# VERY-WELL-POISED HYPERGEOMETRIC SERIES AND THE DENOMINATORS CONJECTURE

TANGUY RIVOAL

Laboratoire de Mathématiques Nicolas Oresme, CNRS UMR 6139,
Université de Caen, BP 5186, 14032 Caen cedex, France
email: rivoal@math.unicaen.fr

This survey deals with the recent appearance of *very-well-poised hypergeometric series* as a tool for studying the diophantine nature of the values of the Riemann zeta function at positive integers. In this context, we give examples of an important and general experimental phenomenon known as *the Denominators Conjecture*, and we explain the ideas behind its proof in the case presented here, recently obtained by C. Krattenthaler and the author.

It is an expanded version of the talk given at the symposium "Analytic Number Theory and Surrounding Areas", held at the RIMS of the University of Kyoto in october 2004. I would like to thank the organizer M. Amou and also N. Hirata-Kohno for their invitation to come to Japan, under the auspicies of the JSPS program.

## 1. Notations

Let us first remind the reader of the definition of hypergeometric series (or functions). These are power series defined by

$$
{}_{q+1}F_q\left[\begin{matrix}\alpha_0, \alpha_1, \ldots, \alpha_q\\ \beta_1, \ldots, \beta_q\end{matrix}; z\right] = \sum_{k=0}^{\infty} \frac{(\alpha_0)_k\,(\alpha_1)_k \cdots (\alpha_q)_k}{k!\,(\beta_1)_k \cdots (\beta_q)_k} z^k,
$$

where $\alpha_j \in \mathbb{C}$, $\beta_j \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ and $(x)_m = x(x+1)\cdots(x+m-1)$ is the Pochhammer symbol. It can be proved that such series converge for all $z \in \mathbb{C}$ such that $|z| < 1$, and for $z = \pm 1$, provided that $\mathrm{Re}(\beta_1 + \cdots + \beta_q) > \mathrm{Re}(\alpha_0 + \alpha_1 + \cdots + \alpha_q)$.

The literature (see [3, 8, 14]) contains various special kind of hypergeometric series whose parameters satisfy particular relations. For example, a hypergeometric series is said to be:

- *balanced* if $\alpha_0 + \cdots + \alpha_q + 1 = \beta_1 + \cdots + \beta_q$;
- *nearly-poised (of the first kind)* if $\alpha_1 + \beta_1 = \cdots = \alpha_q + \beta_q$;
- *well-poised* if $\alpha_0 + 1 = \alpha_1 + \beta_1 = \cdots = \alpha_q + \beta_q$;
- *very-well-poised* if it is well-poised and $\alpha_1 = \frac{1}{2}\,\alpha_0 + 1$.

We will show that the very-well-poised case is of special importance.

## 2. Sketch of some irrationality proofs

In this section, we summarise some results on the irrationality of the values of the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$ at integer values of $s \geq 2$.

### 2.1. **A general scheme.**

A simple way of proving irrationality results for zeta values is to start with a rational function of the form

$$R_n(k) = \frac{Q_n(k)}{(k(k+1)\cdots(k+n))^A} = \frac{Q_n(k)}{(k)_{n+1}^A} \in \mathbb{Q}(k)$$

where $n \geq 0$ and $A \geq 1$ are integers, $Q_n(k) \in \mathbb{Q}[k]$, and then consider the series

$$S_n(z) = \sum_{k=1}^{\infty} R_n(k) z^{-k}$$

which we assume to be convergent for $z = 1$, forcing $\deg(Q_n(k)) \leq A(n+1) - 2$. Then, by partial fractions expansion of $R_n(k)$, it is easy to prove that there exist polynomials $(P_{j,n}(z))_{j=0,\ldots,n}$ in $\mathbb{Q}[z]$, of degree at most $n$ such that

$$S_n(z) = P_{0,n}(z) + \sum_{j=1}^{A} P_{j,n}(z) \operatorname{Li}_j(1/z).$$

Here, we have encountered the polylogarithmic functions defined by $\operatorname{Li}_s(z) = \sum_{k=1}^{\infty} z^k/k^s$ for $s \geq 1$, $|z| \leq 1$ and $(s, z) \neq (1, 1)$. Note that for $s \geq 2$, $\operatorname{Li}_s(1) = \zeta(s)$ and $\operatorname{Li}_s(-1) = (1 - 2^{1-s})\zeta(s)$. Furthermore, under these conditions, it can be proved that $\mathrm{d}_n^{A-j} P_{j,n}(z) \in \mathbb{Z}[z]$, where $\mathrm{d}_n = \text{l.c.m.}\{1, 2, \ldots, n\}$, and $P_{1,n}(1) = 0$. Consequently, we have that

$$\mathrm{d}_n^A S_n(1) = p_{0,n} + \sum_{j=2}^{A} p_{j,n} \zeta(s)$$

where $p_{j,n} = \mathrm{d}^A P_{j,n}(1) \in \mathbb{Z}$ (and a similar expression for $S_n(-1)$).

We also have at our disposal the "differential" trick, which generalises the previous construction: let $C \geq 0$ be an integer and consider the series

$$S_{n,C}(z) = \sum_{k=1}^{\infty} \frac{1}{C!} \frac{\partial^C R_n(k)}{\partial k^C} z^{-k}.$$

Then there exist a polynomial $\tilde{P}_{0,n}(z) \in \mathbb{Q}[z]$, of degree at most $n$, depending on $C$ such that

$$S_{n,C}(z) = \tilde{P}_{0,n}(z) + (-1)^C \sum_{j=1}^{A} \binom{j+C-1}{j-1} P_{j,n}(z) \operatorname{Li}_{j+C}(1/z)$$

where the $P_{j,n}$ are as above and the polylogarithms are shifted by $C$. It can be proved that $\mathrm{d}_n^{A+C}\tilde{P}_{0,n}(1) \in \mathbb{Z}$, hence there exist integers $\tilde{p}_{j,n}$ such that

$$\mathrm{d}_n^{A+C} S_{n,C}(1) = \tilde{p}_{0,n} + \sum_{j=2}^{A} \tilde{p}_{j,n}\zeta(C+j),$$

and a similar expression holds for $S_{n,C}(-1)$.

