\varepsilon=0}\Bigg) \tag{4.15}$$

In this expression, the terms for which $s\in\mathbb{Z}$ and $k=0$ globally contribute 0, and can be omitted. As already said, the form of this expansion (4.15) of u_n is not new, it holds for any solution of a difference operator associated to a Fuchsian differential operator. But in this particular situation where u_n is a solution of a Σ-operator, it has remarkable arithmetical properties: (1) the parameters s are integers or rational numbers equal (mod $\mathbb{Z}$) to local exponents at a non-zero singularity ξ_j of the G-operator $\mathcal{M}(R)$, (2) all the coefficients

$$\binom{k}{r}\binom{r}{p}\binom{r-p}{q}e^{2i\pi s}\rho_{j,k,s}(2i\pi)^{k-r-1}(1/\omega)^{s}\log(1/\omega)^{r-p-q}$$

and

$$\binom{k}{p}\binom{k-p}{q}\rho_{j,k,s}(1/\omega)^{s}\log(1/\omega)^{k-p-q}/(2i\pi)$$

are in the ring $\mathbf{G}$ of G-values, ([6]) and (3) the series $\sum_{m=0}^{\infty}\phi_{j,k,p,s,\omega,m}x^m$ are G-functions. This completes the proof of *(iii)* and of Theorem 1.

5 Examples

In this section, we present some details on difference operators alluded to in the introduction.

5.1 A Σ-operator related to $\ln(2)$

The difference operator $\mathcal{R}_1 := (n-1)\sigma^2-(6n-3)\sigma+n$ is related to $\ln(2)$. It has a basis of solutions $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ (*i.e.*, $m=\mu=2$ with our conventions) with $a_0=0, a_1=2$, $b_0=1, b_1=3$ such that $d_na_n\in\mathbb{Z}$ and $b_n\in\mathbb{Z}$, where $d_n=\mathrm{lcm}\{1,2,\ldots,n\}$. We have in particular $b_n=\sum_{j=0}^{n}\binom{n}{j}\binom{n+j}{n}$. Moreover, not only $a_n/b_n\to\ln(2)$ but $d_n(b_n\log(2)-a_n)\to 0$, and this implies that $\ln(2)$ has an irrationality exponent ≤ 4.623; see [1].

[6] We recall that $1/\pi\in\mathbf{G}$; see [15, §2.2].

The sequences $(a_{-n-1})_{n\leq 0}$ and $(b_{-n-1})_{n\leq 0}$ are backward solutions of $\mathcal{R}_1$ for all $n\leq 0$.

We have

$$\mathcal{L}_1 := \mathcal{M}(\mathcal{R}_1) = (\delta-1)z^2-(6\delta-3)z+\delta = (z^2-6z+1)\delta+z^2-3z = z\big((z^2-6z+1)\partial+(z-3)\big).$$

$\mathcal{L}_1$ a G-operator because it is of minimal order for the G-function $\lambda(z) := 1/\sqrt{1-6z+z^2} = \sum_{n=0}^{\infty} b_n z^n$, and it is self-adjoint: $\mathcal{L}_1 = \mathcal{L}_1^*$. Hence $\mathcal{R}_1$ is a Σ-operator such that $\mathcal{R}_1 = \mathcal{R}_{1,*}$. The non-zero singularities of $\mathcal{L}_1$ are $\xi_1 := (\sqrt{2}-1)^2$ and $\xi_2 := (\sqrt{2}+1)^2 = 1/\xi_1$, both with a local exponent $-\frac{1}{2}$. The generalized exponents of $\mathcal{R}_1$ are thus $\xi_1\cdot(1+(m_1+1/2)/n)$ and $\xi_2\cdot(1+(m_2+1/2)/n)$ for some $m_1,m_2\in\mathbb{Z}$.

