CONVERGENCE AND MODULAR TYPE PROPERTIES OF A TWISTED RIEMANN SERIES

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Abstract. We consider the series $\Phi(\alpha)=\sum_{m=1}^{\infty}\frac{1}{m^2}\sin(2\pi m^2\alpha)\cot(\pi m\alpha)$, a twist of the famous continuous but almost nowhere differentiable sine series defined by Riemann. In a slightly different but equivalent form, this series appeared in the first author's paper [*On the distribution of multiple of real numbers*, Monatsh. Math **164**.3 (2011), 325–360]. We pursue here the study of Φ, which is almost everywhere but not everywhere convergent. We first prove that Φ enjoys a modular type property, in the following sense (with Φ_n the n-th partial sum of Φ): For all $\alpha\in(0,1]$, the sequence $\Phi_N(\alpha)-\alpha\Phi_{\lfloor \alpha N\rfloor}(-1/\alpha)$ has a finite simple limit $\Omega(\alpha)$ as $N\to+\infty$. Using analytic properties of Ω, we then prove that $\Phi(\alpha)$ converges if and only if α is irrational and $\sum_j \log(q_{j+1})/q_j$ converges (Brjuno's condition), where q_j is the j-th denominator in the sequence of convergents to α. This completes the results obtained in the above mentioned paper, where it was proved that $\Phi(\alpha)$ converges absolutely under Brjuno's condition.

1. Introduction

This paper deals with the series

$$\Phi(\alpha)=\sum_{m=1}^{\infty}\frac{1}{m^2}\sin(2\pi m^2\alpha)\cot(\pi m\alpha). \tag{1.1}$$

(The summand is defined for any real number α because $\sin(2\pi m^2 z)\cot(\pi m z)$ is an entire function for any integer $m\geq 1$.) It is a twist of Riemann series $\sum_{m=1}^{\infty}\frac{1}{m^2}\sin(2\pi m^2\alpha)$. It appeared in a slightly different form in the diophantine study in [13] concerning the behavior of the quantity $\frac{1}{n}\sum_{m=1}^{n}||mn\alpha||$, where $||x||$ denotes the distance of $x\in\mathbb{R}$ to $\mathbb{Z}$. Riemann series is continuous on $\mathbb{R}$ and nowhere differentiable, except at the rational numbers of the form p/q with p and q both odd and coprime (Hardy, Gerver, Itatsu; see [5, Chapitre VII]). Duistermaat [6] showed that these facts are simple consequences of the following modular type functional equation, where $R(\alpha)=\sum_{m=1}^{\infty}\frac{e^{2i\pi m^2\alpha}}{m^2}$ and $\alpha>0$:

$$R(\alpha)-e^{i\pi/4}\alpha^{3/2}R(-1/\alpha)=\frac{\pi^2}{6}+i\pi e^{i\pi/4}\sqrt{\alpha}-i\frac{\pi}{2}\alpha-\frac{3}{2}e^{i\pi/4}\int_0^{\alpha}\sqrt{\tau}R(-1/\tau)\mathrm{d}\tau, \tag{1.2}$$

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which itself can be deduced from the classical modular equation satisfied by the theta series $\sum_{m\in\mathbb{Z}} e^{i\pi m^2\alpha}$. The latter was used by Jaffard [7] to compute the multifractal spectrum of Riemann series. The important information in (1.2) is that the right-hand side is much smoother (differentiable with continuous derivative on $(0,+\infty)$) than what is suggested by the left-hand side (continuous but almost nowhere differentiable).

We address here two questions:
– When does the series $\Phi(\alpha)$ converge?
– Does the series $\Phi(\alpha)$ satisfy a modular type functional equation (like (1.2)) that provides some non-trivial analytic information?

Although both questions might seem unrelated, it turns out that our answer to the second one is an important step to answer the first one.

We shall now make a couple of comments concerning the first question. On the one hand, it is easy to prove that the series diverges for all rational number $\alpha = p/q$ with $(p,q) = 1$. Indeed, in this case, for any integer $J \geq 1$,

$$\sum_{m=1}^{Jq-1} \frac{\sin(2\pi m^2\alpha)\cot(\pi m\alpha)}{m^2} = \sum_{k=1}^{q-1} \sin(2\pi k^2 p/q)\cot(\pi kp/q) \sum_{j=1}^{J-1} \frac{1}{(jq+k)^2} + 2\sum_{j=1}^{J-1} \frac{1}{jq},$$

whence a logarithmic divergence of $\Phi(p/q)$. On the other hand, it was proved in [13] that $\Phi(\alpha)$ converges absolutely for any irrational number α satisfying *Brjuno's diophantine condition*

$$\sum_{j=0}^{\infty} \frac{\log(q_{j+1})}{q_j} < \infty, \tag{1.3}$$

where q_j denotes the j-th denominator of the sequence of convergents to α. ([1]) More precisely, the "absolute convergence" result proved in [13] concerns the series of general term $\frac{1}{m^2}\sum_{k=1}^{m} \cos(2\pi km\alpha)$ but this does not change anything since

$$\begin{aligned} \frac{1}{m^2}\sum_{k=1}^{m} \cos(2\pi km\alpha) &= \frac{\cos(\pi m(m+1)\alpha)\sin(\pi m^2\alpha)}{m^2\sin(\pi m\alpha)} \\ &= \frac{\sin(2\pi m^2\alpha)\cot(\pi m\alpha)}{2m^2} - \frac{\sin(\pi m^2\alpha)^2}{m^2} \end{aligned} \tag{1.4}$$

and the second fraction on the right of (1.4) is the term of an absolutely convergent series for any α. The proof uses the fact that $|\sin(2\pi m^2\alpha)\cot(\pi m\alpha)| \ll \frac{||m^2\alpha||}{||m\alpha||}$ and then estimates in terms of the continued fraction expansion of α for the growth of the sums $\sum_{m=1}^{N} \frac{||m^2\alpha||}{m^2||m\alpha||}$,

[1]The results proved in [13] concerns the series $\sum_{n=1}^{\infty} \frac{\cos(\pi n(n+1)\alpha)\sin(\pi n^2\alpha)}{n^2\sin(\pi n\alpha)}$ and its simple relation with $\Phi(\alpha)$ is given by Eq. (1.4) below. Furthermore, Brjuno's condition is replaced in [14] by the more complicated condition $\sum_j \frac{\log(\max(q_{j+1}/q_j, q_j))}{q_j} < \infty$. It was not stated in [14] that both conditions are equivalent: indeed, we have $q_{j+1}/q_j \leq \max(q_{j+1}/q_j, q_j) \leq 2q_{j+1}$ and the series $\sum_j \log(q_j)/q_j$ is convergent for any irrational number α. It is well-known that only Liouville numbers may fail to satisfy Brjuno's condition, which thus holds almost surely.

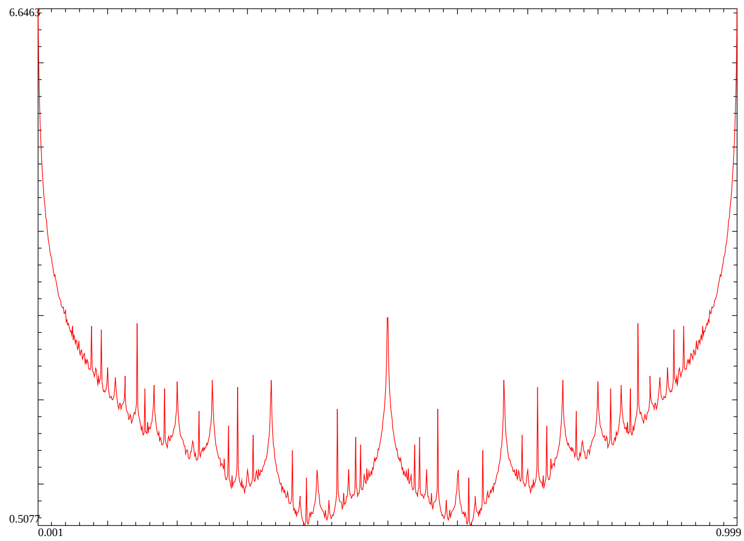

FIGURE 1. Plot of Φ_{500} on [0,1]

inspired by the estimates found in Kruse's paper [9] for the sums $\sum_{m=1}^{N} \frac{1}{m^s ||m\alpha||}$. See also [14].

In the present paper, a modular type property of $\Phi(\alpha)$ is established and used in order to understand more precisely the convergence properties of $\Phi(\alpha)$. For any integer N and any real number α, we denote by $\Phi_N(\alpha)$ the N-th partial sum of $\Phi(\alpha)$:

$$\Phi_N(\alpha) = \sum_{m=1}^{N} \frac{\sin(2\pi m^2 \alpha) \cot(\pi m \alpha)}{m^2}.$$

We then consider

$$\Omega_N(\alpha) = \Phi_N(\alpha) - \alpha \Phi_{\lfloor \alpha N \rfloor}(-1/\alpha)$$

where $\lfloor \cdot \rfloor$ denotes the floor function. We observe that the limit of $\Omega_N(\alpha)$ is *a priori* defined almost everywhere on $(0,1]$ but not everywhere. The first part of the paper is devoted to the proof of the following result, which in particular shows that the limit of $\Omega_N(\alpha)$ exists and is defined everywhere on $(0,1]$. For other instances of such a phenomenon, see [1, 2, 3, 15, 18].