Given this very general construction, the problem is now to choose suitably $A$ and $Q_n(k)$ in order to apply the following criteria for linear independence, due to Nesterenko [10].

**Theorem 1.** *Given $N$ real numbers $\theta_1, \theta_2, \ldots, \theta_N$, suppose there exist $N$ sequences $(p_{\ell,n})_{n\geq 0}$ of integers such that (i) $\left|\sum_{\ell=1}^{N} p_{\ell,n}\theta_\ell\right| = \alpha^{n+o(n)}$ and (ii) $\forall \ell = 1, \ldots, N$, $|p_{\ell,n}| \leq \beta^{n+o(n)}$, for some reals $\alpha, \beta > 0$. Then,*

$$\dim_{\mathbb{Q}}(\mathbb{Q}\,\theta_1 + \mathbb{Q}\,\theta_2 + \cdots + \mathbb{Q}\,\theta_N) \geq 1 - \frac{\log(\alpha)}{\log(\beta)}\,.$$

If we are only interested in proving the irrationality of one of the numbers $\theta_\ell$, then we don't have to check the condition $(ii)$, but only to prove that $\alpha < 1$ to get a dimension $> 1$. Furthermore, in this case, the proof of Theorem 1 is straightforward.

Finally, to get asymptotic values which are as small as possible for $|S_{n,C}(\pm 1)|^{1/n}$, heuristically, the first terms of the sum should be cancelled, *i.e.* $Q_n(k)$ should have a factor $((k-1)\ldots(k-m))^B = (k-m)_m^B$, where $B > C$ (to cancel the effect of the $C^{\text{th}}$-derivative on $R_n(k)$). The parameter $m$ will always be of the form $rn$ for a suitable integer $r \geq 1$.

2.2. **Irrationality of $\zeta(2)$.** It is well-known that $\zeta(2) = \pi^2/6$, a result due to Euler, and Legendre proved that $\pi^2$ is irrational. Hence, one concludes that $\zeta(2)$ is also irrational. But this proof uses a shortcut and it is an interesting problem to prove the irrationality of $\zeta(2)$ without using it. This problem was first solved by Apéry [2], who showed that there exist two sequences $(\alpha_n)_{n\geq 0}$ and $(\beta_n)_{n\geq 0}$ such that $\alpha_n \in \mathbb{Z}$, $\mathrm{d}_n^2\beta_n \in \mathbb{Z}$, where $\mathrm{d}_n = \text{l.c.m}\{1, 2, \ldots, n\} = e^{n+o(n)}$, and

$$\lim_{n\to+\infty} |\mathrm{d}_n^2(\alpha_n\zeta(2) - \beta_n)|^{1/n} = e^2\left(\frac{\sqrt{5}-1}{2}\right)^5 < 1. \tag{1}$$

These properties immediately imply that:

**Theorem 2.** $\zeta(2)$ *is irrational.*

There are many different ways of constructing these sequences and the following "hypergeometric" one is particularly simple:

$$\begin{aligned}
\alpha_n\zeta(2) - \beta_n &= (-1)^n \frac{n!^4}{(2n+1)!^2}\, {}_3F_2\left[\begin{matrix} n+1, n+1, n+1 \\ 2n+2, 2n+2 \end{matrix}; 1\right] \\
&= (-1)^n n! \sum_{k=1}^{\infty} \frac{(k-n)_n}{(k)_{n+1}^2} = (-1)^n \int_0^1\int_0^1 \frac{x^n(1-x)^n y^n(1-y)^n}{(1-(1-x)y)^{n+1}}\,\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

This hypergeometric series is nearly poised of the first kind: the series on the second line is suitable for proving, after just expanding in partial fractions the rational function $n!(k-n)_n/(k)_{n+1}^2$, the existence of $\alpha_n$ and $\beta_n$, while the integral, due to Beukers [5], immediately gives (1). The equality between the series and the integral is a straightforward computation. Furthermore, we obtain

$$\alpha_n = \sum_{j=0}^{n} \binom{n}{j}^2 \binom{n+j}{n} = {}_3F_2\left[\begin{matrix}-n, -n, n+1 \\ 1,\ 1\end{matrix}; 1\right].$$

2.3. **Irrationality of $\zeta(3)$.** Contrary to $\zeta(2)$, there is no known shortcut for proving the irrationality of $\zeta(3)$ (and conjecturally, this number has no algebraic relation with $\pi$) and Apéry's great achievement was to give the first proof of this fact (see [2]). In fact, he proved that there exist two sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ such that $a_n \in \mathbb{Z}$, $\mathrm{d}_n^3 b_n \in \mathbb{Z}$ and

$$\lim_{n\to+\infty} |\mathrm{d}_n^3(2a_n\zeta(3) - b_n)|^{1/n} = e^3(\sqrt{2}-1)^4 < 1.$$

These properties imply that:

