REMAINDER PADÉ APPROXIMANTS FOR HYPERGEOMETRIC SERIES

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Abstract. Remainder Padé Approximation (RPA) consists in adding to the n-th partial sum of a series a suitable Padé approximant of the asymptotic expansion in the variable $1/n$ of the remainder term. In a previous paper, we proved the non-trivial property that the RPA of the exponential function is identical to the Padé approximant to the function e^z. In this paper, we extend this property to the hypergeometric series ${}_1F_1(1;a;z)$ and ${}_2F_0(b,1;z)$.

1. Introduction

1.1. **Remainder Padé Approximants.** In [8], the first author introduced a new kind of rational approximation, the Remainder Padé Approximants (RPA). Let $f(z)=\sum_{k=0}^{\infty} a_k z^k \in \mathbb{C}[[z]]$; for simplicity of the exposition, in the discussion below f is supposed to be convergent in a disk D of positive radius ([1]). For any integer $n\geq 0$, we assume the existence of some functions $b_n(z)\in\mathbb{C}(z)$ such that the normalized remainder series $\Phi(z,n) := \frac{1}{b_n(z)}\sum_{k=n}^{\infty} a_k z^k$ admits, for any fixed $z\in D\setminus\{\text{zeros of all the } b_n\}$, an asymptotic expansion as $n\to+\infty$ of the form

$$\Phi(z,n)\sim \widehat{\Phi}_z\Big(\frac{1}{n}\Big) := \sum_{k=0}^{\infty}\frac{\phi_k(z)}{n^k}\in\mathbb{C}(z)\big[\big[\frac{1}{n}\big]\big].$$

In practice, $b_n(z)$ is simply $a_n z^n$, though this might not be the only possibility in principle. Let $[p/q]_{\widehat{\Phi}_z(t)}\in\mathbb{C}(z,t)$ denote the ordinary Padé approximant of $\widehat{\Phi}_z(t)\in\mathbb{C}(z)[[t]]$ at $t=0$: we thus have

$$f(z)=\sum_{k=0}^{n-1} a_k z^k + b_n(z)\Phi(z,n)\approx\sum_{k=0}^{n} a_k z^k + b_n(z)\big([p/q]_{\widehat{\Phi}_z(t)}\big)_{t=1/u(n)}. \tag{1.1}$$

where the sequence of complex numbers $u(n)$ is $n+o(n)$. We say that the rational fraction

$$\frac{A(z)}{B(z)} := \sum_{k=0}^{n-1} a_k z^k + b_n(z)\big([p/q]_{\widehat{\Phi}_z(t)}\big)_{t=1/u(n)}$$

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[1] It is possible to extend the notion of RPA when f diverges everywhere but is summable in some sense; we present an example of this situation in Theorem 2.

is a Remainder Padé Approximant of $f(z)$, where $A(z), B(z) \in \mathbb{C}[z]$ are of lowest degrees. Sometimes, there exists a sequence $u(n)$ such that the corresponding RPA of $f(z)$ coincides with an ordinary Padé approximant of $f(z)$ at $z = 0$, say $[N/D]_{f(z)}$. When this happens, the $\approx$ sign in (1.1) can then be understood as follows: the order at $z = 0$ of the remainder $B(z)f(z) - A(z)$ is larger than or equal to $N + D + 1$, the order of Padé approximation. It is not yet understood for which class of functions this remarkable coincidence happens. We use the term *RPA phenomenon* to cover such a non-trivial fact.

1.2. **The results.** The goal of this paper is to show that the RPA phenomenon occurs (in various generality) for the hypergeometric series

$$E_a(z) := {}_1F_1(1; a; z) = \sum_{k=0}^{\infty} \frac{z^k}{(a)_k} \quad \text{and} \quad {}_2F_0(b, 1; z) = \sum_{k=0}^{\infty} (b)_k z^k.$$

We assume that $a \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. The former series generalizes the exponential function ${}_1F_1(1; 1; z) = \exp(z)$, while the latter generalizes Euler's series ${}_2F_0(1, 1; z) = \sum_{k=0}^{\infty} k! z^k$. The variable z is a formal one in Theorems 1 and 2 below. However, it is important to have in mind that, though the series $E_a(z)$ is an entire function of z, the ${}_2F_0(b, 1; z)$ diverges for every z. To overcome this divergence, we shall first work with the function

$$\mathcal{E}_b(z) := \frac{1}{\Gamma(b)} \int_0^{\infty} \frac{u^{b-1}}{1 - uz} e^{-u} du$$

defined for $\Re(b) > 0$ and analytic for $z \in \mathbb{C} \setminus [0, +\infty)$. It admits an asymptotic expansion $\widehat{\mathcal{E}}_b(z)$ as $z \to 0$ in any angular sector centered at 0 that does not contain $[0, +\infty)$. It turns out that $\widehat{\mathcal{E}}_b(z) = {}_2F_0(b, 1; z)$ in such sectors.