We now follow the proof of Theorem 1(iii). Let $\alpha_1 := \xi_2/\xi_1$ and $\alpha_2 := 1/\alpha_1$. We have for all $n\geq 0$

$$\begin{aligned}b_n &= \frac{1}{2i\pi}\int_{\xi_1\widetilde{\gamma}_1}\frac{\lambda(z)}{z^{n+1}}dz + \frac{1}{2i\pi}\int_{\xi_2\widetilde{\gamma}_2}\frac{\lambda(z)}{z^{n+1}}dz = \frac{\xi_1^{-n}}{2i\pi}\int_{\widetilde{\gamma}_1}\frac{\lambda(\xi_1 z)}{z^{n+1}}dz + \frac{\xi_2^{-n}}{2i\pi}\int_{\widetilde{\gamma}_2}\frac{\lambda(\xi_2 z)}{z^{n+1}}dz\\ &= \frac{\xi_1^{-n-1}}{\pi\omega^{1/2}}\int_0^1(1-x)^{n/\omega-1}x^{-1/2}\Phi_{1,\omega}(x)dx + \frac{\xi_2^{-n-1}}{\pi\omega^{1/2}}\int_0^1(1-x)^{n/\omega-1}x^{-1/2}\Phi_{2,\omega}(x)dx,\end{aligned}$$

where $\Phi_{j,\omega}(x) = \sum_{m=0}^{\infty}\phi_{j,\omega,m}x^m$ $(j=1,2)$ are G-functions such that $\phi_{1,\omega,0} = (\alpha_1-1)^{-1/2}$ and $\phi_{2,\omega,0} = (1-\alpha_2)^{-1/2}$, and ω is large enough. Therefore by (4.13) (with $s=-1/2$ and $k=0$), we have for all $n\geq 0$

$$b_n = \frac{\xi_1^{-n-1}}{\pi\omega^{1/2}}\sum_{m=0}^{\infty}\phi_{1,\omega,m}\frac{\Gamma(m+1/2)\Gamma(n/\omega)}{\Gamma(m+1/2+n/\omega)} + \frac{\xi_2^{-n-1}}{\pi\omega^{1/2}}\sum_{m=0}^{\infty}\phi_{2,\omega,m}\frac{\Gamma(m+1/2)\Gamma(n/\omega)}{\Gamma(m+1/2+n/\omega)}.$$

Since

$$\frac{\Gamma(1/2)\Gamma(n/\omega)}{\Gamma(1/2+n/\omega)}\sim\frac{(\pi\omega)^{1/2}}{n^{1/2}}\quad\text{and}\quad\frac{\Gamma(m+1/2)\Gamma(n/\omega)}{\Gamma(m+1/2+n/\omega)} = \mathcal{O}\Big(\frac{1}{n^{m+1/2}}\Big)\quad(m\geq 1)$$

we conclude that ([7])

$$b_n\sim\frac{\xi_1^{-n-1}}{\sqrt{\pi(\alpha_1-1)n}} = \frac{(\sqrt{2}+1))^{2n+1}}{\sqrt{4\sqrt{2}\pi n}}.$$

It is believed that $1/\sqrt{\pi}$ is not in $\mathbf{G}$, so that it is probably not true that the coefficients of the asymptotic expansion of b_n are in $\mathbf{G}$. On the other hand, they are in the $\mathbf{G}$-module $\mathbf{S}$ generated by all the values of the derivatives of the Gamma function at rational points. $\mathbf{S}$ is also the $\mathbf{G}[\gamma]$-module generated by all the values of the Gamma function at rational points, where γ is Euler's constant, and it is a ring; see [16, §2].

It is proved in [1] that

$$b_n\log(2)-a_n = \int_0^1\frac{(x(1-x))^n}{(1+x)^{n+1}}dx\to 0$$

[7] A general result of Mcintosh [21] gives the full asymptotic expansion of b_n, as well as of the sequences v_n and q_n in §5.2, using their explicit "binomial sum" expressions.

hence

$$b_n \log(2) - a_n = \frac{\xi_2^{-n-1}}{\pi\omega^{1/2}} \sum_{m=0}^{\infty} \phi_{\omega,m} \frac{\Gamma(m+1/2)\Gamma(n/\omega)}{\Gamma(m+1/2+n/\omega)} \sim c_0 \frac{(\sqrt{2}-1))^{2n+1}}{\sqrt{n}},$$

where $c_0 \in \mathbb{C}^*$ and $\phi_{\omega,m} \in \mathbb{Q} + \mathbb{Q}\log(2)$.