**Theorem 3.** *$\zeta(3)$ is irrational.*

Once again, there exist many ways of constructing these sequences. Gutnik [7] and Beukers [6] independently essentially proposed the following:

$$\begin{aligned} 2a_n\zeta(3) - b_n &= -\sum_{k=1}^{\infty} \frac{\partial}{\partial k}\left(\frac{(k-n)_n^2}{(k)_{n+1}^2}\right) \\ &= \int_0^1\int_0^1\int_0^1 \frac{x^n(1-x)^n y^n(1-y)^n z^n(1-z)^n}{(1-(1-(1-x)y)z)^{n+1}}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z. \end{aligned}$$

Strictly speaking, the series is not hypergeometric but is linked to solutions of a certain hypergeometric differential equation. Here, the equality between the series and the triple Beukers' integral [5] is not completely straightforward. Finally, we have

$$a_n = \sum_{j=0}^{n} \binom{n}{j}^2 \binom{n+j}{n}^2 = {}_4F_3\left[\begin{matrix}-n, -n, n+1, n+1 \\ 1,\ 1,\ 1\end{matrix}; 1\right].$$

2.4. **A very-well-poised series related to $\zeta(2)$.** In this section, we try to give a new and more complicated proof of the irrationality of $\zeta(2)$: the reason for this will become clear later.

Let's consider the following very-well-poised hypergeometric series

$$\begin{aligned} \mathbf{S}_n &= n!\sum_{k=1}^{\infty}\left(k+\frac{n}{2}\right)\frac{(k-n)_n(k+n+1)_n}{(k)_{n+1}^3}(-1)^k \\ &= \frac{n!^5(3n+2)!}{2(2n+1)!^4}\,{}_6F_5\left[\begin{matrix}3n+2, \frac{3}{2}n+2, n+1, \ldots, n+1 \\ \frac{3}{2}n+1, 2n+2, \ldots, 2n+2\end{matrix}; -1\right], \end{aligned}$$

which fits into our general scheme. *A priori*, $\mathbf{S}_n \in \mathbb{Q} + \mathbb{Q}\zeta(2) + \mathbb{Q}\zeta(3)$ but due to the *very* special form of the numerator (we will explain this later), we have $\mathbf{S}_n = -\mathbf{p}_n \frac{1}{2}\zeta(2) - \mathbf{q}_n$, where $\mathrm{d}_n \mathbf{p}_n$ and $\mathrm{d}_n^3 \mathbf{q}_n$ are integers. Unfortunately, such estimates are not enough to give a new proof of the irrationality of $\zeta(2)$, but this would be the case if we could "throw away" one power of $\mathrm{d}_n$ for $\mathbf{p}_n$ and $\mathbf{q}_n$.

It is possible to give the following expression for $\mathbf{p}_n$:

$$\mathbf{p}_n = (-1)^{n+1} \sum_{j=0}^{n} \left(\frac{n}{2} - j\right) \binom{n}{j}^3 \binom{n+j}{n} \binom{2n-j}{n} \cdot \left(4H_{n-j} - 4H_j + H_{n+j} - H_{2n-j} - \frac{1}{\frac{n}{2} - j}\right),$$

where $H_m = 1 + 1/2 + \cdots + 1/m$ by definition. Hence, there is no reason to expect anything better than a denominator $\mathrm{d}_n$ for $\mathbf{p}_n$. But, surprisingly, numerical computations suggest the following:

**Claim 1.** *The number $\mathbf{p}_n$ and $\mathrm{d}_n^2 \mathbf{q}_n$ appear to be integers and furthermore, $\mathbf{p}_n$ and $\mathbf{q}_n$ are the same as Apéry's $\alpha_n$ and $-\beta_n/2$ for $\zeta(2)$.*

Thus, the proof of this claim would be exactly what we need to give a new proof of $\zeta(2) \notin \mathbb{Q}$.

2.5. **A very-well-poised series related to $\zeta(3)$.** K. Ball introduced the following series (see the introduction of [11])

$$\begin{aligned}\mathbf{B}_n &= n!^2 \sum_{k=1}^{\infty} \left(k + \frac{n}{2}\right) \frac{(k-n)_n (k+n+1)_n}{(k)_{n+1}^4} \\ &= \frac{n!^7 (3n+2)!}{2(2n+1)!^5} \, {}_7F_6 \left[\begin{matrix} 3n+2, \frac{3}{2}n+2, n+1, \ldots, n+1 \\ \frac{3}{2}n+1, 2n+2, \ldots, 2n+2 \end{matrix}; 1\right]\end{aligned}$$

in the hope that it would give a completely elementary proof of the irrationality of $\zeta(3)$, in the style of the usual irrationality proof of $\exp(1)$. The similarity of $\mathbf{S}_n$ and $\mathbf{B}_n$ is not an accident, and one has $\mathbf{B}_n = \mathbf{a}_n \zeta(3) - \mathbf{b}_n$ with $\mathrm{d}_n \mathbf{a}_n$ and $\mathrm{d}_n^4 \mathbf{b}_n$ integers, whereas one would have expected *a priori* $\mathbf{B}_n \in \mathbb{Q} + \mathbb{Q}\zeta(2) + \mathbb{Q}\zeta(3) + \mathbb{Q}\zeta(4)$. Note that, once again, the given denominators are to large to get a new proof of $\zeta(3) \notin \mathbb{Q}$.

The expression for $\mathbf{a}_n$ is

$$\mathbf{a}_n = (-1)^{n+1} \sum_{j=0}^{n} \left(\frac{n}{2} - j\right) \binom{n}{j}^4 \binom{n+j}{n} \binom{2n-j}{n} \cdot \left(5H_{n-j} - 5H_j + H_{n+j} - H_{2n-j} - \frac{1}{\frac{n}{2} - j}\right)$$

and numerical computations suggest the following improvements:

**Claim 2.** *The numbers* $\mathbf{a}_n$ *and* $\mathrm{d}_n^3\mathbf{b}_n$ *appear to be integers and furthermore* $\mathbf{a}_n$ *and* $\mathbf{b}_n$ *are the same as Apéry's* $a_n$ *and* $b_n/2$ *for* $\zeta(3)$.