To state our results, we need to introduce some notations. The parameters $\alpha_j, j = 1, \ldots, r$, are pairwise distinct non-zero complex numbers, and we set $\underline{\alpha} := (\alpha_j)_{j=1,\ldots,r}$. For any $z \in \mathbb{C}$ and any $t \in \mathbb{C}$ such that $1/t \notin \mathbb{N}$, we define

$$\Phi_z(t) := \sum_{k=0}^{\infty} \frac{z^k}{(1 - 1/t)_k}.$$

For any $j = 1, \ldots, r$ and any $n \geq 0$, we have the trivial identity

$$\sum_{k=0}^{\infty} \frac{(\alpha_j z)^k}{(a)_k} = \sum_{k=0}^{n-1} \frac{(\alpha_j z)^k}{(a)_k} + \frac{(\alpha_j z)^n}{(a)_n} \Phi_{\alpha_j z}\left(\frac{-1}{a - 1 + n}\right). \tag{1.2}$$

In [9], it has been proved that, for any fixed $z \in \mathbb{C}$, $\Phi_z(t)$ admits an asymptotic expansion $\widehat{\Phi}_z(t) := \sum_{k=0}^{\infty} \varphi_k(-z) t^k$ as $t \to 0, t < 0$, where the coefficients $\varphi_k(z)$ are *Touchard exponential polynomials of degree k*. The latter are defined by the exponential generating function $e^{z(e^X - 1)} = \sum_{k=0}^{\infty} \varphi_k(z) \frac{X^k}{k!}$ (see [13]).

We provided in [9] explicit expressions for the type II Padé approximants at $t = 0$ of parameters $(rp - 1, p)$ for the formal power series $\widehat{\Phi}_{\alpha_j z}(t)$ in $\mathbb{C}[z][[t]]$ for $j = 1, \ldots, r$: we denote them by $P_{j,\underline{\alpha},p}(t,z)/Q_{\underline{\alpha},p}(t,z)$, and the expressions of these polynomials in t are given in §2. It turns out that they are also polynomials in z.

We now "replace" $\Phi_{\alpha_j z}\left(\frac{-1}{a-1+n}\right)$ in (1.2) by the corresponding simultaneous Padé approximant of parameters $(rp-1,p)$ of $\widehat{\Phi}_{\alpha_j z}(t)$ evaluated at $t=-1/(a-1+n)$ (this substitution is possible because $Q_{\underline{\alpha},p}(-1/(a-1+n),z)$ does not vanish identically as a function of z). This alteration of (1.2) provides the formal Laurent series

$$R_{j,n,\underline{\alpha},p}(z) := \sum_{k=0}^{\infty}\frac{(\alpha_j z)^k}{(a)_k} - \sum_{k=0}^{n-1}\frac{(\alpha_j z)^k}{(a)_k} - \frac{(\alpha_j z)^n}{(a)_n}\frac{P_{j,\underline{\alpha},p}(\frac{-1}{a-1+n},z)}{Q_{\underline{\alpha},p}(\frac{-1}{a-1+n},z)}, \quad j=1,\dots,r,$$

and we want to know in which sense they are ≈ 0. By construction, for every j, the rational fraction

$$\sum_{k=0}^{n-1}\frac{(\alpha_j z)^k}{(a)_k} + \frac{(\alpha_j z)^n}{(a)_n}\frac{P_{j,\underline{\alpha},p}(\frac{-1}{a-1+n},z)}{Q_{\underline{\alpha},p}(\frac{-1}{a-1+n},z)} \tag{1.3}$$

is a RPA of $E_a(\alpha_j z)$. The degrees of its numerator and denominator are *a priori* bounded by $n+rp-1$ and rp respectively. A better bound holds for the numerator ((i) below) and it is crucial to prove (ii). (Throughout the paper, $\mathcal{O}(z^c)$ denotes a Laurent series in z with order equal to c at $z=0$.)

Theorem 1. *Let us fix the integers $r\geq 1$, $n\geq 1$, $p\geq 0$, such that $n\geq p$, $a\in\mathbb{C}\setminus\mathbb{Z}_{\leq 0}$, $\alpha_1,\dots,\alpha_r\in\mathbb{C}$ pairwise distinct.*

- (i) *The degrees of the numerator and denominator of the RPA in* (1.3) *are bounded by $n+(r-1)p-1$ and rp respectively, and moreover $R_{j,n,\underline{\alpha},p}(z)=\mathcal{O}(z^{n+rp})$.*
- (ii) *The collection of the RPA in* (1.3)*, $j=1,\dots,r$, is the type II Padé approximant of $(E_a(\alpha_j z))_{j=1,\dots,r}$ of parameters $(n+(r-1)p-1,p)$.*
- (iii) *For $r=1$ and $\alpha_1=1$,* (ii) *gives the following equality of rational fractions:*

$$\sum_{k=0}^{n-1}\frac{z^k}{(a)_k} + \frac{z^n}{(a)_n}\big([p-1/p]_{\widehat{\Phi}_z(t)}\big)_{t=-1/(a-1+n)} = [n-1/p]_{E_a(z)},$$

For a generic value of a, our proof needs the assumption that $n\geq p$. However, if $a=1$, it can be removed in the statement of Theorem 1, and we simply recover the results from [9]. The explanation of this fact is Identity (2.5) in §2.