Theorem 1. *The sequence of functions Ω_N has a simple limit Ω on $(0,1]$ as $N \to +\infty$. Moreover,*

$$\Omega_N(\alpha) = \frac{1}{\pi\alpha} \sum_{m=1}^{N} \frac{\sin(2\pi m^2 \alpha)}{m^3} + G_N(\alpha) \tag{1.5}$$

where G_N has a simple limit G on $[0,1]$ as $N \to +\infty$ and $|G_N(\alpha)|$ is bounded by an absolute constant for all $\alpha \in [0,1]$ and all $N \geq 1$.

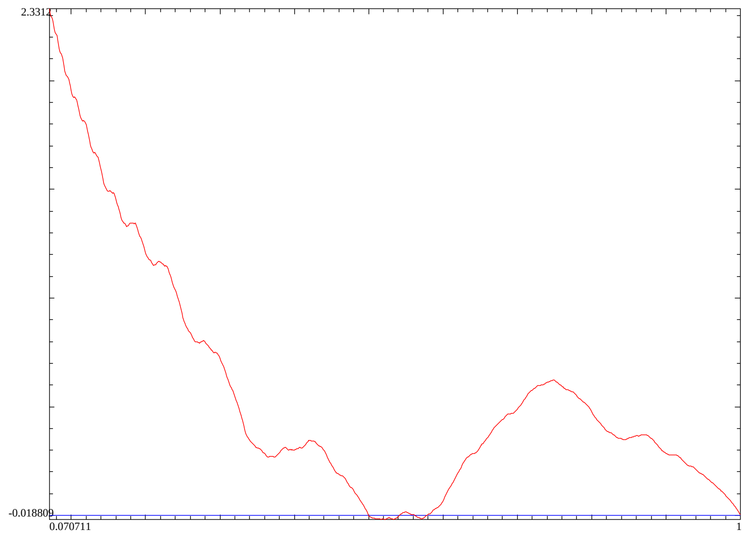

FIGURE 2. Plot of Ω_{500} on $[0.07, 1]$

In particular, the function

$$\Omega(\alpha) - \frac{1}{\pi\alpha}\sum_{m=1}^{\infty}\frac{\sin(2\pi m^2\alpha)}{m^3} \tag{1.6}$$

is defined and bounded on $[0,1]$. The Riemann-like sine series in (1.6) is continuous and behaves like $\pi\alpha\log(1/\alpha)$ as $\alpha \to 0^+$ (see Lemma 3). These facts will be important for the proof of Theorem 2 stated below. We will provide explicit expressions for the limits $G(\alpha)$ and $\Omega(\alpha)$ but they are not easy to study precisely. Graphical experiments (see Figure 2) done with the computer algebra system PARI/GP lead us to formulate the following conjecture:

Conjecture 1. *The function G is continuous on* $[0,1]$.

Fortunately, Theorem 1 provides us enough control on Ω and this conjecture is not required for the proof of our next result.

Theorem 2. *For any irrational number $\alpha \in (0,1)$, the series $\Phi(\alpha)$ converges if and only if Brjuno's condition* (1.3) *holds.*

In case of convergence, we have

$$\Phi(\alpha) = \sum_{j=0}^{\infty} \alpha T(\alpha)\cdots T^{j-1}(\alpha)\Omega(T^j(\alpha)) \tag{1.7}$$

where $T^k(\alpha)$ denotes the k-th iterate of α by the Gauss map $T(\alpha) = \{1/\alpha\}$.

Remark. By the classical properties of continued fraction expansions (See Section 3.1), we have $T^j(\alpha) \approx \frac{q_j}{q_{j+1}}$ and $\alpha T(\alpha)\cdots T^{j-1}(\alpha) = |q_{j-1}\alpha - p_{j-1}| \approx \frac{1}{q_j}$. Hence, since Ω has a logarithmic behavior at the origin, identity (1.7) can be viewed as the quantitative version of the first (qualitative) part of Theorem 2. Similar expansions can be found in [15, 16] for instance and implicitly in [17].

As mentioned above, it was proved in [13] that $\Phi(\alpha)$ converges absolutely if Brjuno's condition holds. Together with Theorem 2, this leads to the following result.

Corollary 1. *The series $\Phi(\alpha)$ converges if and only if it converges absolutely.*

We observe here that Φ could be related with "false" Lerch' sums and "false" theta functions ("false" means here that we replace summations over $\mathbb{Z}$ by summations over $\mathbb{N}^*$ in the usual definitions of Lerch' sums and theta functions). For Lerch' sums, we refer the reader to Mordell's papers [11, 12] and Zwegers' thesis [19] on mock theta functions. It would be interesting to know if an alternative expression for Ω could be deduced from this relationship which would prove Conjecture 1.

Beside Conjecture 1, let us conclude this introduction with a few other problems. Formally, the Fourier series expansion of $\Phi(\alpha)$ is $S(\Phi)(\alpha) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{2\phi_n}{n^2} \cos(2\pi n\alpha)$ where

$$\phi_n = \sum_{\substack{d|n \\ 1\le d\le\sqrt{n}}} d^2 - \frac{n}{2} \quad \text{if } n \text{ is a square,} \quad \phi_n = \sum_{\substack{d|n \\ 1\le d\le\sqrt{n}}} d^2 \quad \text{otherwise.}$$

This is an easy consequence of (1.4). Two problems are to determine for which α's the Fourier series $S(\Phi)(\alpha)$ converges ([2]) and for which ones we have $S(\Phi)(\alpha) = \Phi(\alpha)$. In the spirit of Davenport-like problems (see [4, 10]), the natural answers would be "if and only if Brjuno's condition holds" for both problems but we don't know what to expect here. It was proved in [13] that $S(\Phi)(\alpha) = \Phi(\alpha)$ if $\sum_{j=0}^{\infty} q_{j+1}/q_j^2$ converges, which is an almost sure condition but stronger than Brjuno's condition. Another problem is the determination of the minima of Φ on $[0, 1]$, which seems to be at $\alpha = \frac{\sqrt{5}-1}{2}$ and $\alpha = \frac{3-\sqrt{5}}{2}$. Finally, it would be interesting to know if our method can be adapted to study the convergence of the series

$$\sum_{m=1}^{\infty} \frac{\sin(2\pi m^2\alpha)\cot(\pi m\alpha)}{m^s}$$

for any given $s \in (1, 2)$. (For obvious reasons, this series converges, respectively diverges, for every $\alpha \in \mathbb{R}$ if $s > 2$, respectively if $s \le 1$.)

We will frequently work with analytic functions h on an open subset Ω of $\mathbb{C}$ defined as the quotient of analytic functions $h = f/g$ on Ω. For any $z \in \Omega$, the quotient $f(z)/g(z)$ (which is not well defined if $g(z) = 0$) will mean $h(z)$. This will be implicit in the whole paper. We will still denote by $\lfloor\cdot\rfloor$ a modified floor function: on $[0, +\infty)$ it coincides with the usual floor function while on $(-\infty, 0)$ it is set to zero. We will also often treat labels given to certain quantities as mathematical expressions; for instance, if (10.1) and (10.2) are such labels, we will freely write things like $|(10.1)| \le 1$ or $(10.2) = 0$ when the meaning is obvious.

[2]Since $\Phi \in L^2(0, 1)$, $S(\Phi)$ converges to Φ almost everywhere by Carleson's theorem.

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2. Proof of Theorem 1

2.1. **Strategy of the proof; a basic identity.** Let $f:\mathbb{R}\to\mathbb{C}$ be a map. For all $N\in\mathbb{N}$ and all $\alpha\in\mathbb{R}\setminus\mathbb{Q}$, we set

$$\Psi_N(\alpha)=\sum_{m=1}^{N}\frac{f(2\pi m^2\alpha)\cot(\pi m\alpha)}{m^2}.$$

(The value of an empty sum is set to 0.) Using the classical expansion

$$\cot(\pi z)=\frac{1}{\pi z}+2z\sum_{n=1}^{\infty}\frac{1}{z^2-n^2},$$

which is valid and uniform on any compact subset of $\mathbb{C}\setminus\mathbb{Z}$, we get, for all $M,N\in\mathbb{N}$ and all $\alpha\in(0,+\infty)\setminus\mathbb{Q}$,

$$\begin{aligned}\Psi_N(\alpha)-\alpha\Psi_M(-1/\alpha)&=\frac{1}{\pi\alpha}\sum_{m=1}^{N}\frac{f(2\pi m^2\alpha)}{m^3}+\frac{\alpha^2}{\pi}\sum_{m=1}^{M}\frac{f(-2\pi m^2/\alpha)}{m^3}\\&+2\alpha\sum_{m=1}^{N}\sum_{n=1}^{M}\left(\frac{f(2\pi m^2\alpha)}{m}-\alpha\frac{f(-2\pi n^2/\alpha)}{n}\right)\frac{1}{m^2\alpha^2-n^2}\\&+2\alpha\sum_{m=1}^{N}\frac{1}{m}\sum_{n=M+1}^{+\infty}\frac{f(2\pi m^2\alpha)}{m^2\alpha^2-n^2}-2\alpha^2\sum_{n=1}^{M}\frac{1}{n}\sum_{m=N+1}^{+\infty}\frac{f(-2\pi n^2/\alpha)}{m^2\alpha^2-n^2}.\end{aligned}$$