This would be enough to give an elementary proof of the irrationality of $\zeta(3)$, since a simple application of Stirling's formula proves that

$$\lim_{n\to+\infty} |\mathbf{B}_n|^{1/n} = (\sqrt{2}-1)^4.$$

2.6. **Irrationality of infinitely many values of $\zeta$ at the odd integers.** The following very-well-poised hypergeometric series

$$\begin{aligned}\mathbf{S}_{n,A,r} &= n!^{A-2r}\sum_{k=1}^{\infty}\Big(k+\frac{n}{2}\Big)\frac{(k-rn)_{rn}(k+n+1)_{rn}}{(k)_{n+1}^A}(-1)^{kA}\\ &= n!^{A-2r}\frac{(rn)!^{A+1}((2r+1)n+2)!}{2((r+1)n+1)!^{A+1}}\\ &\quad\times {}_{A+3}F_{A+2}\left[\begin{matrix}(2r+1)n+2, \frac{2r+1}{2}n+2, rn+1, \ldots, rn+1\\ \frac{2r+1}{2}n+1, (r+1)n+2, \ldots, (r+1)n+2\end{matrix}; (-1)^A\right]\end{aligned}$$

was introduced in [11] and [4] to prove that

**Theorem 4.** *For any even* $A \geq 4$,

$$\dim_{\mathbb{Q}}(\mathbb{Q}+\mathbb{Q}\zeta(3)+\mathbb{Q}\zeta(5)+\cdots+\mathbb{Q}\zeta(A-1)) \geq \frac{1+o(1)}{1+\log(2)}\log(A). \tag{2}$$

(In fact, it was a similar series without the factor $k+n/2$, which is useless in this particular case.) The new parameter $r$ is an integer such that $1 \leq r \leq A/2$. We now briefly indicate the steps in the proof of (2).

The form of the numerator of the summand first implies that

$$\mathbf{S}_{n,A,r} = p_{0,n} + \sum_{\substack{j=2\\ j\equiv A-1[2]}}^{A} p_{j,n}\zeta(j),$$

which involves only odd zeta values if $A$ is even, and $\mathrm{d}_n^{A-j}$ is a denominator of $p_{j,n}$. Using an explicit expression for the $p_{j,n}$'s, we can prove that

$$\limsup_{n\to+\infty} |p_{j,n}|^{1/n} \leq 2^{A-2r}(2r+1)^{2r+1}.$$

Furthermore, we have a Euler type integral representation

$$\mathbf{S}_{n,A,r} = \frac{((2r+1)n+1)!}{n!^{2r+1}}\int_{[0,1]^{A+1}}\frac{\prod_{j=1}^{A+1}x_j^{rn}(1-x_j)^n\mathrm{d}x_j}{(1-x_1\cdots x_{A+1})^{(2r+1)n}}\frac{1+x_1\cdots x_{A+1}}{(1-x_1\cdots x_{A+1})^3}$$

from which we deduce that

$$0 < \lim_{n\to+\infty} |\mathbf{S}_{n,A,r}|^{1/n} \leq 2^{2r+1} r^{2r-A}.$$

It remains to apply Nesterenko's linear independence criteria, with the optimal choice $r = [A/\log^2(A)]$, to get (2).

Although this is not needed here, we note that, as the reader might have already suspected, numerical computations suggest that:

**Claim 3.** $\mathrm{d}_n^{A-1}$ *seems to be a denominator for the* $p_{j,n}$.

2.7. **Irrationality of one of the nine numbers $\zeta(5), \zeta(7), \ldots, \zeta(21)$.** Let $A \geq 6$ be an even integer and consider the series

$$\tilde{\mathbf{S}}_{A,n} = n!^{A-6} \sum_{k=1}^{\infty} \frac{1}{2} \frac{\partial^2}{\partial k^2} \left( \left(k + \frac{n}{2}\right) \frac{(k-n)_n^3 (k+n+1)_n^3}{(k)_{n+1}^A} \right),$$

which is not exactly hypergeometric, but has enough properties in common with very-well-poised series to give that

$$\tilde{\mathbf{S}}_{A,n} = \tilde{p}_{0,n} + \sum_{\substack{j=5 \\ \mathrm{odd}\, j}}^{A+1} \tilde{p}_{j,n} \zeta(j).$$

Note that differentiating twice together with the numerator of the summand enable us to have a linear form only in odd zeta values from $\zeta(5)$, and not $\zeta(3)$. Furthermore, $\mathrm{d}_n^{A+2}$ is a denominator of $\tilde{p}_{j,n}$ and we now seek the smallest possible $A$ such that $\mathrm{d}_n^{A+2}\tilde{\mathbf{S}}_{A,n}$ tends to 0 as $n$ tends to infinity.

This can be done by first noting that $\tilde{\mathbf{S}}_{A,n}$ is the real part of the integral

$$\frac{(-1)^n n!^{A-6}}{2i\pi} \int_{c+i\infty}^{c-i\infty} \left(z + \frac{n}{2}\right) \frac{\Gamma(z)^{A+3}\Gamma(n-z+1)^3\Gamma(z+2n+1)^3}{\Gamma(z+n+1)^{A+3}} e^{i\pi z} \mathrm{d}z,$$

(where $c$ is any real in $(0,1)$) and then by applying the saddle point method to estimate the asymptotic behavior of this integral as $n$ tends to infinity. One finds that 20 is the smallest such $A$, yielding the following result proved by the author in [12].