Let us now describe our results for the function $\mathcal{E}_b(z)$. For any $n\geq 0$, b such that $\Re(b)>0$ and $z\in\mathbb{C}\setminus[0,+\infty)$, we have the identity (proved in §3)

$$\mathcal{E}_b(z) = \sum_{k=0}^{n-1}(b)_k z^k + (b)_n z^n \Psi_z\left(\frac{1}{b-1+n}\right) \tag{1.4}$$

where

$$\Psi_z(t) = \frac{1}{\Gamma(1+1/t)}\int_0^{\infty}\frac{u^{1/t}}{1-uz}e^{-u}du$$

is defined for any $z \in \mathbb{C} \setminus (0, +\infty)$ and any t such that $\Re(1/t) > -1$ (in particular, for any $t > 0$). We shall prove in §3 that when $z < 0$, $\Psi_z(t)$ admits an asymptotic expansion as $t \to 0$, $t > 0$, given by ([2])

$$\widehat{\Psi}_z(t) := -\frac{1}{z}\sum_{k=0}^{\infty} \varphi_k(1/z)t^{k+1} = -\frac{t}{z}\widehat{\Phi}_{-1/z}(t). \tag{1.5}$$

It then immediately follows that the Padé approximant $[p/p]$ at $t = 0$ of the formal power series $\widehat{\Psi}_z(t) \in \mathbb{C}[1/z][[t]]$ is $-\frac{t}{z}\frac{P_{j,1,p}(t,-1/z)}{Q_{1,p}(t,-1/z)}$ where the polynomials are those used to construct the Padé approximants $[p-1/p]$ of $\widehat{\Phi}_z(t)$ when $r = 1$.

In (1.4), we now "replace" $\mathcal{E}_b(z)$ by $\widehat{\mathcal{E}}_b(z)$, and $\Psi_z\left(\frac{1}{b-1+n}\right)$ by the Padé approximant $[p/p]$ of $\widehat{\Psi}_z(t)$ evaluated at $t = 1/(b-1+n)$ (this substitution is possible because $Q_{1,p}(1/(b-1+n), 1/z)$ does not vanish identically as a function of z). This alteration of (1.4) provides the formal Laurent series

$$S_{n,p}(z) := \widehat{\mathcal{E}}_b(z) - \sum_{k=0}^{n-1}(b)_k z^k + \frac{(b)_n z^{n-1}}{(b-1+n)}\frac{P_{j,1,p}(\frac{1}{b-1+n}, -\frac{1}{z})}{Q_{1,p}(\frac{1}{b-1+n}, -\frac{1}{z})},$$

and we want to know in which sense it is ≈ 0. Because of (1.4), the rational fraction

$$\sum_{k=0}^{n-1}(b)_k z^k - \frac{(b)_n z^{n-1}}{(b-1+n)}\frac{P_{j,1,p}(\frac{1}{b-1+n}, -\frac{1}{z})}{Q_{1,p}(\frac{1}{b-1+n}, -\frac{1}{z})} \tag{1.6}$$

can be viewed as a RPA of $\mathcal{E}_b(z)$ in an extended sense. The degrees of its numerator and denominator are *a priori* bounded by $n + p - 1$ and p respectively. Again, a better bound holds for the numerator ((*i*) below) and again it is crucial to prove (*ii*).

Theorem 2. *Let us fix $b \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, and the integers $n \geq 1$, $p \geq 0$ such that $n \geq p$.*

(*i*) *The degrees of the numerator and denominator of the RPA in* (1.6) *are bounded by $n-1$ and p respectively, and moreover $S_{n,p}(z) = \mathcal{O}(z^{n+p})$ exists and is finite.*

(*ii*) *The RPA in* (1.6) *coincides with the Padé approximant $[n-1/p]$ of $\widehat{\mathcal{E}}_b(z)$. In other words,*

$$\sum_{k=0}^{n-1}(b)_k z^k + (b)_n z^n\big([p/p]_{\widehat{\Psi}_z(t)}\big)_{t=1/(b-1+n)} = [n-1/p]_{\widehat{\mathcal{E}}_b(z)}.$$

It seems that a RPA phenomenon similar to (*ii*) in Theorem 1 does not hold for any family $(\widehat{\mathcal{E}}_b(\beta_j))_{j=1,\ldots,r}$ with $r \geq 2$.