If f is the restriction of an analytic function vanishing on $2\pi\mathbb{Z}$, then the above equality is meaningful and valid on $(0,+\infty)$. Therefore, in the particular case $f=\sin$, we obtain, for all $\alpha>0$,

$$\Omega_N(\alpha)=\frac{1}{\pi\alpha}\sum_{m=1}^{N}\frac{\sin(2\pi m^2\alpha)}{m^3}-\frac{\alpha^2}{\pi}\sum_{m=1}^{\lfloor\alpha N\rfloor}\frac{\sin(2\pi m^2/\alpha)}{m^3}\tag{2.1}$$

$$+2\alpha\sum_{m=1}^{N}\sum_{n=1}^{\lfloor\alpha N\rfloor}\left(\frac{\sin(2\pi m^2\alpha)}{m}+\alpha\frac{\sin(2\pi n^2/\alpha)}{n}\right)\frac{1}{m^2\alpha^2-n^2}\tag{2.2}$$

$$+2\alpha\sum_{m=1}^{N}\frac{1}{m}\sum_{n=\lfloor\alpha N\rfloor+1}^{+\infty}\frac{\sin(2\pi m^2\alpha)}{m^2\alpha^2-n^2}\tag{2.3}$$

$$+2\alpha^2\sum_{n=1}^{\lfloor\alpha N\rfloor}\frac{1}{n}\sum_{m=N+1}^{+\infty}\frac{\sin(2\pi n^2/\alpha)}{m^2\alpha^2-n^2}.\tag{2.4}$$

In order to prove Theorem 1, it is sufficient to show that the three sequences (2.2), (2.3) and (2.4) converge as N tends to $+\infty$ and that their moduli are bounded by an absolute constant for all $\alpha \in [0,1]$ and $N \geq 1$. This will be proved in Sections 2.2, 2.3 and 2.4 respectively.

2.2. **Study of** (2.2).

Proposition 1. *The double series*

$$\alpha \sum_{m,n\geq 1} \frac{1}{|m^2\alpha^2 - n^2|} \left| \frac{\sin(2\pi m^2\alpha)}{m} + \frac{\alpha \sin(2\pi n^2/\alpha)}{n} \right| \tag{2.5}$$

converges and defines a bounded function of α on $[0,1]$. Therefore, (2.2) converges as $N \to +\infty$ and its modulus is bounded by an absolute constant for all $\alpha \in [0,1]$ and all $N \geq 1$.

Proof. For all integers $m,n \geq 1$, for all $\alpha > 0$, we have

$$\begin{aligned}
&\frac{1}{m^2\alpha^2 - n^2}\left(\frac{\sin(2\pi m^2\alpha)}{m} + \frac{\alpha\sin(2\pi n^2/\alpha)}{n}\right) \\
&\qquad = \left(\frac{1}{m} - \frac{\alpha}{n}\right)\frac{\sin(2\pi m^2\alpha)}{(m^2\alpha^2 - n^2)} + \alpha\frac{\sin(2\pi n^2/\alpha) + \sin(2\pi m^2\alpha)}{(m^2\alpha^2 - n^2)n} \\
&\qquad = \frac{n - \alpha m}{mn}\cdot\frac{\sin(2\pi m^2\alpha)}{(m^2\alpha^2 - n^2)} + 2\alpha\frac{\cos\left(\pi(m^2\alpha - n^2/\alpha)\right)\sin\left(\pi(m^2\alpha + n^2/\alpha)\right)}{(m^2\alpha^2 - n^2)n}.
\end{aligned}$$

Therefore, for all $\alpha > 0$,

$$(2.5) \leq \alpha \sum_{m,n\geq 1} \left| \frac{n - \alpha m}{mn(m^2\alpha^2 - n^2)} \right| \tag{2.6}$$

$$+ 2\alpha^2 \sum_{m,n\geq 1} \left| \frac{\sin\left(\pi(m^2\alpha + n^2/\alpha)\right)}{(m^2\alpha^2 - n^2)n} \right|. \tag{2.7}$$

We first study (2.6). We have $\left|\frac{1}{m\alpha+n}\right| \leq \frac{1}{2\sqrt{\alpha mn}}$. So

$$\alpha \left| \frac{m\alpha - n}{mn(m^2\alpha^2 - n^2)} \right| \leq \frac{\sqrt{\alpha}}{2(mn)^{3/2}}.$$

It follows that (2.6) is bounded in modulus by an absolute constant for all $\alpha \in [0,1]$.

In order to study (2.7), we use:

$$(2.7) \leq 2\alpha^2 \sum_{\substack{m,n\geq 1 \\ |m\alpha - n|\leq 1/2}} \left| \frac{\sin\left(\pi(m^2\alpha + n^2/\alpha)\right)}{(m^2\alpha^2 - n^2)n} \right| \tag{2.8}$$

$$+ 2\alpha^2 \sum_{\substack{m,n\geq 1 \\ |m\alpha - n| > 1/2}} \left| \frac{1}{(m^2\alpha^2 - n^2)n} \right|. \tag{2.9}$$

Let us first consider the integers $m, n \geq 1$ such that $|m\alpha - n| \leq 1/2$. Let us assume that α is irrational. We set $\varepsilon = \alpha - \frac{n}{m}$; we have $|\varepsilon| \leq \frac{1}{2m}$. Note that

$$m^2\alpha + n^2/\alpha - 2mn = (m\sqrt{\alpha} - n/\sqrt{\alpha})^2 = m^2\varepsilon^2/\alpha.$$

Therefore,

$$\alpha^2 \left| \frac{\sin\left(\pi(m^2\alpha + n^2/\alpha)\right)}{n(m^2\alpha^2 - n^2)} \right| = \alpha^2 \left| \frac{\sin\left(\pi m^2\varepsilon^2/\alpha\right)}{n(m\alpha + n)m\varepsilon} \right|$$
$$\leq \alpha^2 \left| \frac{\pi m^2\varepsilon^2/\alpha}{n(m\alpha + n)m\varepsilon} \right| = \left| \frac{\alpha\pi m\varepsilon}{n(m\alpha + n)} \right| \leq \left| \frac{\alpha\pi}{2n(m\alpha + n)} \right|$$

where we have used the inequalities $|\sin(x)| \leq |x|$ valid for all real number x and $m|\epsilon| \leq 1/2$. By continuity, this estimate remains valid if α is rational. We now observe that

$$\sum_{\substack{m,n\geq 1 \\ |m\alpha - n| \leq 1/2}} \frac{1}{n(m\alpha + n)} = \sum_{n\geq 1} \frac{1}{n} \Bigg(\sum_{\substack{m\geq 1 \\ \frac{n-1/2}{\alpha} \leq m \leq \frac{n+1/2}{\alpha}}} \frac{1}{m\alpha + n} \Bigg)$$

and that

$$\sum_{\substack{m\geq 1 \\ \frac{n-1/2}{\alpha} \leq m \leq \frac{n+1/2}{\alpha}}} \frac{1}{m\alpha + n} \leq \frac{1}{2n - 1/2} + \int_{\frac{n-1/2}{\alpha}}^{\frac{n+1/2}{\alpha}} \frac{dt}{t\alpha + n}$$
$$= \frac{1}{2n - 1/2} + \frac{1}{\alpha} \log \frac{2n + 1/2}{2n - 1/2} \leq \frac{1 + \frac{1}{\alpha}}{2n - 1/2}.$$

It follows that the right hand side of (2.8) is bounded by an absolute constant for all $\alpha \in [0, 1]$. Let us now study (2.9). We have

$$(2.9) = 2\alpha^2 \sum_{n\geq 1} \frac{1}{n} \Bigg(\sum_{\substack{m\geq 1 \\ m > \frac{2n+1}{2\alpha}}} + \sum_{\substack{m\geq 1 \\ m < \frac{2n-1}{2\alpha}}} \Bigg) \left| \frac{1}{m^2\alpha^2 - n^2} \right|.$$

On the one hand, for $n \geq 1$,

$$\sum_{m > \frac{2n+1}{2\alpha}} \frac{1}{|m^2\alpha^2 - n^2|} \leq \frac{1}{|\left(\frac{2n+1}{2\alpha}\right)^2 \alpha^2 - n^2|} + \sum_{m > \frac{2n+1}{2\alpha} + 1} \frac{1}{m^2\alpha^2 - n^2}$$
$$\leq \frac{4}{4n + 1} + \int_{\frac{2n+1}{2\alpha}}^{+\infty} \frac{dx}{x^2\alpha^2 - n^2} = \frac{4}{4n + 1} + \frac{\log(4n + 1)}{2n\alpha}.$$

