Selberg's integral and linear forms in zeta values

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Abstract

Using Selberg's integral, we present some new Euler-type integral representations of certain nearly-poised hypergeometric series. These integrals are also shown to produce linear forms in odd and/or even zeta values that generalize previous work of the author.

1 Selberg's integral and nearly-poised series

Much work has been devoted to evaluating multiple hypergeometric integrals after Beukers' proof of Apéry's theorem "$\zeta(3)$ is irrational", in which he used the following integrals equations [Be]:

$$\int_0^1\int_0^1 \frac{x^n(1-x)^n y^n(1-y)^n}{(1-(1-x)y)^{n+1}}\,\mathrm{d}x\mathrm{d}y = a_n\zeta(2)+b_n$$

and

$$\int_0^1\int_0^1\int_0^1 \frac{x^n(1-x)^n y^n(1-y)^n z^n(1-z)^n}{(1-(1-(1-x)y)z)^{n+1}}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = A_n\zeta(3)+B_n$$

for some (explicitly computable) rational numbers a_n b_n, A_n and B_n. See in particular the work of Hata [Ha], Rhin and Viola [RV1, RV2], Vasilyev [Va], Sorokin [So], Zudilin [Zu]. Similarly, in order to prove that infinitely many odd zeta values are irrational, the following multiple integral was used in [Ri] and [BR] :

$$J_a^{r,n} := \int_{[0,1]^{a+1}} \frac{\prod_{j=0}^a x_j^{rn}(1-x_j)^n}{(1-x_0\cdots x_a)^{(2r+1)n+2}}\,\mathrm{d}x_0\cdots\mathrm{d}x_a\ , \qquad (1)$$

where $n\geq 0$, $a,r\geq 1$ are integers such that $(a+1)n>(2r+1)n+2$. This is interesting principally because there exist explicitly computable rational numbers $p_{j,a}^{r,n}$ (for $j=0,\ldots,a$) such that

$$J_a^{r,n} = p_{0,a}^{r,n} + \sum_{j=2}^{a} p_{j,a}^{r,n}\zeta(j)$$

and for $j \geq 2$, $p_{j,a}^{r,n} = 0$ if $a(n+1)+j$ is odd (the *parity phenomenon*). In [Zu], the following integral, which generalizes Beukers' integrals above,

$$\int_{[0,1]^a} \frac{\prod_{j=1}^a x_j^n (1-x_j)^n}{Q_a(x_1, x_2, \ldots, x_a)^{n+1}} \, \mathrm{d}x_1 \cdots \mathrm{d}x_a \ ,$$

with $Q_a(x_1, x_2, \ldots, x_a) = 1 - (1 - (1 - \cdots x_3)x_2)x_1$, is proved to produce linear forms in odd or even zeta values. Zudilin gives an identity between such integrals and a slight modification of (1), which is not at all obvious. In this note, we construct another kind of hypergeometric integral, based on Selberg's integral, that also generalizes (1) and produces linear forms in odd or even zeta values (Theorem 1 in §2). We will also show how it is related to more usual Euler-type integrals (Propositions 1 and 2 below).

We first remind the reader of the definition of a hypergeometric series ${}_pF_q(z)$:

$${}_pF_q \left[\begin{matrix} \alpha_1, \, \alpha_2, \, \ldots, \, \alpha_p \\ \beta_1, \, \beta_2, \, \ldots, \, \beta_q \end{matrix} \, ; \, z \right] := \sum_{k=0}^{\infty} \frac{(\alpha_1)_k (\alpha_2)_k \cdots (\alpha_p)_k}{(1)_k (\beta_1)_k \cdots (\beta_q)_k} z^k \, .$$