We conclude this introduction with other examples of the RPA phenomenon in the literature. It may happens that, instead of Padé approximants, we recover some previously known functional approximations of $f(z)$, or even numerical approximations of certain

[2]This identity is reminiscent of André's duality between solutions of E-operators at $z = 0$ and $z = \infty$; see [1]. The former involve function like $E_a(z)$, while the latter involve functions like $\widehat{\mathcal{E}}_b(1/z)$, with $a, b \in \mathbb{Q}$.

of its values. For instance, the first author constructed certain sequences of RPA for $\zeta(2) = \sum_{k=1}^{\infty} 1/k^2$ and $\zeta(3) = \sum_{k=1}^{\infty} 1/k^3$ (both can be viewed as values of the polylogarithms $\sum_{k=1}^{\infty} z^k/k^s$ at $z = 1$) and proved that they are exactly the sequences of rationals numbers used by Apéry [2] to prove the irrationality of $\zeta(2)$ and $\zeta(3)$. In [10], the second author used a modification of these RPA to produce a sequence of fast converging rational approximations for Catalan's constant $G = \sum_{k=0}^{\infty}(-1)^k/(2k+1)^2$. In [9], using certain discrete multiple orthogonal polynomials introduced in [3], which generalize the classical Charlier orthogonal polynomials, we proved that certain RPA of e^z coincide with the Padé approximants of the function e^z. The notion of RPA is also pertinent in the context of q-hypergeometric series and we refer to [4, 5, 7] for details.

The proofs of Theorems 1 and 2 are given in §2 and §3 respectively.

2. Proof of Theorem 1

We first recall the notion of type II (diagonal) Padé approximants. For a given family $(F_j(X))_{j=1,\dots,r}$ of formal series in $\mathbb{C}[[X]]$ and any integers p and q such that $p \geq (r-1)q \geq 0$, there exist (by linear algebra) some polynomials $P_1(x), \dots, P_r(x)$ and $Q(X)$ in $\mathbb{C}[X]$, not all 0, such that $\deg(P_j) \leq p$, $\deg(Q) \leq r \cdot q$ and

$$Q(X)F_j(X) - P_j(X) = \mathcal{O}(X^{p+q+1}), \quad 1 \leq j \leq r.$$

The fractions P_j/Q are by definition the type II Padé approximants of $(F_j(X))_{j=1,\dots,r}$ of parameters (p, q). When $r = 1$, we recover the ordinary Padé approximants $[p/q]$ of $F_1(X)$.

Proof of (i). We recall the expressions of the type II approximants of the functions $\widehat{\Phi}_{\alpha_j z}(t)$ (obtained in [9]):

$$P_{j,\underline{\alpha},p}(t,z) = (-t)^{rp-1} \sum_{k_1,\dots,k_r=0}^{p} \Bigg(\prod_{i=1}^{r} \binom{p}{k_i} (-\alpha_i z)^{p-k_i}\Bigg) (-1/t)_{k_1+\dots+k_r} \sum_{i=1}^{k_1+\dots+k_r} \frac{(\alpha_j z)^{i-1}}{(-1/t)_i}, \tag{2.1}$$

and

$$Q_{\underline{\alpha},p}(t,z) = (-t)^{rp} \sum_{k_1,\dots,k_r=0}^{p} \Bigg(\prod_{i=1}^{r} \binom{p}{k_i} (-\alpha_i z)^{p-k_i}\Bigg) (-1/t)_{k_1+\dots+k_r}. \tag{2.2}$$

The P's have degree $rp-1$ in both t and z, while the Q's have degree rp in both t and z. Moreover, $Q_{\underline{\alpha},p}(0,1) = 1$.

For the sake of simplicity, we set $K_r = k_1 + k_2 + \dots + k_r$. Substituting $-1/(a-1+n)$ for t (which is possible because $a \notin \mathbb{Z}_{\leq 0}$ and $n \geq 1$), we get

$$Q_{\underline{\alpha},p}\Big(\frac{-1}{a-1+n}, z\Big) = (a+n-1)^{-rp} \sum_{k_1,\dots,k_r=0}^{p} \Bigg(\prod_{i=1}^{r} \binom{p}{k_i} (-\alpha_i z)^{p-k_i}\Bigg) (a-1+n)_{K_r}$$

and

$$P_{j,\underline{\alpha},p}\Big(\frac{-1}{a-1+n},z\Big)=\frac{(a-1+n)^{1-rp}(a)_{n-1}}{(\alpha_j z)^n}$$
$$\times\sum_{k_1,\ldots,k_r=0}^{p}\bigg(\prod_{i=1}^{r}\binom{p}{k_i}(-\alpha_i z)^{p-k_i}\bigg)(a-1+n)_{K_r}\big(S_{n+K_r-1}(\alpha_j z)-S_{n-1}(\alpha_j z)\big)$$

where $S_n(z)=\sum_{k=0}^{n}z^k/(a)_k$. (Note that $Q_{\underline{\alpha},p}\big(\frac{-1}{a-1+n},z\big)$ is clearly not identically zero, as stated in the Introduction.)