**Theorem 5.** *At least one of the nine numbers $\zeta(5), \zeta(7), \ldots, \zeta(21)$ is irrational.*

Numerical computations suggest that:

**Claim 4.** *For any even $A \geq 6$, $\mathrm{d}_n^{A+1}$ seems to be a denominator of the $\tilde{p}_{j,n}$.*

Here, the consequences would very important since the same argument shows that $\mathrm{d}_n^{18+1}\tilde{\mathbf{S}}_{18,n}$ tends to 0 (while $\mathrm{d}_n^{18+2}\tilde{\mathbf{S}}_{18,n}$ does not), thus proving, conjecturally, the irrationality of one of the eight numbers $\zeta(5), \zeta(7), \ldots, \zeta(19)$.

## 3. A very general phenomenon

In the light of the previous examples, it is time to adopt a general approach to very-well-poised series of hypergeometric kind.

Let $z$ be a complex number such that $|z| \geq 1$, and $A$, $B$, $C$, $r$ be positive integers such that $1 \leq 2Br \leq A$. Consider the series

$$\mathbf{S}_{n,A,B,C,r}(z) = n!^{A-2Br} \sum_{k=1}^{\infty} \frac{1}{C!} \frac{\partial^C}{\partial k^C} \left( \left(k + \frac{n}{2}\right) \frac{(k-rn)_{rn}^B (k+n+1)_{rn}^B}{(k)_{n+1}^A} \right) z^{-k},$$

which is really hypergeometric when $C = 0$. According to the general scheme developed in section 2.1, we have that

$$\mathbf{S}_{n,A,B,C,r}(z) = \mathbf{p}_{0,C,n}(z) + (-1)^C \sum_{m=1}^{A} \binom{C+m-1}{m-1} \mathbf{p}_{m,n}(z) \operatorname{Li}_{C+m}(1/z),$$

where the polynomials $\mathbf{p}_{m,n}(X)$ also depend on $A$,$B$ and $r$, but not $C$, except $\mathbf{p}_{0,C,n}(X)$.

Using the trivial but important relation $(\alpha)_m = (-1)^m(-\alpha - m + 1)_m$ (for any $\alpha \in \mathbb{C}$), one immediately proves that the rational summand of $\mathbf{S}_{n,A,B,C,r}(z)$

$$R_n(k) = \left(k + \frac{n}{2}\right) \frac{(k-rn)_{rn}^B (k+n+1)_{rn}^B}{(k)_{n+1}^A}$$

satisfies the symmetry $R_n(-n-k) = (-1)^{A(n+1)+1} R_n(k)$, from which one deduces that $z^n \mathbf{p}_{j,n}(1/z) = (-1)^{A+j+1} \mathbf{p}_{j,n}(z)$ $(j \geq 1)$. Consequently, when $A$ is even, $\mathbf{S}_{n,A,B,C,r}((-1)^A)$ is a rational linear combination of $1, \zeta(C+3), \zeta(C+5), \ldots, \zeta(C+A-1)$, whereas when $A$ is odd, $\mathbf{S}_{n,A,B,C,r}((-1)^A)$ is a rational linear combination of $1, \zeta(C+2), \zeta(C+4), \ldots, \zeta(C+A-1)$.

The coefficients of these linear forms satisfy $\mathrm{d}_n^{A+C} \mathbf{p}_{0,C,n}(X) \in \mathbb{Z}[X]$ and $\mathrm{d}_n^{A-m} \mathbf{p}_{m,n}(X) \in \mathbb{Z}[X]$, but the evidence above, along with many other numerical computations, led the author to formulate the following conjecture in [13], which contains the previous Claims 1–4.

**Denominators Conjecture.** *Fix integers $A \geq 2$, $B \geq 0$, $r \geq 0$ and $n \geq 0$ (with $0 \leq 2Br \leq A$). Then the rational numbers $\mathrm{d}_n^{A+C-1} \mathbf{p}_{0,C,n}\left((-1)^A\right)$ and $\mathrm{d}_n^{A-m-1} \mathbf{p}_{m,n}\left((-1)^A\right)$ (for all $m \in \{1, \ldots, A\}$) are integers.*

Nothing similar holds if the factor $k + n/2$ is omitted from the series: it corresponds to the word *very* in very-well-poised and its presence is crucial.

## 4. Idea of the proof of the Denominators Conjecture

After many partial steps towards the proof of the conjecture (as witnessed by the different versions posted in the arXiv), C. Krattenthaler and the author finally proved it completely in [9].

**Theorem 6.** *The Denominators Conjecture is true.*

It follows that Claims 1 to 4 are all true. We now explain the ideas behind this result.

4.1. **A refined Denominators Conjecture.** We will only consider the case of the "leading" coefficient, that is to say the coefficient $\mathbf{p}_{A-1,n}\left((-1)^A\right)$ of $\zeta(C+A-1)$, which depends on $A$, $B$ and $r$ but not on $C$. From now on, we assume that $r=1$ and set $P_n(A,B)=(-1)^{B(n+1)}\mathbf{p}_{A-1,n}\left((-1)^A\right)$. Then,

$$P_n(A,B)=\sum_{j=0}^{n}\left(\frac{n}{2}-j\right)\binom{n}{j}^A\binom{n+j}{n}^B\binom{2n-j}{n}^B \cdot\left((A+B)H_{n-j}-(A+B)H_j+BH_{n+j}-BH_{2n-j}-\frac{1}{\frac{n}{2}-j}\right).$$

The general estimates prove that $\mathrm{d}_n P_n(A,B)\in\mathbb{Z}$.