On the other hand, for $n \geq 2$,

$$\sum_{1 \leq m < \frac{2n-1}{2\alpha}} \frac{1}{|m^2\alpha^2 - n^2|} \leq \frac{1}{|\left(\frac{2n-1}{2\alpha}\right)^2 \alpha^2 - n^2|} + \sum_{1 \leq m < \frac{2n-1}{2\alpha} - 1} \frac{1}{n^2 - (m\alpha)^2}$$
$$\leq \frac{4}{4n-1} + \int_1^{\frac{2n-1}{2\alpha}} \frac{dx}{n^2 - x^2\alpha^2}$$
$$= \frac{4}{4n-1} + \frac{1}{2n\alpha}\left(\log(4n-1) - \log\left(\frac{n+\alpha}{n-\alpha}\right)\right)$$
$$\leq \frac{4}{4n-1} + \frac{\log(4n-1)}{2n\alpha}.$$

If $n = 1$ and $\alpha > 1/2$ then $\sum_{1 \leq m < \frac{2n-1}{2\alpha}} \frac{1}{|m^2\alpha^2 - n^2|} = 0$. If $n = 1$ and $\alpha \leq 1/2$ then arguing as above we get

$$\sum_{1 \leq m < \frac{2n-1}{2\alpha}} \frac{1}{|m^2\alpha^2 - n^2|} \leq \frac{4}{4n-1} + \frac{\log(4n-1)}{2n\alpha}.$$

These inequalities show that (2.9) is bounded in modulus by an absolute constant for all $\alpha \in [0, 1]$. □

2.3. **Study of** (2.3)**.**

Lemma 1. *The sequence* (2.3) *tends to* 0 *as* $N \to +\infty$ *for all* $\alpha \in]0, 1[$. *Its modulus is bounded by an absolute constant for all* $\alpha \in [0, 1]$ *and all* $N \geq 1$.

Proof. We have

$$|(2.3)| \leq 2\alpha \sum_{m=1}^{\lfloor N - \frac{1}{\alpha} - 2 \rfloor} \frac{1}{m} \sum_{n=\lfloor \alpha N \rfloor + 1}^{\infty} \frac{1}{|n^2 - \alpha^2 m^2|} \tag{2.10}$$
$$+ 2\alpha \sum_{m=\lfloor N - \frac{1}{\alpha} - 2 \rfloor + 1}^{N} \frac{1}{m} \sum_{n=\lfloor \alpha N \rfloor + 3}^{\infty} \frac{1}{|n^2 - \alpha^2 m^2|} \tag{2.11}$$
$$+ 2\alpha \sum_{m=\lfloor N - \frac{1}{\alpha} - 2 \rfloor + 1}^{N} \frac{1}{m} \sum_{n=\lfloor \alpha N \rfloor + 1}^{\lfloor \alpha N \rfloor + 2} \left| \frac{\sin(2\pi m^2 \alpha)}{n^2 - \alpha^2 m^2} \right|. \tag{2.12}$$

We now proceed to bound the three terms of the right hand side of this inequality.

We first study (2.10). If $N - 1/\alpha - 2 < 1$ then $(2.10) = 0$. We now assume that $N - \frac{1}{\alpha} - 2 \geq 1$. Consider $1 \leq m \leq \lfloor N - \frac{1}{\alpha} - 2 \rfloor$. For $x \geq \lfloor \alpha N \rfloor$, $x - \alpha m \geq \lfloor \alpha N \rfloor - \alpha(N - \frac{1}{\alpha} - 2) = -\{\alpha N\} + 1 + 2\alpha \geq 2\alpha$ and hence the map $x \mapsto \frac{1}{x^2 - \alpha^2 m^2}$ is continuous, positive and decreasing on $[\lfloor \alpha N \rfloor, +\infty)$. Therefore,

$$\sum_{n=\lfloor \alpha N \rfloor+1}^{\infty} \frac{1}{|n^2-\alpha^2 m^2|} \leq \int_{\lfloor \alpha N \rfloor}^{\infty} \frac{dx}{x^2-\alpha^2 m^2} = \frac{1}{2\alpha m} \log\Big(\frac{\lfloor \alpha N \rfloor + \alpha m}{\lfloor \alpha N \rfloor - \alpha m}\Big)$$
$$\leq \frac{1}{2\alpha m} \frac{2\alpha m}{\lfloor \alpha N \rfloor - \alpha m} = \frac{1}{\lfloor \alpha N \rfloor - \alpha m}.$$

Consequently,

$$(2.10) \leq 2\alpha \sum_{m=1}^{\lfloor N-\frac{1}{\alpha}-2 \rfloor} \frac{1}{m(\lfloor \alpha N \rfloor - \alpha m)} = \frac{2\alpha^2}{\lfloor \alpha N \rfloor} \sum_{m=1}^{\lfloor N-\frac{1}{\alpha}-2 \rfloor} \Big(\frac{1}{\alpha m} + \frac{1}{\lfloor \alpha N \rfloor - \alpha m}\Big)$$
$$\leq 2\alpha \frac{1+\log \lfloor N-\frac{1}{\alpha}-2 \rfloor}{\lfloor \alpha N \rfloor} + \frac{2\alpha^2}{\lfloor \alpha N \rfloor} \sum_{m=1}^{\lfloor N-\frac{1}{\alpha}-2 \rfloor} \frac{1}{\lfloor \alpha N \rfloor - \alpha m}.$$

But, for $y \leq \lfloor N-\frac{1}{\alpha}-2 \rfloor + 1$, $\lfloor \alpha N \rfloor - \alpha y \geq \lfloor \alpha N \rfloor - \alpha(N-\frac{1}{\alpha}-1) = -\{\alpha N\}+1+\alpha \geq \alpha$ and hence the map $y \mapsto \frac{1}{\lfloor \alpha N \rfloor - \alpha y}$ is continuous, positive and increasing on $(-\infty, \lfloor N-\frac{1}{\alpha}-2 \rfloor + 1]$. Hence,

$$\sum_{m=1}^{\lfloor N-\frac{1}{\alpha}-2 \rfloor} \frac{1}{\lfloor \alpha N \rfloor - \alpha m} \leq \int_{1}^{\lfloor N-\frac{1}{\alpha}-2 \rfloor+1} \frac{dy}{\lfloor \alpha N \rfloor - \alpha y}$$
$$= \frac{1}{\alpha} \log \frac{\lfloor \alpha N \rfloor - \alpha}{\lfloor \alpha N \rfloor - \alpha\big(\lfloor N-\frac{1}{\alpha}-2 \rfloor + 1\big)} \leq \frac{1}{\alpha} \log\Big(\frac{\lfloor \alpha N \rfloor}{\alpha} - 1\Big).$$

Therefore,

$$(2.10) \leq 2\alpha \frac{1+\log \lfloor N-\frac{1}{\alpha}-2 \rfloor}{\lfloor \alpha N \rfloor} + 2\alpha \frac{\log\big(\frac{\lfloor \alpha N \rfloor}{\alpha} - 1\big)}{\lfloor \alpha N \rfloor}.$$

It follows that (2.10) tends to 0 as $N \to +\infty$ for any $\alpha \in [0,1]$. Moreover, we deduce the inequality

$$(2.10) \leq \frac{2\alpha}{\lfloor \alpha N \rfloor} + 2\frac{\log(N-\frac{1}{\alpha})}{N-\frac{1}{\alpha}} + 2\alpha \frac{\log \lfloor \alpha N \rfloor}{\lfloor \alpha N \rfloor} + 2|\alpha \log \alpha|.$$

Since $N-\frac{1}{\alpha} \geq 3$ and $\lfloor \alpha N \rfloor \geq 1$, we get that (2.10) is bounded by an absolute constant for all $\alpha \in [0,1]$ and all $N \geq 1$.

We now study (2.11). We assume that $\lfloor N-\frac{1}{\alpha}-2 \rfloor + 1 \leq m \leq N$. For $x \geq \lfloor \alpha N \rfloor + 2$, $x-\alpha m \geq \lfloor \alpha N \rfloor + 2 - \alpha N = -\{\alpha N\}+2 \geq 1$ and hence the map $x \mapsto \frac{1}{x^2-\alpha^2 m^2}$ is continuous, positive and decreasing on $[\lfloor \alpha N \rfloor + 2, +\infty)$. Therefore,

$$\sum_{n=\lfloor \alpha N \rfloor+3}^{+\infty} \frac{1}{|n^2-\alpha^2 m^2|} \leq \int_{\lfloor \alpha N \rfloor+2}^{+\infty} \frac{dx}{x^2-\alpha^2m^2} = \frac{1}{2\alpha m} \log\left(\frac{\lfloor \alpha N \rfloor + 2 + \alpha m}{\lfloor \alpha N \rfloor + 2 - \alpha m}\right)$$
$$\leq \frac{1}{2\alpha m} \frac{2\alpha m}{\lfloor \alpha N \rfloor + 2 - \alpha m} = \frac{1}{\lfloor \alpha N \rfloor + 2 - \alpha m}.$$