Here, p and q are positive integers, α_j and β_j are complex numbers such that $\beta_j \notin -\mathbb{N}$, and $(x)_n := x(x+1)\cdots(x+n-1)$ is the Pochhammer symbol. We also recall a well-known result of Selberg, whose proof can be found in [Se]: if the complex numbers α, β and γ satisfy

$$\mathrm{Re}(\alpha) > -1, \, \mathrm{Re}(\beta) > -1, \, \mathrm{Re}(\gamma) > -\min\left(\frac{1}{a+1}, \frac{\mathrm{Re}(\alpha)+1}{a}, \frac{\mathrm{Re}(\beta)+1}{a} \right) \ ,$$

we have that

$$\begin{aligned} \mathrm{Sel}_a^{\alpha,\beta,\gamma} &:= \int_{[0,1]^{a+1}} \prod_{j=0}^{a} x_j^\alpha (1-x_j)^\beta \prod_{0 \leq j < \ell \leq a} |x_j - x_\ell|^{2\gamma} \, \mathrm{d}x_0 \cdots \mathrm{d}x_a \\ &= \prod_{j=0}^{a} \frac{\Gamma(\gamma + j\gamma + 1)\Gamma(\alpha + j\gamma + 1)\Gamma(\beta + j\gamma + 1)}{\Gamma(\gamma+1)\Gamma(\alpha + \beta + (a+j)\gamma + 2)} \, . \end{aligned} \tag{2}$$

Let us now define the integral

$$I_a^{\alpha,\beta,\gamma,\delta}(z) := \int_{[0,1]^{a+1}} \frac{\prod_{j=0}^a x_j^\alpha (1-x_j)^\beta \prod_{0 \leq j < \ell \leq a} |x_j - x_\ell|^{2\gamma}}{(1 - z x_0 \cdots x_a)^{\delta+1}} \, \mathrm{d}x_0 \cdots \mathrm{d}x_a \ , \tag{3}$$

where z is a complex number, $|z| \leq 1$ and $\alpha, \beta, \gamma, \delta \geq 0$, $a \geq 1$ are integers ([1]) such that $(a+1)\beta > \delta$ (which ensures convergence on the circle $|z| = 1$). Then the following holds.

[1]More generally, α, β and γ could be taken to be suitable complex numbers.

Proposition 1. *Under the above conditions,*

$$I_a^{\alpha,\beta,\gamma,\delta}(z) = \mathrm{Sel}_a^{\alpha,\beta,\gamma} \times$$
$$ {}_{a+2}F_{a+1}\left[\begin{matrix}\delta+1, & \alpha+1, & \alpha+\gamma+1, & \alpha+2\gamma+1, & \ldots, & \alpha+a\gamma+1 \\ \alpha+\beta+2a\gamma+2, & \alpha+\beta+(2a-1)\gamma+2, & \ldots, & \alpha+\beta+a\gamma+2 \end{matrix} ; z\right] . \quad (4)$$

The hypergeometric function ${}_{a+2}F_{a+1}$ on the RHS of (4) is said to be *nearly-poised* because the sum of an upper parameter (other than $\delta+1$) and a suitable lower parameter is invariant : for all $j = 0, \ldots, a$, $(\alpha + j\gamma + 1) + (\alpha + \beta + (2a - j)\gamma + 2) = 2\alpha + \beta + 2a\gamma + 3$.

Proof. This can be can thought of as a "generating function" rewriting of Selberg's identity. To simplify, we write

$$c_a^{\alpha,\beta,\gamma,\delta} := \frac{\prod_{j=0}^{a} \Gamma(\gamma + j\gamma + 1)\Gamma(\beta + j\gamma + 1)}{\Gamma(\delta + 1)\Gamma(\gamma + 1)^{a+1}} .$$