The Denominators Conjecture claims that $P_n(A,B)$ is always an integer. In the case of $\zeta(2)$ ($A=3$, $B=1$) and $\zeta(3)$ ($A=4$, $B=1$), we could even identify these coefficients as Apéry's numbers:

$$P_n(3,1)=\alpha_n \quad\text{and}\quad P_n(4,1)=a_n. \tag{3}$$

The advantage of (3) is that it is much easier to prove an identity and, in fact, (3) can be proved with the help of Zeilberger's program Ekhad: both sides are shown to satisfy the same recurrences relations, with the same initial values (see the introduction of [9] for references).

But in the general situation, we do not have an identity to prove and the first thing to do is to find one. This can be done as follows. In [15], Vasilyev introduced the following generalisation of Beukers' integrals for $\zeta(2)$ and $\zeta(3)$: let $E\geq 2$ and

$$J_{n,E}=\int_{[0,1]^E}\frac{\prod_{j=1}^{E}x_j^n(1-x_j)^n\,\mathrm{d}x_j}{Q_E(x_1,x_2,\dots,x_E)^{n+1}}$$

where $Q_E(x_1,x_2,\dots,x_E)=1-(\cdots(1-(1-x_1)x_2)\cdots)x_E$. He proved that $J_{n,4}\in\mathbb{Q}+\mathbb{Q}\zeta(2)+\mathbb{Q}\zeta(4)$ and $J_{n,5}\in\mathbb{Q}+\mathbb{Q}\zeta(3)+\mathbb{Q}\zeta(5)$, while Beukers showed that $J_{n,2}\in\mathbb{Q}+\mathbb{Q}\zeta(2)$ and $J_{n,3}\in\mathbb{Q}+\mathbb{Q}\zeta(3)$. Vasilyev conjectured that this dichotomy is valid for all $E$, depending on the parity of $E$ and this was proved by Zudilin in [16].

**Theorem 7.** *For all $E\geq 2$, we have that*

$$J_{n,E}=\frac{n!^{2E+1}(3n+2)!}{2(2n+1)!^{E+2}}\;{}_{E+4}F_{E+3}\left[\begin{matrix}3n+2,\frac{3}{2}n+2,n+1,\dots,n+1\\ \frac{3}{2}n+1,2n+2,\dots,2n+2\end{matrix};(-1)^{E+1}\right]. \tag{4}$$

Indeed, the hypergeometric series (4) is a special case of the very-well-poised series from section 2.6 and consequently it can be represented as a linear form in odd/even zeta values, where the coefficient of $\zeta(E)$ is exactly $(-1)^{n+1}P_n(E+1,1)$. But it is also possible, though quite difficult, to expand the integral on the left hand side of (4) as a linear combinations of zeta values (and also some multiple zeta values), and then to isolate the coefficient of $\zeta(E)$. Assuming the reasonable, but still conjectural, fact that the values $\zeta(n)$, $n\geq 2$, are

$\mathbb{Q}$–linearly independent, the comparison of both sides of (4) led to the following

**Refined Denominators Conjecture in the case** $B=1$**.** *For* $A=2m+1\geq 3$ *odd, set*

$$p_n(A,1)=\sum_{0\leq i_1\leq i_2\leq\cdots\leq i_m\leq n}\binom{n}{i_m}^2\binom{n+i_m}{n}\prod_{k=1}^{m-1}\binom{n}{i_k}^2\binom{n+i_{k+1}-i_k}{n},$$

*and for* $A=2m\geq 2$ *even, set*

$$p_n(A,1)=\sum_{0\leq i_1\leq i_2\leq\cdots\leq i_m\leq n}(-1)^{i_m}\binom{n}{i_m}\binom{n+i_m}{n}\prod_{k=1}^{m-1}\binom{n}{i_k}^2\binom{n+i_{k+1}-i_k}{n},$$

*Then, for all integers* $A\geq 2$, $n\geq 0$, *we have that* $P_n(A,1)=(-1)^{An+1}p_n(A,1)$.

A similar integral identity can be used to obtain the:

**Refined Denominators Conjecture in the case** $B=0$**.** *For* $A=2m+3\geq 5$ *odd, set*

$$p_n(A,0)=\sum_{0\leq i_1\leq i_2\leq\cdots\leq i_m\leq n}\prod_{k=1}^{m}\binom{n}{i_k}^2\binom{n+i_{k+1}-i_k}{n},$$

*where, by definition,* $i_{m+1}=n$ *and for* $A=2m+2\geq 4$ *even, set*

$$p_n(A,0)=\sum_{0\leq i_1\leq i_2\leq\cdots\leq i_m\leq n}\binom{n}{i_m}^2\prod_{k=1}^{m-1}\binom{n}{i_k}^2\binom{n+i_{k+1}-i_k}{n}.$$

*Then, for all integers* $A\geq 4$, $n\geq 0$, *we have that* $P_n(A,0)=(-1)^{n+1}p_n(A,0)$.

It is now quite clear why $P_n(A,0)$ and $P_n(A,1)$ should be integers, since these refined conjectures express them as multiple sums of products of binomial coefficients. It is also possible to give similar refinements for the values of $A$ not covered by these conjectural identities (and to prove them "by hand").

4.2. **The key identity.** C. Krattenthaler has produced an extremely useful electronic version, which he called HYP, of Gasper & Rahman's book [8] that lists the almost infinitely many identities between ($q$–)hypergeometric series: HYP is not only an electronic library but, more importantly, a "tool box", that is to say one can feed it a hypergeometric series, ask it to perform a certain transform and output the result.