Hence,

$$\begin{aligned}(2.11) &\leq 2\alpha \sum_{m=\left\lfloor N-\frac{1}{\alpha}-2\right\rfloor+1}^{N} \frac{1}{m(\lfloor \alpha N \rfloor + 2 - \alpha m)} \\ &= \frac{2\alpha^2}{\lfloor \alpha N \rfloor + 2} \sum_{m=\left\lfloor N-\frac{1}{\alpha}-2\right\rfloor+1}^{N} \left(\frac{1}{\alpha m} + \frac{1}{\lfloor \alpha N \rfloor + 2 - \alpha m}\right) \\ &\leq 2\alpha \left(\frac{1}{\left\lfloor N - \frac{1}{\alpha} - 2\right\rfloor + 1} + \frac{\log(N) - \log(\left\lfloor N - \frac{1}{\alpha} - 2\right\rfloor + 1)}{\lfloor \alpha N \rfloor + 2}\right) \\ &+ \frac{2\alpha^2}{\lfloor \alpha N \rfloor + 2} \sum_{m=\left\lfloor N-\frac{1}{\alpha}-2\right\rfloor+1}^{N} \frac{1}{\lfloor \alpha N \rfloor + 2 - \alpha m}.\end{aligned}$$

But, for $y \leq N$, $\lfloor \alpha N \rfloor + 2 - \alpha y \geq \lfloor \alpha N \rfloor + 2 - \alpha N = -\{\alpha N\} + 2 \geq 1$ and hence the map $y \mapsto \frac{1}{\lfloor \alpha N \rfloor + 2 - \alpha y}$ is continuous, positive and increasing on $(-\infty, N]$. Hence,

$$\begin{aligned}\sum_{m=\left\lfloor N-\frac{1}{\alpha}-2\right\rfloor+1}^{N} \frac{1}{\lfloor \alpha N \rfloor + 2 - \alpha m} &\leq \frac{1}{\lfloor \alpha N \rfloor + 2 - \alpha N} + \int_{\left\lfloor N-\frac{1}{\alpha}-2\right\rfloor+1}^{N} \frac{dy}{\lfloor \alpha N \rfloor + 2 - \alpha y} \\ &\leq 1 + \frac{1}{\alpha} \log \frac{\lfloor \alpha N \rfloor + 2 - \alpha\left(\left\lfloor N - \frac{1}{\alpha} - 2\right\rfloor + 1\right)}{\lfloor \alpha N \rfloor + 2 - \alpha N} \\ &\leq 1 + \frac{1}{\alpha}\log(3 + 2\alpha).\end{aligned}$$

Therefore,

$$\begin{aligned}(2.11) &\leq \frac{2\alpha}{\left\lfloor N - \frac{1}{\alpha} - 2\right\rfloor + 1} + 2\alpha \frac{\log(N) - \log(\left\lfloor N - \frac{1}{\alpha} - 2\right\rfloor + 1))}{\lfloor \alpha N \rfloor + 2} + \frac{2\alpha(\alpha + \log(3+2\alpha))}{\lfloor \alpha N \rfloor + 2} \\ &\leq \frac{2\alpha}{\left\lfloor N - \frac{1}{\alpha} - 2\right\rfloor + 1} + 2\frac{\log(N + \frac{1}{\alpha})}{N + \frac{1}{\alpha}} + \frac{2\alpha(\alpha + \log(3+2\alpha))}{\lfloor \alpha N \rfloor + 2}.\end{aligned}$$

It follows that (2.11) tends to 0 as $N \to +\infty$ and that its modulus is bounded by an absolute constant for all $\alpha \in [0, 1]$ and all $N \geq 1$. Finally, for any integers m, n, we have

$|\sin(2\pi m^2\alpha)| = |\sin(2\pi m(n-\alpha m))| \leq 2\pi|m(n-\alpha m)|$. Hence,

$$\begin{aligned}
(2.12) &\leq \sum_{m=\lfloor N-\frac{1}{\alpha}-2\rfloor+1}^{N} \frac{2\alpha}{m} \sum_{n=\lfloor \alpha N\rfloor+1}^{\lfloor \alpha N\rfloor+2} \left|\frac{2\pi m(n-\alpha m)}{n^2-\alpha^2m^2}\right| \\
&= 4\alpha\pi \sum_{m=\lfloor N-\frac{1}{\alpha}-2\rfloor+1}^{N} \sum_{n=\lfloor \alpha N\rfloor+1}^{\lfloor \alpha N\rfloor+2} \frac{1}{n+\alpha m} \\
&\leq 4\alpha\pi \frac{2(N-\lfloor N-\frac{1}{\alpha}-2\rfloor)}{\lfloor \alpha N\rfloor+1+\alpha} \\
&\leq 8\alpha\pi \frac{(\frac{1}{\alpha}+3)}{\lfloor \alpha N\rfloor+1+\alpha}. \qquad (2.13)
\end{aligned}$$

It follows that (2.12) tends to 0 as $N \to +\infty$ and that its modulus is bounded by an absolute constant for all $\alpha \in [0,1]$ and all $N \geq 1$. □

2.4. **Study of** (2.4)**.**

Lemma 2. *The sequence* (2.4) *tends to zero as* $N \to +\infty$ *for any* $\alpha \in]0,1]$. *Its modulus is bounded by an absolute constant for all* $\alpha \in [0,1]$ *and all* $N \geq 1$.

Proof. We obviously have

$$\begin{aligned}
(2.4) \leq 2\alpha^2 &\sum_{n=1}^{\lfloor \alpha N\rfloor-3} \frac{1}{n} \sum_{m=N+1}^{+\infty} \frac{1}{|n^2-\alpha^2m^2|} \qquad (2.14)\\
&+ 2\alpha^2 \sum_{n=\lfloor \alpha N\rfloor-2}^{\lfloor \alpha N\rfloor} \frac{1}{n} \sum_{m=\lfloor N+1/\alpha+1\rfloor+1}^{+\infty} \frac{1}{|n^2-\alpha^2m^2|} \qquad (2.15)\\
&+ 2\alpha^2 \sum_{n=\lfloor \alpha N\rfloor-2}^{\lfloor \alpha N\rfloor} \frac{1}{n} \sum_{m=N+1}^{\lfloor N+1/\alpha+1\rfloor} \left|\frac{\sin(2\pi n^2/\alpha)}{n^2-\alpha^2m^2}\right|. \qquad (2.16)
\end{aligned}$$

We now proceed to bound the three terms of the right hand side of this inequality.

If $\alpha N < 4$ then (2.14) is zero. Assume that $\alpha N \geq 4$ and consider $1 \leq n \leq \lfloor \alpha N\rfloor - 3$. For $y \geq N$, $\alpha y - n \geq \alpha N - (\lfloor \alpha N\rfloor - 3) = \{\alpha N\} + 3 > 1$ and hence the map $y \mapsto \frac{1}{\alpha^2y^2-n^2}$ is continuous, positive and decreasing on $[N,+\infty)$. Therefore,

$$\sum_{m=N+1}^{+\infty} \frac{1}{|\alpha^2m^2-n^2|} \leq \int_N^{+\infty} \frac{dy}{\alpha^2y^2-n^2} = \frac{1}{2\alpha n}\log\left(\frac{\alpha N+n}{\alpha N-n}\right) \leq \frac{1}{\alpha(\alpha N-n)}.$$

Hence,

$$(2.14) \leq 2\alpha \sum_{n=1}^{\lfloor \alpha N \rfloor - 3} \frac{1}{n(\alpha N - n)} = \frac{2}{N} \sum_{n=1}^{\lfloor \alpha N \rfloor - 3} \left(\frac{1}{n} + \frac{1}{\alpha N - n} \right)$$
$$\leq 2 \frac{1 + \log(\lfloor \alpha N \rfloor - 3)}{\lfloor \alpha N \rfloor} + \frac{2}{N} \sum_{n=1}^{\lfloor \alpha N \rfloor - 3} \frac{1}{\alpha N - n}.$$

But, for $x \leq \lfloor \alpha N \rfloor - 2$, $\alpha N - x \geq \alpha N - \lfloor \alpha N \rfloor + 2 = \{\alpha N\} + 2 > 1$ and hence the map $x \mapsto \frac{1}{\alpha N - x}$ is continuous, positive and increasing on $(-\infty, \lfloor \alpha N \rfloor - 2]$. Therefore,

$$\sum_{n=1}^{\lfloor \alpha N \rfloor - 3} \frac{1}{\alpha N - n} \leq \int_1^{\lfloor \alpha N \rfloor - 2} \frac{dx}{\alpha N - x} = \log \left(\frac{\alpha N - 1}{\alpha N - \lfloor \alpha N \rfloor + 2} \right) \leq \log(\alpha N - 1).$$

Consequently,

$$(2.14) \leq 2 \frac{1 + \log(\lfloor \alpha N \rfloor - 3)}{\lfloor \alpha N \rfloor} + 2 \frac{\log(\alpha N - 1)}{N} \tag{2.17}$$
$$\leq 2 \frac{1 + \log(\alpha N - 1)}{\alpha N - 1} + 2 \frac{\log(\alpha N - 1)}{N} \tag{2.18}$$

It is now clear that (2.14) tends to 0 as $N \to +\infty$ and that its modulus is bounded by an absolute constant for all $\alpha \in [0, 1]$ and all $N \geq 1$.