Hence, after some simplifications, the RPAs of the functions $E_{\alpha_j}(z)$, $j=1,\ldots,r$, are given by

$$\sum_{k=0}^{n-1}\frac{(\alpha_j z)^k}{(a)_k}+\frac{(\alpha_j z)^n}{(a)_n}\frac{P_{j,\underline{\alpha},p}(\frac{-1}{a-1+n},z)}{Q_{\underline{\alpha},p}(\frac{-1}{a-1+n},z)}$$
$$=\frac{\displaystyle\sum_{k_1,\ldots,k_r=0}^{p}\bigg(\prod_{i=1}^{r}\binom{p}{k_i}(-\alpha_i z)^{p-k_i}\bigg)(a-1+n)_{K_r}S_{n+K_r-1}(\alpha_j z)}{\displaystyle\sum_{k_1,\ldots,k_r=0}^{p}\bigg(\prod_{i=1}^{r}\binom{p}{k_i}(-\alpha_i z)^{p-k_i}\bigg)(a-1+n)_{K_r}}. \quad (2.3)$$

At first sight, the numerator and denominator of this rational fraction have degrees bounded by $n+rp-1$ and rp in z respectively, but a better estimate holds for the numerator. Indeed, we shall now prove that its degree is bounded by $n+(r-1)p-1$. Indeed, with $K'_{j,r}=k_1+\cdots+k_{j-1}+k_{j+1}+\cdots+k_r$ and $\underline{k}'_{j,r}=(k_1,\ldots,k_{j-1},k_{j+1},\ldots,k_r)$, the numerator of (2.3) can be expressed as

$$\sum_{k_1,\ldots,k_{j-1},k_{j+1},\ldots,k_r=0}^{p}\bigg(\prod_{i=1,i\neq j}^{r}\binom{p}{k_i}(-\alpha_i z)^{p-k_i}\bigg)(a-1+n)_{K'_{j,r}}A_{j,\underline{k}'_{j,r}}(z) \quad (2.4)$$

with

$$A_{j,\underline{k}'_{j,r}}(z)=\sum_{k_j=0}^{p}\binom{p}{k_j}(-\alpha_j z)^{p-k_j}(a-1+n+K'_{j,r})_{k_j}S_{n+K_r-1}(\alpha_j z).$$

We claim that the degree in z of the polynomial $A_{j,\underline{k}'_{j,r}}(z)$ is at most $\max(p,n+K'_{j,r})-1$. To prove this observation, which is non-trivial, we first observe that, for any integers m,q such that $m\geq q\geq 0$,

$$\big(t^{a-1}S_m(xt)\big)^{(q)}_{|t=1}=x^q S_{m-q}(x)+U_q(x) \quad (2.5)$$

where

$$U_q(x):=\sum_{j=0}^{q-1}(-1)^{q-j}(1-a)_{q-j}x^j.$$

(Above and below in this section, all the differentiations are with respect to t.) Observe that $U_q(x) \equiv 0$ if $a = 1$, but in general $U_q(x)$ has degree $q - 1$ in x. Since $(a-1+m)_k = (-1)^k (t^{-a+1-m})^{(k)}_{|t=1}$, it follows that

$$
\begin{aligned}
A_{j,\underline{k}'_{j,r}}(z) &= (-1)^p \sum_{k=0}^{p} \binom{p}{k} \big(t^{-a+1-n-K'_{j,r}}\big)^{(k)}_{|t=1} \big(t^{a-1} S_{n+K'_{j,r}+p-1}(\alpha_j z t)\big)^{(p-k)}_{|t=1} \\
&\quad + \sum_{k=0}^{p} (-1)^{p-k} \binom{p}{k} (a-1+n+K'_{j,r})_k U_{p-k}(\alpha_j z) \\
&= (-1)^p \big(t^{-n-K'_{j,r}} S_{n+K'_{j,r}+p-1}(\alpha_j z t)\big)^{(p)}_{|t=1} \qquad (2.6) \\
&\quad + \sum_{k=0}^{p} (-1)^{p-k} \binom{p}{k} (a-1+n+K'_{j,r})_k U_{p-k}(\alpha_j z). \qquad (2.7)
\end{aligned}
$$

The polynomial in (2.7) obviously has degree at most $p - 1$ in z. On the other hand

$$
\begin{aligned}
\big(t^{-n-K'_{j,r}} S_{n+K'_{j,r}+p-1}(\alpha_j z t)\big)^{(p)}_{|t=1} &= \sum_{j=0}^{n+K'_{j,r}+p-1} \frac{(\alpha_j z)^j}{(a)_j} \big(t^{j-n-K'_{j,r}}\big)^{(p)}_{|t=1} \\
&= (-1)^p \sum_{j=0}^{n+K'_{j,r}+p-1} \frac{(\alpha_j z)^j}{(a)_j} (n+K'_{j,r}-j)_p.
\end{aligned}
$$

Since $(n + K'_{j,r} - j)_p = 0$ for any $j \in \{n + K'_{j,r}, \dots, n + K'_{j,r} + p - 1\}$, it follows that the polynomial in (2.6) has degree at most $n + K'_{j,r} - 1$, so that the degree of $A_{j,\underline{k}'_{j,r}}(z)$ is at most $\max(p, n + K'_{j,r}) - 1$ as claimed. Since $n \geq p$ by assumption, we deduce that the degree of the numerator of the RPAs given by (2.4) is at most $n+(r-1)p-1$, as expected.