Thanks to this software, it becomes easier (but still very difficult in our situation) to handle hypergeometric sums and prove the "refined conjectural identities" above, which are in fact special cases of the following key identity, where one can recognize the very-well-poised factor $\dfrac{n}{2}+\dfrac{a_1}{2}-j$ in the second simple sum:

$$
\begin{aligned}
(-1)^n \sum_{0\le i_1\le i_2\le\cdots\le i_m\le n} & \binom{a_2-b_1}{n-i_m}\binom{-a_1+a_2+i_m-i_{m-1}}{i_m-i_{m-1}}\\
&\cdot \frac{\Gamma(n+b_1+b_2-a_1+1)}{\Gamma(a_2-a_3+i_m+1)\,\Gamma(n+a_3-a_1-a_2+b_1+b_2-i_m+1)}\\
&\cdot \frac{\Gamma(a_2+i_m+1)}{\Gamma(a_2-b_2-n+i_m+1)\,\Gamma(n+b_2+1)}\\
&\cdot \frac{\Gamma(n+a_3-a_1-a_2+b_1+b_2-i_{m-1}+1)}{\Gamma(-a_1+a_3-i_{m-1}+1)\,\Gamma(n-a_2+b_1+b_2+1)}\\
&\cdot \left(\prod_{k=1}^{m-1}\binom{n-a_1+b_{2m-2k+1}+b_{2m-2k+2}+i_k-i_{k-1}}{i_k-i_{k-1}}\right.\\
&\cdot \frac{\Gamma(n-a_1+b_{2m-2k-1}+b_{2m-2k+1}+1)}{\Gamma(b_{2m-2k+1}+i_k+1)\,\Gamma(n-a_1+b_{2m-2k-1}-i_k+1)}\\
&\left.\cdot \frac{\Gamma(n-a_1+b_{2m-2k}+b_{2m-2k+2}+1)}{\Gamma(b_{2m-2k+2}+i_k+1)\,\Gamma(n-a_1+b_{2m-2k}-i_k+1)}\right)
\end{aligned}
$$

$$=$$

$$
\begin{aligned}
&\frac{2\pi}{\sin(\pi a_1)}\cdot\frac{\Gamma(n+b_1+b_2-a_1+1)\,\Gamma(n+b_1+1)}{n!\,\Gamma(n-a_2+b_1+b_2+1)\,\Gamma(a_2-b_2+1)}\\
&\times\sum_{j=0}^{n}\left(\left(\frac{n}{2}+\frac{a_1}{2}-j\right)\binom{n}{j}\cdot\frac{\Gamma(a_3+1)}{\Gamma(-a_1+j+1)\,\Gamma(n+a_1-j+1)}\right.\\
&\cdot\frac{\Gamma(-a_1+a_2+j+1)}{\Gamma(-n-a_1+a_3+j+1)\,\Gamma(n+a_2-a_3+1)}\\
&\cdot\frac{\Gamma(n+a_2-j+1)}{\Gamma(a_3-j+1)\,\Gamma(a_2-a_1+1)}\\
&\left.\cdot\left(\prod_{k=1}^{2m}\frac{\Gamma(n-a_1+b_k+b_{k+2}+1)}{\Gamma(n+b_k-j+1)\,\Gamma(-a_1+b_k+j+1)}\right)\right),
\end{aligned}
$$

where, by definition, $b_{2m+1}=b_{2m+2}=0$, $i_0=0$ and where for $m=1$, an empty product is given the value 1. In fact, this identity is more or less the same as a formula proved by Andrews [1] in the seventies, but without HYP...

4.3. **The case $B\ge 2$.** From the key identity, it is not only possible to prove the refined Denominators Conjecture for $P_n(A,0)$ and $P_n(A,1)$, but also to handle the case of $B\ge 2$:

**Refined Denominators Conjecture in the case $B \geq 2$.** *For $B \geq 2$ and $A = 2m+1 \geq 3$ odd, set*

$$p_n(A,B) = \sum_{0 \leq i_1 \leq i_2 \leq \cdots \leq i_{m+B-1} \leq n} (-1)^{i_{m+B-1}} \binom{n}{i_{m+B-1}} \binom{n+i_{m+B-1}}{n} \binom{n+i_{m+B-1}-i_{m+B-2}}{n} \cdot \left( \prod_{k=B}^{m+B-2} \binom{n}{i_k}^2 \binom{n+i_k-i_{k-1}}{n} \right) \binom{n}{i_{B-1}} \binom{2n-i_{B-1}}{n} \binom{n}{i_{B-1}-i_{B-2}} \cdot \left( \prod_{k=1}^{B-2} \binom{n+i_k}{n} \binom{2n-i_k}{n} \binom{n}{i_k - i_{k-1}} \right),$$

*where, by definition, $i_0 = 0$.*

*For $B \geq 2$ and $A = 2m+2 \geq 2$ even, set*

$$p_n(A,B) = \sum_{0 \leq i_1 \leq i_2 \leq \cdots \leq i_{m+B-1} \leq n} \binom{n+i_{m+B-1}}{n} \left( \prod_{k=B}^{m+B-1} \binom{n}{i_k}^2 \binom{n+i_k-i_{k-1}}{n} \right) \cdot \binom{n}{i_{B-1}} \binom{2n-i_{B-1}}{n} \binom{n}{i_{B-1}-i_{B-2}} \left( \prod_{k=1}^{B-2} \binom{n+i_k}{n} \binom{2n-i_k}{n} \binom{n}{i_k - i_{k-1}} \right).$$

*Then, for all integers $A \geq 2$ and $B \geq 2$, we have that $P_n(A,B) = (-1)^{(A+1)n+1} p_n(A,B)$.*

Thus the Denominators Conjecture is true for the "leading" coefficients. More "human" work is required to extract from the key identity the conjectured denominators of the other coefficients: the interested reader is refered to [9] for the details, which are far from trivial. For completeness, the conjecture is proved for $2\mathbf{p}_{0,C,n}(\pm 1)$ rather than for $\mathbf{p}_{0,C,n}(\pm 1)$: this is the most difficult case of the conjecture, and also the most important since its denominator is always the one of the linear forms in zeta values.