If $\alpha N < 1$ then $(2.15) = 0$. Assume that $\alpha N \geq 1$ and consider $n \in \mathbb{N}^*$ such that $\lfloor \alpha N \rfloor - 2 \leq n \leq \lfloor \alpha N \rfloor$. For $y \geq \left\lfloor N + \frac{1}{\alpha} + 1 \right\rfloor$, $\alpha y - n \geq \alpha N + 1 - \lfloor \alpha N \rfloor = \{\alpha N\} + 1 \geq 1$ and hence the map $y \mapsto \frac{1}{\alpha^2 y^2 - n^2}$ is continuous, positive and decreasing on $[\left\lfloor N + \frac{1}{\alpha} + 1 \right\rfloor, +\infty)$. Therefore, we have

$$\sum_{m=\left\lfloor N + \frac{1}{\alpha} + 1 \right\rfloor + 1}^{\infty} \frac{1}{\alpha^2 m^2 - n^2} \leq \int_{\left\lfloor N + \frac{1}{\alpha} + 1 \right\rfloor}^{\infty} \frac{dy}{\alpha^2 y^2 - n^2} = \frac{1}{2\alpha n} \log \left(\frac{\alpha \left\lfloor N + \frac{1}{\alpha} + 1 \right\rfloor + n}{\alpha \left\lfloor N + \frac{1}{\alpha} + 1 \right\rfloor - n} \right)$$
$$\leq \frac{1}{\alpha} \cdot \frac{1}{\alpha \lfloor N + 1/\alpha + 1 \rfloor - n} \leq \frac{1}{\alpha}.$$

Therefore,

$$(2.15) \leq 2\alpha \sum_{n=\lfloor \alpha N \rfloor - 2}^{\lfloor \alpha N \rfloor} \frac{1}{n}. \tag{2.19}$$

It follows that (2.15) tends to 0 as $N \to +\infty$ and that is modulus is bounded by an absolute constant for all $\alpha \in [0, 1]$ and all $N \geq 1$.

Finally, for all integers m, n, $|\sin(2\pi n^2/\alpha)| = |\sin(2\pi n(m - n/\alpha))| \leq 2\pi|n(n/\alpha - m)|$. Hence,

$$(2.16) \leq 2\alpha^2 \sum_{n=\lfloor \alpha N \rfloor - 2}^{\lfloor \alpha N \rfloor} \frac{1}{n} \sum_{m=N+1}^{\lfloor N+\frac{1}{\alpha}+1 \rfloor} \left| \frac{2\pi n(n/\alpha - m)}{n^2 - \alpha^2 m^2} \right| = 4\alpha\pi \sum_{n=\lfloor \alpha N \rfloor - 2}^{\lfloor \alpha N \rfloor} \sum_{m=N+1}^{\lfloor N+\frac{1}{\alpha}+1 \rfloor} \frac{1}{n + \alpha m}$$
$$\leq 12\alpha\pi \frac{\lfloor N + \frac{1}{\alpha} + 1 \rfloor - N}{1 + \alpha(N+1)} \leq 12\alpha\pi \frac{\frac{1}{\alpha} + 1}{1 + \alpha(N+1)}$$

It follows that (2.16) tends to 0 as $N \to +\infty$ and that is modulus is bounded by an absolute constant for all $\alpha \in [0, 1]$ and all $N \geq 1$.

The expected result follows. □

3. Proof of Theorem 2

The proof of Theorem 2 will be decomposed in many steps. Before that, we recall some facts about continued fraction expansions.

3.1. **Basics of continued fraction expansions.** We refer to [8] for more details and proofs.

Let us consider an irrational number $\alpha \in (0, 1)$. We denote by $[a_0; a_1, a_2, \ldots]$ the regular continued fraction of α, where $a_0 = 0$. For all $k \geq 1$, we have $a_k = \lfloor 1/T^{k-1}(\alpha) \rfloor \geq 1$ where $T(x) = \{1/x\}$ and $\{\cdot\}$ denotes the fractional part function. For any $k \geq 0$, we denote by $[a_0; a_1, a_2, \ldots, a_k]$ the k-th convergent to α and by $p_k = p_k(\alpha)$ and $q_k = q_k(\alpha)$ its numerator and denominator respectively (note that $p_0 = 0$ and $q_0 = 1$). It is convenient to set $p_{-1} = 1$ and $q_{-1} = 0$. For all $k \geq 1$, $p_k = a_k p_{k-1} + p_{k-2}$ and $q_k = a_k q_{k-1} + q_{k-2}$.

In the following proposition, we list a number of basic properties of continued fraction expansions which will be used freely in this paper.

Proposition 2. *(i) For all $k \geq 0$, $q_k \geq F_k \geq 2^{k/2}$ where $(F_k)_{k\geq 0}$ denotes the Fibonacci sequence defined by $F_0 = 0$, $F_1 = 1$ and $F_{k+1} = F_{k-1} + F_k$.*

(ii) Both sequences $(p_k)_{k\geq 0}$ and $(q_k)_{k\geq 0}$ are increasing.

(iii) For all $k \geq 0$, $T^k(\alpha) = \frac{q_k\alpha - p_k}{p_{k-1} - q_{k-1}\alpha} = \frac{|q_k\alpha - p_k|}{|q_{k-1}\alpha - p_{k-1}|}$ and $\alpha T(\alpha) \cdots T^k(\alpha) = (-1)^k(q_k\alpha - p_k) = |q_k\alpha - p_k|$.

(iv) For all $k \geq 0$, $\frac{1}{2} < q_{k+1}|q_k\alpha - p_k| < 1$, and $T^k(\alpha) \asymp \frac{q_k}{q_{k+1}}$ and $\alpha T(\alpha) \cdots T^k(\alpha) \asymp \frac{1}{q_{k+1}}$.

(v) The series $\sum_{k=0}^{\infty} \alpha T(\alpha) \cdots T^k(\alpha)$ converges for all irrational number α.

3.2. **An iterative procedure.** Theorem 1 implies the following decomposition:

$$\Omega_N(\alpha) = A(\alpha) + B_N(\alpha) + C_N(\alpha) \tag{3.1}$$

where $A(\alpha) = A_1(\alpha) + A_2(\alpha)$ with

$$A_1(\alpha) = \frac{1}{\pi\alpha} \sum_{m=1}^{+\infty} \frac{\sin(2\pi m^2\alpha)}{m^3}, \qquad A_2(\alpha) = G(\alpha),$$

$$B_N(\alpha) = -\frac{1}{\pi\alpha}\sum_{m=N+1}^{+\infty}\frac{\sin(2\pi m^2\alpha)}{m^3}, \qquad C_N(\alpha) = G_N(\alpha) - G(\alpha).$$

We observe that $|C_N(\alpha)|$ is bounded by an absolute constant for all $\alpha \in [0,1]$ and all $N \geq 1$, and that, for all $\alpha \in [0,1]$, $\lim_{N\to\infty} C_N(\alpha) = 0$.

From now on, α is a fixed irrational number in $[0,1]$. We define a sequence $(u_j)_{j\geq -1}$ (that also depends on α) by $u_{-1} = N$ and $u_j = \lfloor u_{j-1}(\alpha)T^j(\alpha)\rfloor$ for $j \geq 0$, so that

$$u_j = \left\lfloor \cdots \lfloor\lfloor N\alpha\rfloor T(\alpha)\rfloor \cdots T^j(\alpha)\right\rfloor.$$

For any integer N, we define the integer ℓ_N (that depends on α) by

$$\ell_N = \min\{j \in \mathbb{N} \mid u_j = 0\}.$$

It is clear that $\ell_N \to +\infty$ as $N \to +\infty$. Using (3.1), we get that, for all $N \geq 1$,

$$\begin{aligned}\Phi_N(\alpha) \;=\; & \sum_{j=0}^{\ell_N} \alpha T(\alpha)\cdots T^{j-1}(\alpha)A(T^j(\alpha)) && (3.2)\\ & + \sum_{j=0}^{\ell_N} \alpha T(\alpha)\cdots T^{j-1}(\alpha)B_{u_{j-1}}(T^j(\alpha)) && (3.3)\\ & + \sum_{j=0}^{\ell_N} \alpha T(\alpha)\cdots T^{j-1}(\alpha)C_{u_{j-1}}(T^j(\alpha)) && (3.4)\end{aligned}$$

where, for $j = 0$, $\alpha T(\alpha)\cdots T^{j-1}(\alpha) = 1$ and $u_{j-1} = N$.

We first remark that (3.4) converges to 0 as $N \to +\infty$: this follows from Lebesgue dominated convergence theorem. Indeed, the sequence of elements of $\ell^1(\mathbb{N},\mathbb{C})$ defined by

$$j \mapsto \begin{cases}\alpha T(\alpha)\cdots T^{j-1}(\alpha)C_{u_{j-1}}(T^j(\alpha)) \text{ if } j \leq \ell_N\\ 0 \text{ if } j > \ell_N\end{cases}$$

tends simply to 0 and its absolute value is dominated by an element of $\ell^1(\mathbb{N},\mathbb{C})$ (because $|C_M(\beta)|$ is bounded by an absolute constant for all $\beta \in [0,1]$ and all $N \geq 1$).