5.2 Σ-operators related $\zeta(2)$ and $\zeta(3)$

Let us consider now the operators $\mathcal{R}_2 := (n-1)^2\sigma^2 + (11n^2 - 11n + 3)\sigma - n^2$ and $\mathcal{R}_3 := (n-1)^3\sigma^2 - (34n^3 - 51n^2 + 27n - 5)\sigma + n^3$. They are the celebrated difference operators used by Apéry [4] to prove the irrationality of $\zeta(2)$ and $\zeta(3)$ respectively. See [4, 23].

• $\mathcal{R}_2$ has a basis of solutions $(u_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$ (*i.e.*, $m = \mu = 2$ with our conventions) with $u_0 = 0, u_1 = 5$, $v_0 = 1, v_1 = 3$ such that $d_n^2 u_n \in \mathbb{Z}$ and $v_n \in \mathbb{Z}$. We have in particular $v_n = \sum_{j=0}^{n} \binom{n}{j}^2 \binom{n+j}{n}$. Then $d_n^2(v_n\zeta(2) - u_n) \to 0$, and this implies that $\zeta(2)$ has an irrationality exponent ≤ 11.8508.

The sequences $((-1)^n u_{-n-1})_{n\leq 0}$ and $((-1)^n v_{-n-1})_{n\leq 0}$ are backward solutions of $\mathcal{R}_2$ for all $n \leq 0$.

The operator

$$\begin{aligned}\mathcal{L}_2 := \mathcal{M}(\mathcal{R}_2) &= (\delta-1)^2 z^2 + (11\delta^2 - 11\delta + 3)z - \delta^2 \\ &= z(z^2 + 11z - 1)\partial^2 + (3z^2 + 22z - 1)\partial + (z+3)\end{aligned}$$

is a G-operator (because it is a Fuchsian operator of geometric origin [7, 30] hence a G-operator by a theorem of André [2, p. 111]) and it is self-adjoint. Hence, $\mathcal{R}_2$ is a Σ-operator. We also have $\mathcal{L}_2(\sum_{n\geq 0} v_n z^n) = 0$.

The non-zero singularities of $\mathcal{L}_2$ are $\alpha_1 := ((-\sqrt{5}-1)/2)^5$ and $\alpha_2 := ((\sqrt{5}-1)/2)^5 = -1/\alpha_1$, and the local exponents at both singularities are $0, 0$. Hence, the generalized exponents of $\mathcal{R}_2$ are $\alpha_1 \cdot (1 + m_1/n)$ and $\alpha_2 \cdot (1 + m_2/n)$ for some $m_1, m_2 \in \mathbb{Z}$.

For all n large enough, v_n is a $\mathbf{G}$-linear combination of the four series

$$\alpha_j^n \cdot \sum_{m=0}^{\infty} \phi_{j,1,\omega,m} \frac{\Gamma(m+1)\Gamma(n/\omega)}{\Gamma(m+n/\omega+1)}, \quad \alpha_j^n \cdot \sum_{m=0}^{\infty} \phi_{j,2,\omega,m} \frac{\partial}{\partial\varepsilon}\Big(\frac{\Gamma(m+1+\varepsilon)\Gamma(n/\omega)}{\Gamma(m+n/\omega+\varepsilon+1)}\Big)_{\varepsilon=0}, \quad (j=1,2) \tag{5.1}$$

where for $j = 1, 2$, $k = 1, 2$, the series $\sum_{m=0}^{\infty} \phi_{j,k,\omega,m} z^m$ are G-functions.

It is known [6] that

$$v_n\zeta(2) - u_n = \int_0^1 \int_0^1 \frac{\big(x(1-x)y(1-y)\big)^n}{(1-xy)^{n+1}} dxdy \to 0$$

so that $v_n\zeta(2) - u_n$ is a $\mathbb{C}$-linear combination of the series in (5.1) for $j = 2$ only.