## 5. Arithmetical consequences

As we have already mentioned in subsections 2.4, 2.5 and 2.7, the Denominators Conjecture enables us to give new proofs that $\zeta(2)$, $\zeta(3)$ are irrational and, more importantly, that at least one of the eight numbers $\zeta(5), \zeta(7), \ldots, \zeta(19)$ is irrational. This is proved *via* the series (corresponding to $A = 18$, $B = 3$, $C = 2$ and $r = 1$)

$$n!^{12} \sum_{k=1}^{\infty} \frac{1}{2} \frac{\partial^2}{\partial k^2} \left( \left( k + \frac{n}{2} \right) \frac{(k-n)_n^3 (k+n+1)_n^3}{(k)_{n+1}^{18}} \right) = \tilde{p}_{0,n} + \sum_{\substack{j=5 \\ j \text{ odd}}}^{19} \tilde{p}_{j,n} \zeta(j),$$

where the *a priori* denominator $\mathrm{d}_n^{20}$ of $\tilde{p}_{j,n}$ can now be replaced by $\mathrm{d}_n^{19}$. But this refinement is useless since W. Zudilin proved the following much better result in [17].

**Theorem 8.** *At least one of the four numbers $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational.*

To prove this, he uses more complicated very-well-poised series (in a sense, those considered in this text are the simplest of this kind) and he also formulates a "super" Denominators Conjecture for his linear forms, which is still open. Hence there might be room to prove that at least one of three numbers $\zeta(5), \zeta(7), \zeta(9)$ is irrational.

## Bibliography


[1] G. E. Andrews, *Problems and prospects for basic hypergeometric functions*, Theory and application of special functions, R. A. Askey, ed., Math. Res. Center, Univ. Wisconsin, Publ. No. 35, Academic Press, New York, pp. 191–224, 1975.

[2] R. Apéry, *Irrationalité de $\zeta(2)$ et $\zeta(3)$*, Astérisque **61** (1979), 11–13.

[3] W. N. Bailey, *Generalized hypergeometric series*, Cambridge University Press, Cambridge, 1935.

[4] K. Ball et T. Rivoal, *Irrationalité d'une infinité de valeurs de la fonction zêta aux entiers impairs*, Invent. Math. **146**.1 (2001), 193–207.

[5] F. Beukers, *A note on the irrationality of $\zeta(2)$ and $\zeta(3)$*, Bull. London Math. Soc. **11** (1979), 268–272.

[6] F. Beukers, *Padé approximations in Number Theory* in "Padé approximation and its applications", Amsterdam 1980, LNM 888, 90–99, Springer (1981).

[7] L.A. Gutnik, *The irrationality of certain quantities involving $\zeta(3)$*, Russ. Math. Surv. **34**.3 (1979), 200. In russian in Acta Arith. **42**.3 (1983), 255–264.

[8] G. Gasper et M. Rahman, *Basic hypergeometric series*, Encyclopedia of Mathematics And Its Applications 35, Cambridge University Press, Cambridge, 1990.

[9] C. Krattenthaler and T. Rivoal, *Hypergéométrie et fonction zêta de Riemann*, preprint (2004). `http://arXiv.org/abs/math.NT/0311114`.

[10] Yu.V. Nesterenko, *On the linear independence of numbers*, Mosc. Univ. Math. Bull.**40**.1 (1985), 69–74, translation of Vest. Mosk. Univ., Ser. I (1985), no. 1, 46–54.

[11] T. Rivoal, *La fonction zêta de Riemann prend une infinité de valeurs irrationnelles aux entiers impairs*, C. R. Acad. Sci. Paris, Série I Math. **331**.4 (2000), 267–270. `http://arXiv.org/abs/math.NT/0008051`.

[12] T. Rivoal, *Irrationalité d'au moins un des neuf nombres $\zeta(5)$, $\zeta(7), \dots, \zeta(21)$*, Acta Arith. **103**.2 (2002), 157–167.

[13] T. Rivoal, *Séries hypergéométriques et irrationalité des valeurs de la fonction zêta de Riemann*, J. Théor. Nombres Bordeaux **15**.1 (2003), 351–365.

[14] L. J. Slater, *Generalized hypergeometric functions*, Cambridge University Press, Cambridge, 1966.

[15] D. V. Vasilyev, *On small linear forms for the values of the Riemann zeta-function at odd points*, prépublication no.1 (558), Nat. Acad. Sci. Belarus, Institute Math., Minsk (2001).

[16] W. Zudilin, *Well-poised hypergeometric service for diophantine problems of zeta values*, J. Théor. Nombres Bordeaux **15**.2 (2003), 593–626; see also: *Multiple integral-representations of very-well-poised hypergeometric series*, e-print available at `http://arXiv.org/abs/math.NT/0206177`.

[17] W. Zudilin, *Arithmetic of linear forms involving odd zeta values*, à paraître au J. Théor. Nombres Bordeaux. `http://arXiv.org/abs/math.NT/0206176`.