Then, using the expansion

$$\frac{1}{(1-t)^{\delta+1}} = \sum_{k=0}^{\infty} \binom{\delta+k}{k} t^k$$

and interverting the $\sum - \int$ signs in (3), we get

$$\begin{aligned} & I_a^{\alpha,\beta,\gamma,\delta}(z) \\ &= \sum_{k=0}^{\infty} \binom{\delta+k}{k} z^k \int_{[0,1]^{a+1}} \prod_{j=0}^{a} x_j^{\alpha+k}(1-x_j)^{\beta} \prod_{0\le j<\ell\le a} |x_j - x_\ell|^{2\gamma} \, \mathrm{d}x_0 \cdots \mathrm{d}x_a \\ &= c_a^{\alpha,\beta,\gamma,\delta} \sum_{k=0}^{\infty} \frac{\Gamma(k+\delta+1)}{\Gamma(k+1)} \prod_{j=0}^{a} \left(\frac{\Gamma(k+\alpha+j\gamma+1)}{\Gamma(k+\alpha+\beta+(a+j)\gamma+2)} \right) z^k , \end{aligned} \quad (5)$$

where we have used Selberg's identity (2) with $\alpha + k$ replacing α in the last step. We now note that, thanks to the identity $(x)_n = \Gamma(x+n)/\Gamma(x)$, the equation (5) is simply the RHS of (4).

An interesting consequence of Proposition 1 is the construction of new integral representations for $I_a^{\alpha,\beta,\gamma,\delta}(z)$ in which the discriminant $\prod_{0\le j<\ell\le a} |x_j - x_\ell|$ does not appear.

Proposition 2. *Let σ be any permutation of the set $\{0, \ldots, a\}$. Then*

$$I_a^{\alpha,\beta,\gamma,\delta}(z) = \prod_{j=0}^{a} \left(\frac{\Gamma(\gamma + j\gamma + 1)\Gamma(\beta + j\gamma + 1)}{\Gamma(\gamma+1)\Gamma(\beta + (a - j + \sigma(j))\gamma + 1)} \right)$$
$$\cdot \int_{[0,1]^{a+1}} \frac{\prod_{j=0}^{a} x_j^{\alpha+j\gamma}(1-x_j)^{\beta+(a-j+\sigma(j))\gamma}}{(1 - zx_0 \cdots x_a)^{\delta+1}} \, \mathrm{d}x_0 \cdots \mathrm{d}x_a . \quad (6)$$

Proof. We first note that in (4) an upper parameter (other than $\delta+1$) is always less than a lower parameter, that is to say, for any $0 \le j,\, k \le a$, $\alpha + j\gamma + 1 < \alpha + \beta + (a+k)\gamma + 2$. This inequality can be reformulated as follows : for any permutation σ of the set $\{0, \ldots, a\}$ and for any $0 \le j \le a$, the inequality $\alpha + j\gamma + 1 < \alpha + \beta + (a + \sigma(j))\gamma + 2$ holds. Hence, we can apply a classical identity of Euler which expresses a hypergeometric series as a multiple integral (*cf.* [Sl], p. 108)

$$I_a^{\alpha,\beta,\gamma,\delta}(z) = \prod_{j=0}^{a} \left(\frac{\Gamma(\gamma + j\gamma + 1)\Gamma(\beta + j\gamma + 1)\Gamma(\alpha + \beta + (a + \sigma(j))\gamma + 2)}{\Gamma(\gamma+1)\Gamma(\beta + (a - j + \sigma(j))\gamma + 1)\Gamma(\alpha + \beta + (a + j)\gamma + 2)} \right) \cdot \int_{[0,1]^{a+1}} \frac{\prod_{j=0}^{a} x_j^{\alpha + j\gamma}(1 - x_j)^{\beta + (a - j + \sigma(j))\gamma}}{(1 - z x_0 \cdots x_a)^{\delta+1}} \mathrm{d}x_0 \cdots \mathrm{d}x_a \,. \quad (7)$$

To conclude, we note that, for any permutation σ,

$$\prod_{j=0}^{a} \Gamma(\alpha + \beta + (a + \sigma(j))\gamma + 2) = \prod_{j=0}^{a} \Gamma(\alpha + \beta + (a + j)\gamma + 2)\ ,$$

which simplifies the Gamma quotient in (7) and proves the identity (6).