• $\mathcal{R}_3$ has a basis of solutions $(p_n)_{n\geq 0}$ and $(q_n)_{n\geq 0}$ (*i.e.*, $m = \mu = 2$ with our conventions) with $p_0 = 0, p_1 = 6$, $q_0 = 1, q_1 = 5$ such that $d_n^3 p_n \in \mathbb{Z}$ and $q_n \in \mathbb{Z}$. We have in particular

$q_n = \sum_{j=0}^n \binom{n}{j}^2\binom{n+j}{n}^2$. Then $d_n^3(q_n\zeta(2) - p_n) \to 0$, and this implies that $\zeta(3)$ has an irrationality exponent ≤ 13.4179.

The sequences $(p_{-n-1})_{n\le 0}$ and $(q_{-n-1})_{n\le 0}$ are backward solutions of $\mathcal{R}_3$ for all $n \le 0$.

The operator

$$\begin{aligned}\mathcal{L}_3 := \mathcal{M}(\mathcal{R}_3) &= (\delta-1)^3 z^2 - (34\delta^3 - 51\delta^2 + 27\delta - 5)z + \delta^3 \\ &= z^2(z^2 - 34z + 1)\partial^3 + (6z^3 - 153z^2 + 3z)\partial^2 + (7z^2 - 112z + 1)\partial + (z-5)\end{aligned}$$

is a G-operator (because it is a Fuchsian operator of geometric origin [7, 13]) and it is self-adjoint. Hence, $\mathcal{R}_3$ is a Σ-operator. We also have $\mathcal{L}_3(\sum_{n\ge 0} q_n z^n) = 0$.

The non-zero singularities of $\mathcal{L}_3$ are $\beta_1 := (\sqrt{2}+1)^4$ and $\beta_2 := (\sqrt{2}-1)^4 = 1/\beta_1$, and the local exponents at both singularities are $0, 1, 1/2$. Hence, the generalized exponents of $\mathcal{R}_3$ are $\beta_1 \cdot (1 + (m_1 + s_1)/n)$ and $\beta_2 \cdot (1 + (m_2 + s_2)/n)$ for some $m_1, m_2 \in \mathbb{Z}$, and $s_1, s_2 \in \{0, 1/2\}$.

For all n large enough, q_n is a $\mathbf{G}$-linear combination of the six series

$$\beta_j^n \cdot \sum_{m=0}^{\infty} \phi_{j,1,\omega,m} \frac{\Gamma(m+1)\Gamma(n/\omega)}{\Gamma(m+n/\omega+1)}, \quad \beta_j^n \cdot \sum_{m=0}^{\infty} \phi_{j,2,\omega,m} \frac{\partial}{\partial\varepsilon}\Big(\frac{\Gamma(m+1+\varepsilon)\Gamma(n/\omega)}{\Gamma(m+n/\omega+\varepsilon+1)}\Big)_{\varepsilon=0},$$
$$\beta_j^n \cdot \sum_{m=0}^{\infty} \phi_{j,3,\omega,m} \frac{\Gamma(m+3/2)\Gamma(n/\omega)}{\Gamma(m+n/\omega+3/2)} \qquad (j=1,2) \quad (5.2)$$

where for $j = 1, 2$, $k = 1, 2, 3$, the series $\sum_{m=0}^{\infty} \phi_{j,k,\omega,m} z^m$ are G-functions.

It is known [6] that

$$q_n\zeta(3) - p_n = \frac{1}{2}\int_0^1\int_0^1 \frac{\big(x(1-x)y(1-y)z(1-z)\big)^n}{(1-(1-xy)z)^{n+1}} dxdydz \to 0$$

so that $v_n\zeta(2) - u_n$ is a $\mathbb{C}$-linear combination of the series in (5.2) for $j = 2$ only.