Moreover, we have $|B_M(\beta)| \ll \frac{1}{\beta(M+1)^2}$ for all $\beta \in [0,1]$ and all $N \geq 1$, where the implicit constant is absolute. Hence,

$$\left|B_{u_{j-1}}(T^j(\alpha))\right| \ll \frac{1}{T^j(\alpha)(u_{j-1}+1)^2}$$

uniformly for all $N \geq 1$ and all $j \in \{0,...,\ell_N - 1\}$. Note that this implies that, for all $j \in \mathbb{N}$, $B_{u_{j-1}}(T^j(\alpha)) \to 0$ as $N \to +\infty$. Moreover, for all integer $j < \ell_N$,

$$T^j(\alpha)u_{j-1} \geq 1$$

because $u_j > 0$. Thus,

$$\left|B_{u_{j-1}}(T^j(\alpha))\right| \ll 1$$

uniformly with respect to $N \in \mathbb{N}^*$ and $j \in \{0, ..., \ell_N - 1\}$. Using Lebesgue dominated convergence theorem as above, we get that

$$\sum_{j=0}^{\ell_N - 1} \alpha T(\alpha) \cdots T^{j-1}(\alpha) B_{u_{j-1}}(T^j(\alpha))$$

tends to 0 as $N \to +\infty$.

Consequently, we obtain that

$$\phi_N(\alpha) = \sum_{j=0}^{\ell_N} \alpha T(\alpha) \cdots T^{j-1}(\alpha) A(T^j(\alpha)) + \alpha T(\alpha) \cdots T^{\ell_N - 1}(\alpha) B_{u_{\ell_N - 1}}(T^{\ell_N}(\alpha)) + \mathrm{o}(1) \quad (3.5)$$

as N tends to $+\infty$.

3.3. **Some intermediate results.** We need to prove simple analytic results.

Lemma 3. *We have*

$$\sum_{n=1}^{\infty} \left| \frac{\sin(2\pi n^2 \beta)}{n^3} \right| \ll \beta(1 + \log(1/\beta)) \quad (3.6)$$

for all $\beta \in [0, 1]$ and for some absolute constant.

Moreover,

$$\sum_{n=1}^{\infty} \frac{\sin(2\pi n^2 \beta)}{n^3} = \pi\beta \log(1/\beta) \cdot \big(1 + o(1)\big), \quad \beta \to 0^+. \quad (3.7)$$

Proof. We have

$$\begin{aligned} \sum_{n=1}^{\infty} \left| \frac{\sin(n^2 \beta)}{n^3} \right| &\le \beta \sum_{n=1}^{\lfloor \beta^{-1/2} \rfloor} \frac{1}{n} + \sum_{n=\lfloor \beta^{-1/2} \rfloor + 1}^{\infty} \frac{1}{n^3} \\ &\le \beta \left(\log \lfloor \beta^{-1/2} \rfloor + 1 \right) + \frac{1}{\left(\lfloor \beta^{-1/2} \rfloor + 1 \right)^3} + \frac{1}{2 \left(\lfloor \beta^{-1/2} \rfloor + 1 \right)^2} \\ &\le \frac{1}{2} \beta \left(\log(1/\beta) + 1 \right) + \beta^{3/2} + \beta \end{aligned} \quad (3.8)$$

and the first part of the lemma follows.

For any $x \ge 0$, we have $x - \frac{x^3}{6} \le \sin(x)$, so that for any $\varepsilon \in (0, 1)$, $\sin(x) \ge (1 - \varepsilon)x$ provided that $0 \le x \le \sqrt{\varepsilon}$. Therefore, for any $\varepsilon \in (0, 1)$, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\sin(n^2 \beta)}{n^3} &\ge (1 - \varepsilon)\beta \sum_{1 \le n \le \varepsilon^{1/4}/\beta^{1/2}} \frac{1}{n} - \sum_{n > \varepsilon^{1/4}/\beta^{1/2}} \frac{1}{n^3} \\ &\ge (1 - \varepsilon)\beta \left(\log(\varepsilon^{1/4}/\beta^{1/2}) + 1 \right) + \mathcal{O}\big(\beta/\varepsilon^{1/2}\big). \end{aligned}$$

Choosing $\varepsilon = \frac{1}{\log(1/\beta)}$, we get

$$\sum_{n=1}^{\infty} \frac{\sin(n^2\beta)}{n^3} \geq \frac{1}{2}\beta\log(1/\beta) - \frac{1}{2}\beta + \beta\cdot\Big(1-\frac{1}{\log(1/\beta)}\Big)\cdot\Big(1-\frac{1}{4}\log(\log(1/\beta))\Big) + \mathcal{O}\big(\beta\log(1/\beta)^{1/2}\big)$$

$$= \frac{1}{2}\beta\log(1/\beta) + \mathcal{O}(\beta\log\log(1/\beta)) \tag{3.9}$$

as $\beta \to 0^+$. Combining (3.8) and (3.9) provides the second part of the lemma. □

Lemma 4. *There exists $\eta \in (0,1)$ such that for all $N \geq 1$ and all $\beta \in (0,\eta]$, we have $A_1(\beta) + B_N(\beta) \geq 0$ and $A_1(\beta) \geq 0$.*

Proof. Arguing as in Lemma 3, we see that, for any integer N,

$$\sum_{n=1}^{N} \frac{\sin(n^2\beta)}{n^3} \geq \begin{cases} (1-\varepsilon)\beta \displaystyle\sum_{1\leq n\leq N} \frac{1}{n} \geq 0 & \text{if} \quad N \leq \varepsilon^{1/4}/\beta^{1/2} \\ (1-\varepsilon)\beta \displaystyle\sum_{1\leq n\leq \varepsilon^{1/4}/\beta^{1/2}} \frac{1}{n} - \sum_{n>\varepsilon^{1/4}/\beta^{1/2}}^{\infty} \frac{1}{n^3} & \text{if} \quad N > \varepsilon^{1/4}/\beta^{1/2} \end{cases}$$

where $\varepsilon = \frac{1}{\log(1/\beta)}$. The result follows from the fact that

$$(1-\varepsilon)\beta \sum_{1\leq n\leq \varepsilon^{1/4}/\beta^{1/2}} \frac{1}{n} - \sum_{n>\varepsilon^{1/4}/\beta^{1/2}}^{\infty} \frac{1}{n^3} = \frac{1}{2}\beta\log(1/\beta) + \mathcal{O}(\beta\log\log(1/\beta))$$

as $\beta \to 0^+$ and hence is positive for β close to 0. □

For the proof of following proposition, we fix $\eta \in (0,1)$ such that:

- for all $\beta \in (0,\eta]$, $A_1(\beta) + B_N(\beta) \geq 0$ and $A_1(\beta) \geq 0$;
- there exists $c_1, c_2 > 0$ such that, for all $\beta \in (0,\eta]$,

$$c_1 \log(1/\beta) \leq A_1(\beta) \leq c_2 \log(1/\beta). \tag{3.10}$$

The existence of such an η is guaranteed by Lemmas 3 and 4.

Proposition 3. *Let $(N_k)_{k\geq 0}$ be an increasing sequence of integers. The following properties are equivalent:*

(i) $\sum_{j=0}^{N_k} \alpha T(\alpha)\cdots T^{j-1}(\alpha)A(T^j(\alpha))$ *converges as $k \to +\infty$;*
(ii) $\sum_{j=0}^{N_k} \alpha T(\alpha)\cdots T^{j-1}(\alpha)A_1(T^j(\alpha))$ *converges as $k \to +\infty$;*
(iii) $\sum_{\substack{0\leq j\leq N_k \\ T^j(\alpha)\leq\eta}} \alpha T(\alpha)\cdots T^{j-1}(\alpha)A_1(T^j(\alpha))$ *converges as $k \to +\infty$;*
(iv) $\sum_{j=0}^{\infty} \frac{\log(q_{j+1}(\alpha))}{q_j(\alpha)} < \infty$.

Proof. The equivalence of *(i)* and *(ii)* follows from the convergence of the series

$$\sum_{j=0}^{\infty} \alpha T(\alpha)\cdots T^{j-1}(\alpha)A_2(T^j(\alpha))$$

which itself follows from the fact that A_2 is bounded.

The equivalence of (ii) and (iii) follows from the convergence of

$$\sum_{\substack{0\le j\le N_k\\ T^j(\alpha)>\eta}} \alpha T(\alpha)\cdots T^{j-1}(\alpha)A_1(T^j(\alpha))$$

as $k\to+\infty$, which itself follows from the fact that A_1 is bounded on $[\eta,1]$.