2 The well-poised case

Special attention should be paid to a particular case of Proposition 1 : when $\delta = 2\alpha + \beta + 2a\gamma + 1$, then the hypergeometric function on the RHS of (4) is *well-poised*. In this case, the corresponding integral can be evaluated at $z = 1$ in term of odd (resp. even) zeta values, according to the values of the parameters. (In the nearly-poised case, such a decomposition involves both the even and odd zeta values.)

Theorem 1. *For integers $\alpha, \beta, \gamma \ge 0$ and $a \ge 1$ such that $a\beta > 2\alpha + 2a\gamma + 2$, the integral*

$$J_a^{\alpha,\beta,\gamma} := \int_{[0,1]^{a+1}} \frac{\prod_{j=0}^{a} x_j^{\alpha}(1 - x_j)^{\beta} \prod_{0 \le j < \ell \le a} |x_j - x_\ell|^{2\gamma}}{(1 - x_0 \cdots x_a)^{2\alpha + \beta + 2a\gamma + 2}} \mathrm{d}x_0 \cdots \mathrm{d}x_a \qquad (8)$$

is convergent and there exist explicitly computable rational numbers $P_{j,a}^{\alpha,\beta,\gamma}$ (for $j = 0, \ldots, a$) such that

$$J_a^{\alpha,\beta,\gamma} = P_{0,a}^{\alpha,\beta,\gamma} + \sum_{j=2}^{a} P_{j,a}^{\alpha,\beta,\gamma} \zeta(j)\ . \qquad (9)$$

Furthermore, for $j \ge 2$, $P_{j,n}^{\alpha,\beta,\gamma} = 0$ if $a(\beta + 1) + j$ is odd.

Clearly, when $\gamma = 0$, the integral $J_a^{\alpha,\beta,\gamma}$ includes the integral $J_a^{r,n}$ mentioned in the introduction as a particular case.

Proof. It is now well-established that the parity phenomenon is a consequence of the well-poisedness of the underlying hypergeometric function (see [RZ] for more details of this idea). We will therefore simply sketch the reasoning and give references at each step. With the notation used in the proof of Proposition 1 and with $\delta = 2\alpha + \beta + 2a\gamma + 1$, a trivial transformation of the Pochhammer symbols of the hypergeometric series (4) gives the following identity :

$$J_a^{\alpha,\beta,\gamma} = c_a^{\alpha,\beta,\gamma,\delta} \sum_{k=0}^{\infty} \frac{(k+1)_\delta}{\prod_{j=0}^{a}(k+\alpha+j\gamma+1)_{\beta+a\gamma+1}} \, . \tag{10}$$

The expansion in partial fractions of the rational function

$$\mathrm{R}(X) := \frac{(X+1)_\delta}{\prod_{j=0}^{a}(X+\alpha+j\gamma+1)_{\beta+a\gamma+1}}$$

immediately implies (9) (see [Ri], [BR]). Furthermore, applying the trivial identity $(x)_n = (-1)^n(-x-n+1)_n$ to $\mathrm{R}(X)$, we obtain the crucial symmetry relation

$$\mathrm{R}(-X-\delta-1) = (-1)^{a(\beta+1)}\mathrm{R}(X) \, . \tag{11}$$

From (11) (essentially by the uniqueness of the decomposition in partial fractions), we then deduce that $P_{j,a}^{\alpha,\beta,\gamma} = 0$ if $j \geq 2$ and $j + a(\beta+1)$ is odd, which completes the proof (see [Fi], [Co], or [Ri], [BR] for the slightly different original argument).

An interesting problem would be to prove the identity (9) and the parity phenomenom without expanding the integral (8) as the series (10) (the method used in [RV1, RV2] could be relevant here). Furthermore, hopefully we will find new diophantine applications of Theorem 1, other than those already known for $\gamma = 0$.

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