5.3 A Σ-operator related to $L(2, \chi_{-3})$

Let us consider the difference operator $\mathcal{R}_4 := 9(n-1)^2\sigma^2 - (10n^2 - 10n + 3)\sigma + n^2$ which recently appeared in the context of the proof of the linear independence over $\mathbb{Q}$ of $1, \zeta(2)$ and $L(2, \chi_{-3})$; see [10, p. 161]. We associate to $\mathcal{R}_4$ three sequences: $(a_n)_{n\ge 0}$, $(b_n)_{n\ge 0}$ with the initial conditions $a_0 = 1, a_1 = 3$, $b_0 = 0, b_1 = 1$ form a basis of solutions of $\mathcal{R}_4$ (*i.e.*, $m = \mu = 2$ with our conventions), and $(c_n)_{n\ge 0}$ with the initial conditions $c_0 = 0, c_1 = 1$ is solution of the inhomogeneous equation $\mathcal{R}_4 = 1$. ([8]) Then remarkably $b_n/a_n \to L(2, \chi_{-3})/2$ and $c_n/a_n \to \zeta(2)/4$, though not fast enough to imply (directly) the irrationality of one of these numbers. But they are the imput to prove that they are independent. Moreover, we have $a_n = \sum_{j=0}^n \binom{n}{j}^2\binom{2j}{j}$.

[8] This sequence $(c_n)_{n\ge 0}$ is solution of the third order operator $(\sigma-1)\mathcal{R}_4$, which is a Σ-operator because $\mathcal{M}((\sigma-1)\mathcal{R}_4) = (z-1)\mathcal{M}(\mathcal{R}_4)$ is a G-operator. The latter is not self-adjoint.

The sequences $(9^{-n-1}a_{-n-1})_{n\le 0}$ and $(9^{-n-1}b_{-n-1})_{n\le 0}$ are backward solutions of $\mathcal{R}_4$ for all $n \le 0$.

The operator

$$\begin{aligned}\mathcal{L}_4 := \mathcal{M}(\mathcal{R}_4) &= 9(\delta-1)^2z^2 - (10\delta^2-10\delta+3)z+\delta^2\\ &= z(z-1)(9z-1)\partial^2+(27z^2-20z+1)\partial+3(3z-1)\end{aligned}$$

is a G-operator (because it is a Fuchsian operator of geometric origin [30, §7]) and it is self-adjoint. Hence $\mathcal{R}_4$ is a Σ-operator. We also have $\mathcal{L}_4(\sum_{n\ge 0} a_n z^n)=0$.

The non-zero singularities of $\mathcal{L}_4$ are $\xi_1 = 1$ and $\xi_2 = 1/9$, both with local exponents $0, 0$. The generalized exponents of $\mathcal{R}_4$ are thus $1+m_1/n$ and $9(1+m_2/n)$ for some $m_1, m_2 \in \mathbb{Z}$.

For all n large enough, a_n is a $\mathbf{G}$-linear combination of the four series

$$\xi_j^{-n}\sum_{m=0}^{\infty}\phi_{j,1,\omega,m}\frac{\Gamma(m+1)\Gamma(n/\omega)}{\Gamma(m+n/\omega+1)}, \quad \xi_j^{-n}\sum_{m=0}^{\infty}\phi_{j,2,\omega,m}\frac{\partial}{\partial\varepsilon}\Big(\frac{\Gamma(m+1+\varepsilon)\Gamma(n/\omega)}{\Gamma(m+n/\omega+\varepsilon+1)}\Big)_{\varepsilon=0}, \quad (j=1,2) \tag{5.3}$$

where for $j = 1, 2$, $k = 1, 2$, the series $\sum_{m=0}^{\infty}\phi_{j,k,\omega,m}z^m$ are G-functions. It is proved in [10, p. 163] that

$$a_nL(2,\chi_{-3}) - 2b_n = \int_0^1\int_0^1 \frac{\big(9xy(1-x^3)(1-y^3)\big)^n}{\big(1+xy+x^2y^2\big)^{2n+1}}dxdy \to 0$$

so that $a_nL(2,\chi_{-3}) - 2b_n$ is a $\mathbb{C}$-linear combination of the series in (5.3) for $j = 1$ only.