Inequality (3.10) implies that

$$\sum_{\substack{0\le j\le N_k\\ T^j(\alpha)\le\eta}} \alpha T(\alpha)\cdots T^{j-1}(\alpha)A(T^j(\alpha))$$

converges as $k\to+\infty$ if and only if

$$\sum_{\substack{0\le j\le N_k\\ T^j(\alpha)\le\eta}} \alpha T(\alpha)\cdots T^{j-1}(\alpha)\log(1/T^j(\alpha))$$

converges as $k\to+\infty$. This is equivalent to the convergence of

$$\sum_{\substack{0\le j\le N_k\\ T^j(\alpha)\le\eta}} \frac{1}{q_j}\log(q_{j+1}/q_j)$$

(we use Proposition 2 here). This is also equivalent to the convergence of the series $\sum_{j=0}^{\infty}\frac{\log(q_{j+1})}{q_j}$, as follows from the convergence of the series $\sum_{\substack{0\le j\le N\\ T^j(\alpha)>\eta}}\frac{1}{q_j}\log(q_{j+1}/q_j)$ and $\sum_{j=0}^{\infty}\frac{\log(q_j)}{q_j}$ (again by Proposition 2). □

3.4. **End of the proof of Theorem 2.** We can now complete the proof. Let us first assume that $\sum_{j=0}^{\infty}\frac{\log q_{j+1}}{q_j}$ is convergent. Proposition 3 ensures that the series

$$\sum_{j=0}^{\infty}\alpha T(\alpha)\cdots T^{j-1}(\alpha)A(T^j(\alpha))$$

converges. Moreover, Lemma 3 implies that $|B_M(\beta)|\ll\log(1/\beta)$ for all $\beta\in(0,\frac12]$ and $M\ge1$, and that $|B_M(\beta)|\ll1$ for all $\beta\in(\frac12,1]$ and $M\ge1$. (In both cases, the implicit constants are absolute.) Therefore,

$$\begin{aligned}\alpha T(\alpha)\cdots T^{\ell_N-1}(\alpha)&\left|B_{u_{\ell_N-1}}(T^{\ell_N}(\alpha))\right|\\ &\ll\begin{cases}\alpha T(\alpha)\cdots T^{\ell_N-1}(\alpha)\log\left(1/T^{\ell_N}(\alpha)\right) \text{ if } & T^{\ell_N}(\alpha)\le\frac12\\ \alpha T(\alpha)\cdots T^{\ell_N-1}(\alpha) \text{ if } & T^{\ell_N}(\alpha)>\frac12\end{cases}\\ &\ll\begin{cases}\frac{1}{q_{\ell_N}}\log\left(\frac{q_{\ell_N+1}}{q_{\ell_N}}\right) \text{ if } & T^{\ell_N}(\alpha)\le\frac12\\ \frac{1}{q_{\ell_N}} \text{ if } & T^{\ell_N}(\alpha)>\frac12\end{cases}\end{aligned}\tag{3.11}$$

which tends to 0 as $N\to+\infty$. Eq. (3.5) ensures that $\Phi(\alpha)$ converges.

To prove the converse statement, we now assume that $\Phi(\alpha)$ is convergent. We separate two cases.

a) We first assume that there exists $c > 0$ such that, for all N large enough, $T^{\ell_N}(\alpha) > c$. Then $\frac{\log(q_{\ell_N+1})}{q_{\ell_N}} \to 0$ as $N \to +\infty$. So

$$\alpha T(\alpha) \cdots T^{\ell_N - 1}(\alpha) B_{u_{\ell_N - 1}}(T^{\ell_N}(\alpha))$$

tends to 0 as $N \to +\infty$ (recall inequality (3.11)). Hence, (3.5) ensures that

$$\sum_{j=0}^{\ell_N} \alpha T(\alpha) \cdots T^{j-1}(\alpha) A(T^j(\alpha))$$

converges as $N \to +\infty$. This is equivalent to the convergence of $\sum_{j=0}^{\infty} \frac{\log(q_{j+1})}{q_j}$ by Proposition 3.

b) We assume that a) does not hold, i.e. that there exists an increasing sequence of integers $(N_k)_{k\geq 0}$ such that $T^{\ell_{N_k}}(\alpha) \to 0$ as $k \to +\infty$. Since

$$\sum_{j=0}^{\ell_N} \alpha T(\alpha) \cdots T^{j-1}(\alpha) A_2(T^j(\alpha)),$$

converges as $N \to +\infty$ (because A_2 is bounded), (3.5) ensures that

$$\sum_{j=0}^{\ell_N - 1} \alpha T(\alpha) \cdots T^{j-1}(\alpha) A_1(T^j(\alpha)) + \alpha T(\alpha) \cdots T^{\ell_N - 1}(\alpha) \big(A_1(T^{\ell_N}(\alpha)) + B_{u_{\ell_N - 1}}(T^{\ell_N}(\alpha)) \big)$$

is convergent as $N \to +\infty$.

But

$$\sum_{\substack{0 \leq j \leq \ell_N - 1 \\ T^j(\alpha) \geq \eta}} \alpha T(\alpha) \cdots T^{j-1}(\alpha) A_1(T^j(\alpha))$$

converges as $N \to +\infty$ (because A_1 is bounded on $[\eta, 1]$). Hence, setting

$$x_k = \sum_{\substack{0 \leq j \leq \ell_{N_k} - 1 \\ T^j(\alpha) < \eta}} \alpha T(\alpha) \cdots T^{j-1}(\alpha) A_1(T^j(\alpha))$$

and

$$y_k = \alpha T(\alpha) \cdots T^{\ell_{N_k} - 1}(\alpha) \Big(A_1(T^{\ell_{N_k}}(\alpha)) + B_{u_{\ell_{N_k} - 1}}(T^{\ell_{N_k}}(\alpha)) \Big),$$

we obtain that $x_k + y_k$ converges as $k \to +\infty$. But for any integer j such that $T^j(\alpha) < \eta$, we have $A_1(T^j(\alpha)) \geq 0$ and, for all k large enough,

$$A_1(T^{\ell_{N_k}}(\alpha)) + B_{u_{\ell_{N_k} - 1}}(T^{\ell_{N_k}}(\alpha)) \geq 0.$$

Therefore $(x_k)_{k\geq 0}$ is an increasing sequence of positive numbers and y_k is positive for all large enough k. It follows that $(x_k)_{k\geq 0}$ is convergent, providing the desired result by Proposition 3.

This completes the proof of Theorem 2.

Bibliography

[1] L. Báez-Duarte, M. Balazard, B. Landreau, E. Saias, *Étude de l'autocorrélation multiplicative de la fonction "partie fractionnaire"*, Ramanujan J. **9** (2005), no. 1-2, 215–240.
[2] M. Balazard, B. Martin, *Comportement local moyen de la fonction de Brjuno*, preprint (2010).
[3] S. Bettin, B. Conrey, *Period functions and cotangent sums*, http://front.math.ucdavis.edu/1111.0931
[4] R. de la Bretèche, G. Tenenbaum, *Séries trigonométriques à coefficients arithmétiques*, J. Anal. Math. **92** (2004), 1–79.
[5] D. Choimet, H. Queffélec, *Analyse moderne, les grands théorèmes du XXème siècle*, Calvage et Mounet, 2009.
[6] J. J. Duistermaat, *Self-similarity of "Riemann's nondifferentiable function"*, Nieuw Arch. Wisk. (4) **9**.3 (1991) 303–337.
[7] S. Jaffard, *The spectrum of singularities of Riemann's function*, Rev. Math. Iberoamericana **12**.2 (1996), 441–460.
[8] A. Ya. Khintchine, *Continued fractions*, Dover Publications, 1997.
[9] A. H. Kruse, *Estimates of* $\sum_{k=1}^{N} k^{-s} \langle kx \rangle^{-t}$, Acta Arith. **12** (1966/1967), 229–261.
[10] B. Martin, *Nouvelles identités de Davenport*, Funct. Approx. Comment. Math. **37**.2 (2007), 293–327.
[11] L. J. Mordell, *The value of the definite integral integral* $\int_{-\infty}^{\infty} \frac{e^{at^2+bt}}{e^{ct}+d} dt$, Quat. Jour. of Math. **68** (1920), 329–342.
[12] L. J. Mordell, *The definite integral integral* $\int_{-\infty}^{\infty} \frac{e^{at^2+bt}}{e^{ct}+d} dt$ *and the analytic theory of numbers*, Acta Math. **61** (1933), 323–360.
[13] T. Rivoal, *On the distribution of multiple of real numbers*, Monatsh. Math **164**.3 (2011), 325–360.
[14] T. Rivoal, *Extremality properties of some diophantine series*, Experiment. Math. **19**.4 (2010), 481–494.
[15] T. Rivoal, *On the convergence of Diophantine Dirichlet series*, 24 pages, to appear in Proc. Edimb. Math. Soc.
[16] J. Schoißengeier, *On the convergence of a series of Bundschuh*, Uniform Distribution Theory **2**.1 (2007), 107-113.
[17] J. R. Wilton, *An approximate functional equation with applications to a problem of diophantine approximation*, J. reine angew. Math. **169** (1933), 219–237.
[18] Don Zagier, *Quantum modular forms*, Clay Math. Proc., Volume 12, 2010.
[19] S. Zwegers, *Mock Theta Functions*, Ph.D. thesis, Utrecht University, 2002, available at http://front.math.ucdavis.edu/0807.4834

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