To prove that $R_{j,n,\underline{\alpha},p}(z) = \mathcal{O}(z^{n+rp})$, we have to multiply the denominator of (2.3) by the series $\sum_{k=0}^{\infty} \frac{(\alpha_j z)^k}{(a)_k}$ and prove that the product is the sum of a polynomial of degree $n + (r-1)p - 1$ and a function which is $\mathcal{O}(z^{n+rp})$. This amounts to proving that the coefficient c_m of z^m in the Taylor expansion of

$$
\left(\sum_{k_1,\dots,k_r=0}^{p} \left(\prod_{i=1}^{r} \binom{p}{k_i} (-\alpha_i z)^{p-k_i} \right) (a-1+n)_{K_r} \right) \cdot \sum_{k=0}^{\infty} \frac{(\alpha_j z)^k}{(a)_k}
$$

is zero for $m \in \{n + (r-1)p, \dots, n + rp - 1\}$. Let us assume that $m \geq n + (r-1)p$. Then

$$
c_m = \sum_{k_1,\dots,k_r=0}^{p} \left(\prod_{i=1}^{r} \binom{p}{k_i} (-\alpha_i)^{p-k_i} \right) (a-1+n)_{K_r} \frac{\alpha_j^k}{(a)_k} \qquad (2.8)
$$

where in (2.8) the integer $k := m + K_r - rp$ is ≥ 0 (because $m \geq n + (r-1)p \geq rp$).

We now also assume that $m \leq n+rp-1$. In that case, we use the identity

$$\frac{(a-1+n)_{K_r}}{(a)_k} = \frac{1}{(a)_{n-1}}\left(t^{a+n+K_r-2}\right)^{(n+rp-m-1)}_{|t=1}$$

in (2.8) and get

$$\begin{aligned} c_m &= \frac{1}{(a)_{n-1}} \sum_{k_1,\dots,k_r=0}^{p} \left(\prod_{i=1}^{r} \binom{p}{k_i}(-\alpha_i)^{p-k_i}\right)\alpha_j^{m-rp+K_r}\left(t^{a+n+K_r-2}\right)^{(n+rp-m-1)}_{|t=1} \\ &= \frac{1}{(a)_{n-1}}\left(t^{a-1+n-rp}\prod_{i=1}^{r}(\alpha_j-\alpha_i t)^p\right)^{(n+rp-m-1)}_{|t=1}. \end{aligned}$$

Since $t=1$ is a zero of order p of the product $\prod_{i=1}^{r}(\alpha_j-\alpha_i t)^p$, it follows that $c_m=0$ when $n+rp-m-1 \leq p-1$, i.e when $m \geq n+(r-1)p$. This complete the proof of (i).

Proof of (ii). By unicity of type II Padé approximants, the bounds for the degrees of the simultaneous RPAs and the order of $R_{j,n,\underline{\alpha},p}$ at 0 ensure that (2.3) provides the expression of the type II Padé approximants of the family $(E_a(\alpha_j z))_{j=1,\dots,r}$ of parameters $(n+(r-1)p-1,p)$ when $n \geq p$.

3. Proof of Theorem 2

Let b be such that $\Re(b)>0$. We first prove (1.4) of the Introduction, i.e. that for any $n \geq 0$ and any $z \in \mathbb{C}\setminus[0,+\infty)$,

$$\mathcal{E}_b(z) = \sum_{k=0}^{n-1}(b)_k z^k + (b)_n z^n \Psi_z\left(\frac{1}{b-1+n}\right).$$

This follows from

$$\begin{aligned} (b)_n z^n \Psi_z\Big(\frac{1}{b-1+n}\Big) &= \frac{(b)_n z^n}{\Gamma(b+n)}\int_0^\infty \frac{u^{b+n-1}}{1-uz}e^{-u}du \\ &= \frac{(b)_n}{\Gamma(b+n)}\int_0^\infty \frac{u^{b-1}}{1-uz}e^{-u}du + \frac{(b)_n}{\Gamma(b+n)}\int_0^\infty u^{b-1}\frac{(uz)^n-1}{1-uz}e^{-u}du \\ &= \mathcal{E}_b(z) - \frac{(b)_n}{\Gamma(b+n)}\sum_{k=0}^{n-1} z^k \int_0^\infty u^{k+b-1}e^{-u}du \\ &= \mathcal{E}_b(z) - \sum_{k=0}^{n-1}\frac{(b)_n\Gamma(b+k)}{\Gamma(b+n)}z^k = \mathcal{E}_b(z) - \sum_{k=0}^{n-1}(b)_k z^k. \end{aligned}$$

We now prove the following lemma.