5.4 Interpolations of Apéry's numbers for $\zeta(3)$

By Proposition 1(vii) in §3.3, $R_t := \sum_{j=0}^{\mu} p_j(n+t)\sigma^j \in \overline{\mathbb{Q}}[n,\sigma]$ is a Σ-operator for all $t \in \mathbb{Q}$ provided R_0 is a Σ-operator. Therefore, given a solution $(u_n)_{n\ge 0} \in \overline{\mathbb{Q}}^{\mathbb{Z}_{\ge 0}}$ of "G-origin" of R_0 (for instance $\sum_{n\ge 0} u_n z^n$ is a G-function), we could interpret this property as follows: $(u_{n+t})_{n\ge 0}$ is a solution of "G-origin" of R_t in some sense. This interpretation is meaningful if $t \in \mathbb{Z}_{\ge 0}$ but it is meaningless otherwise, because for instance u_t is not defined. However, by Theorem 1(iii), Eq. (1.2) enables us to extend u_n to a meromorphic function $u(n)$ in $\mathbb{C}$ and we still have $Ru(n) = 0$ for complex values of n such that none of $n, n-1, \dots, n-\mu$ is one of the putative poles of $u(n)$; moreover, $u(n)$ has no pole when n is in a suitable half-plane $\Re(n) > \alpha$ ([22, p. 32] applied with $x = -n$). Thus $\sum_{n\ge n_0} u(n+t)z^t$ is a $\mathbb{C}$-linear combination of μ G-functions, where $n_0 \ge 0$ is such that for all $n \ge n_0$, $u(n+t)$ is well-defined and $p_0(n+\mu) \ne 0$. More precisely, the coefficients of the linear combination are the μ initial conditions $u(n_0+t), u(n_0+t+1)\dots, u(n_0+t+\mu-1)$ computed by Eq. (1.2).

Given u_n defined for $n \in \mathbb{Z}_{\ge 0}$ and solution of difference operator, it is sometimes possible to give a meaning to u_n for complex values of n by a different process. For instance, the above mentioned sequence of Apéry numbers $(q_n)_{n\ge 0}$ for $\zeta(3)$, and solution of $\mathcal{R}_3$, can be written for all $n \in \mathbb{Z}_{\ge 0}$

$$q_n := \sum_{j=0}^{n}\binom{n}{j}^2\binom{n+j}{n}^j = \sum_{j=0}^{\infty}\frac{1}{j!^4}(n-j+1)_{2j}^2 \tag{5.4}$$

where by definition $(x)_k := x(x+1)\cdots(x+k-1)$ for $k \geq 0$ and $(x)_0 = 1$ (Pochhammer's symbol). When n is an integer, the summand vanishes for all $j \geq n+1$ and we recover the finite binomial sum that defines q_n. But it has been observed in [30, p. 753, Eq. (7.1)] that in fact the infinite series in (5.4) converges for all $n \in \mathbb{C}$ and defines an entire function in $\mathbb{C}$. Thus, this gives another interpolation of q_n as an entire function $Q(n)$ for all $n \in \mathbb{C}$. However, it does not coincide with the meromorphic interpolation $q(n)$ of q_n given by Theorem 1(iii) and such that $\mathcal{R}_3 q(n) = 0$, as discussed above. Indeed, by [30, p. 753, Proposition 7.1] we have $\mathcal{R}_3 Q(n) = \frac{8}{\pi^2}(2n-1)\sin(\pi n)^2$ for all $n \in \mathbb{C}$ and thus Q and q cannot be the same function. In this identity, let us change n to $n+t$ where $t \in \mathbb{Q}$ is fixed and n now represents any integer. Then

$$\mathcal{R}_3 Q(n+t) = \frac{8}{\pi^2}(2n+2t-1)\sin(\pi t)^2$$

so that $(\sigma-1)^2\mathcal{R}_3 Q(n+t) = 0$. Since $\mathcal{M}((\sigma-1)^2\mathcal{R}_3) = (z-1)^2\mathcal{L}_3$ is a G-operator, $(\sigma-1)^2\mathcal{R}_3$ is a Σ-operator of order 4 and thus $\sum_{n\geq 0} Q(n+t)z^n$ is also a $\mathbf{G}$-linear combination of four G-functions. The coefficients of the linear combination are the initial conditions $Q(t-3)$, $Q(t-2)$, $Q(t-1)$ and $Q(t)$ and they are indeed in $\mathbf{G}$ because for all $t \in \mathbb{Q}$, we have $Q(t) = {}_4F_3[-t,-t,1+t,1+t;1,1,1;1] \in \mathbf{G}$.

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Tanguy Rivoal, Université Grenoble Alpes, CNRS, Institut Fourier, CS 40700, 38058 Grenoble cedex 9, France.

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