Lemma 1. *Let us fix $z \in (-\infty,0)$. The function $\Psi_z(t)$ admits an asymptotic expansion as $t \to 0, t>0$, given by*

$$\widehat{\Psi}_z(t) := -\frac{1}{z}\sum_{k=0}^{\infty}\varphi_k(1/z)t^{k+1} = -\frac{t}{z}\widehat{\Phi}_{-1/z}(t),$$

where the coefficients $\varphi_k(X)$ are Touchard exponential polynomials of degree k in X.

Proof. We first use Proposition 2 in [11] to get the following alternative expression for $\Psi_z(t)$: for any $z \in (-\infty, 0)$ and $t > 0$,

$$\Psi_z(t) = \int_0^\infty \frac{e^{-u}}{(1-zu)^{1+1/t}} du.$$

We now set $v = \ln(1-uz)$, so that

$$\Psi_z(t) = -\frac{1}{z}\int_0^\infty e^{(e^v-1)/z} e^{-v/t} \mathrm{d}v \tag{3.1}$$

for $z \in (-\infty, 0)$ and $t > 0$. Since $v \mapsto e^{(e^v-1)/z}$ is C^∞ at $v = 0$, non-zero at $v = 0$, and

$$\int_0^\infty \left| e^{(e^v-1)/z} e^{-v/t} \right| \mathrm{d}v < \infty,$$

we can apply Watson's lemma ([6, Chapter 2, §2]) to (3.1): the function $\Psi_z(t)$ has the asymptotic expansion

$$\Psi_z(t) \sim -\frac{1}{z}\sum_{k=0}^\infty \left(\frac{d^k}{dv^k} e^{(e^v-1)/z}\right)_{|v=0} t^{k+1} = -\frac{t}{z}\sum_{k=0}^\infty \varphi_k(1/z) t^k, \qquad t \to 0,\, t > 0.$$

This completes the proof of the lemma. □

From now on, we view $\widehat{\Psi}_z(t)$ as a formal series in $\mathbb{C}[1/z][[t]]$. As observed in the Introduction, the identity $\widehat{\Psi}_z(t) = -\frac{t}{z}\widehat{\Phi}_{-1/z}(t)$ enables us to immediately get the Padé approximant $[p/p]$ of $\widehat{\Psi}_z(t)$ at $t = 0$. More generally, let $\underline{\beta} := (\beta_j)_{j=1,\ldots,r}$ denote a family of pairwise distinct complex numbers. Then, without further efforts, we even obtain the type II Padé approximants at $t = 0$ of parameters (rp, p) of the formal power series $(\widehat{\Psi}_{\beta_j z}(t))_{j=1,\ldots,r}$ for $j = 1, \ldots, r$. They are given by

$$-\frac{t}{\beta_j z}\frac{P_{j,1/\underline{\beta},p}(t, -1/z)}{Q_{1/\underline{\beta},p}(t, -1/z)}, \quad j = 1, \ldots, r$$

where

$$\begin{aligned} &P_{j,1/\underline{\beta},p}(t, -1/z) \\ &\qquad = (-t)^{rp-1} \sum_{k_1,\ldots,k_r=0}^{p} \left(\prod_{i=1}^{r} \binom{p}{k_i} (\beta_i z)^{k_i - p}\right) (-1/t)_{k_1+\cdots+k_r} \sum_{i=1}^{k_1+\cdots+k_r} \frac{(-\beta_j z)^{1-i}}{(-1/t)_i}, \end{aligned}$$

$$Q_{1/\underline{\beta},p}(t, -1/z) = (-t)^{rp} \sum_{k_1,\ldots,k_r=0}^{p} \left(\prod_{i=1}^{r} \binom{p}{k_i} (\beta_i z)^{k_i - p}\right) (-1/t)_{k_1+\cdots+k_r}.$$

We now substitute $1/(b-1+n)$ for t in these expressions, which is possible because $\Re(b) > 0$ and $n \geq 1$. Since $Q_{1/\underline{\beta},p}(1/(b-1+n), -1/z)$ is clearly not identically zero, we get simultaneous RPAs of the family $(\mathcal{E}_b(\beta_j z))_{j=1,\dots,r}$ given by the formulas:

$$\sum_{k=0}^{n-1}(b)_k(\beta_j z)^k - (b)_n\frac{(\beta_j z)^{n-1}}{(b-1+n)}\frac{P_{j,1/\underline{\beta},p}(\frac{1}{b-1+n},-\frac{1}{z})}{Q_{1/\underline{\beta},p}(\frac{1}{b-1+n},-\frac{1}{z})}$$
$$=\frac{z^{rp}\displaystyle\sum_{k_1,\dots,k_r=0}^{p}\left(\prod_{i=1}^{r}\binom{p}{k_i}(\beta_i z)^{k_i-p}\right)(-1)^{K_r}T_{n-K_r-1}(\beta_j z)/(b)_{n-K_r}}{z^{rp}\displaystyle\sum_{k_1,\dots,k_r=0}^{p}\left(\prod_{i=1}^{r}\binom{p}{k_i}(\beta_i z)^{k_i-p}\right)(-1)^{K_r}/(b)_{n-K_r}}. \quad (3.2)$$

where $T_n(z) := \sum_{k=0}^{n}(b)_k z^k$.

We are now in position to complete the proof of Theorem 2. We take $r = 1$ and $\beta_1 = 1$ in (3.2).

Proof of (i)**.** It is immediate that the degrees of the numerator and denominator of the rational function in (3.2) are bounded by $n-1$ and p respectively. We now want to prove that, formally,

$$\widehat{\mathcal{E}}_b(z) - \sum_{k=0}^{n-1}(b)_k z^k + \frac{(b)_n z^{n-1}}{(b-1+n)}\frac{P_{j,1,p}(\frac{1}{b-1+n},-\frac{1}{z})}{Q_{1,p}(\frac{1}{b-1+n},-\frac{1}{z})} = \mathcal{O}(z^{n+p}). \quad (3.3)$$

The denominator of the right-hand side of (3.2) is

$$Q(z) := \sum_{k=0}^{p}\frac{(-1)^k\binom{p}{k}}{(b)_{n-k}}z^k \quad (3.4)$$

so that to prove (3.3) it is enough to prove that the formal series expansion of

$$\left(\sum_{k=0}^{p}\frac{(-1)^k\binom{p}{k}}{(b)_{n-k}}z^k\right)\widehat{\mathcal{E}}_b(z)$$

is the sum of a polynomial $P(z)$ of degree $\le n-1$ and a formal power series of order $\ge n+p$ at $z=0$. This in fact means that we want to prove that $P(z)/Q(z)$ is the $[n-1/p]$ Padé approximant of $\widehat{\mathcal{E}}_b(z)$. This is indeed the case because we recognize $Q(z)$ in (3.4) as being of the form given in [12] and unicity of Padé approximant enables us to conclude.

Proof of (ii)**.** This is simply a summary of what was proved in (i). We have

$$\widehat{\mathcal{E}}_b(z) = \sum_{k=0}^{n-1}(b)_k z^k + (b)_n z^n([p/p]_{\Psi_z(t)})_{|t=1/(b-1+n)} + O(z^{n+p}) \quad (3.5)$$

and the bounds for the degrees show that

$$\sum_{k=0}^{n-1}(b)_k z^k + (b)_n z^n([p/p]_{\Psi_z(t)})_{|t=1/(b-1+n)} = [n-1/p]_{\widehat{\mathcal{E}}_b(z)}.$$

The above results are *a priori* proved under the assumption that $\Re(b) > 0$. They hold under the weaker assumption that $b \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ because all these identities between rational fractions extend to this case.

References

[1] Y. André, Séries Gevrey de type arithmétique, I. Théorèmes de pureté et de dualité, *Annals of Math.* **151** (2000), 705–740.

[2] R. Apéry, Irrationality of $\zeta(2)$ and $\zeta(3)$, *Astérisque* **61** (1979), 11–13.

[3] J. Arvesú, J. Coussement and W. Van Assche, Some discrete multiple orthogonal polynomials, *J. Comput. Appl. Math.* **153**, 1-2 (2003), 19–45.

[4] J.-P. Bézivin, Sur les propriétés arithmétiques d'une fonction entière, *Math. Nachr.* **190** (1998), 31–42.

[5] T. Matala-Aho, On q-analogues of divergent and exponential series, *J. Math. Soc. Japan* **61** (2009), 291–313.

[6] P. D. Miller, *Applied asymptotic analysis*, *Graduate Studies in Mathematics* **75**. American Mathematical Society, Providence, RI, 2006.

[7] C. Krattenthaler, I. Rochev, K. Väänänen, W. Zudilin, On the non-quadraticity of values of the q-exponential function and related q-series, *Acta Arith.* **136** (2009), 243–269.

[8] M. Prévost, A new proof of the irrationality of $\zeta(2)$ and $\zeta(3)$ using Padé approximants, *J. Comput. Appl. Math.* **67**.2 (1996), 219–235.

[9] M. Prévost and T. Rivoal, Remainder Padé approximants for the exponential function, *Constr. Approx.* **25**.1 (2007), 109–123.

[10] T. Rivoal, Nombres d'Euler, approximants de Padé et constante de Catalan, *Ramanujan J.* **11**.2 (2006), 199–214.

[11] T. Rivoal, On the arithmetic nature of the values of the Gamma function, Euler's constant and Gompertz's constant, *Michigan Math. J.* **61** (2012), 239–254.

[12] A. Sidi, A new method for deriving Padé approximants for some hypergeometric functions, *J. Comput. Appl. Math.* **7**.1 (1981), 37–40.

[13] J. Touchard, Nombres exponentiels et nombres de Bernoulli, *Canadian J. Math.* **8** (1956), 305–320.

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