REMAINDER PADÉ APPROXIMANTS FOR THE EXPONENTIAL FUNCTION

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Abstract. Following earlier research of ours, we propose a new method for obtaining the complete Padé table of the exponential function. It is based on an explicit construction of certain Padé approximants not for the usual power series for exp at 0 but for a formal power series related in a simple way to the remainder term of the power series for exp. This surprising and non trivial coincidence is proved more generally for type II simultaneous Padé approximants for a family $(\exp(a_j z))_{j=1,\dots,r}$ with distinct complex a's and we recover Hermite's classical formulae. The proof uses certain discrete multiple orthogonal polynomials recently introduced by Arvesú, Coussement and van Assche, which generalise the classical Charlier orthogonal polynomials.

1. Introduction and motivation

In [10], the first author gave a new proof of the irrationality of $\zeta(2) = \sum_{k=1}^{\infty} 1/k^2$ (and also of $\zeta(3) = \sum_{k=1}^{\infty} 1/k^3$) based on an explicit construction of certain Padé approximants of the remainder term $R_2(1/n) = \sum_{k=n}^{\infty} 1/k^2$. More precisely, we have that $\zeta(2) = \sum_{k=1}^{n-1} 1/k^2 + R_2(1/n)$ with

$$R_2(z) = \sum_{k=0}^{\infty} \frac{z^2}{(zk+1)^2}.$$

The function R_2 is meromorphic on $\mathbb{C} \setminus \{0, -1, -1/2, -1/3, \dots\}$ and consequently cannot be holomorphic at 0. However, it is C^∞ at $z = 0$ and admits a Taylor expansion (with radius of convergence zero) $\widehat{R}_2(z) = \sum_{k=0}^{\infty} B_k z^{k+1}$, where B_k is the k-th Bernoulli's number. The keystone of the method is an explicit computation of the diagonal Padé approximant $P_n(z)/Q_n(z) = [n/n]_{\widehat{R}_2}(z) \in \mathbb{Q}(z)$ of the series $\widehat{R}_2(z)$, with a good estimate of the error term $E_n(z) = R_2(z) - [n/n]_{\widehat{R}_2}(z)$. In the final step, one finds that

$$Q_n(1/n)E_n(1/n) = Q_n(1/n)\zeta(2) - Q_n(1/n)\sum_{k=1}^{n-1} 1/k^2 - P_n(1/n) = q_n\zeta(2) - p_n$$

provides a sequence of rational approximations p_n/q_n good enough to imply the irrationality of $\zeta(2)$. Surprisingly, it turns out that the rational numbers p_n and q_n are well known: they are exactly those used by Apéry [1] for the same purpose.

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A similar phenomenon occurs for $\zeta(3)$, where one recovers Apéry's celebrated approximations, also given in [1]. Furthermore, in [12], the second author adapted this method to produce a sequence of fast converging rational approximations u_n/v_n for Catalan's constant $G = \sum_{k=0}^{\infty}(-1)^k/(2k+1)^2$ (the rôle of the Bernoulli's numbers being played by Euler's numbers); although the irrationality of G could not be deduced from these approximations, they were found to be the same as the approximations $\hat{u}_n/\hat{v}_n$ to G previously obtained in [13] by means of a completely different method, based on hypergeometric series (the proof of this coincidence is quite long and intricate). The hypergeometric method is central to recent progress on the problem of the arithmetical nature of the values of the Riemann zeta function $\zeta(s)$ at odd integers $s \geq 3$; it also provides new proofs of the irrationality of $\zeta(2)$ and $\zeta(3)$ (see [8] for a description of this method).

In [6], very general Padé type approximants problems were proposed and solved in order to generalise (by a functional approach) the hypergeometric method. This time, the involved functions were not remainder functions like R_2 but the polylogarithmic functions $\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} z^k/k^s$, defined for $s \geq 1$ and $|z| \leq 1$, with $(s, z) \neq (1, 1)$. For example, the rationals numbers p_n and q_n for $\zeta(2)$ are explicitly produced by the solutions of the following two point Padé type problem (first proposed and solved by Beukers in [4]): find polynomials $A_n(z), B_n(z)$ and $C_n(z) \in \mathbb{Q}[z]$, of degree at most n, such that

$$\begin{cases} A_n(z)\,\mathrm{Li}_2(z) + B_n(z)\,\mathrm{Li}_1(z) + C_n(z) = \mathcal{O}(z^{2n+1}); \\ A_n(z)\log(z) + B_n(z) = \mathcal{O}((1-z)^{n+1}). \end{cases}$$

Indeed, this system admits a unique solution (up to a multiplicative factor) and $A_n(1) = q_n$ and $C_n(1) = -p_n$. A similar system is given at the end of [12] that produces the approximation $\hat{u}_n/\hat{v}_n$ for G alluded to above and thus provides another Padé interpretation for the numbers u_n/v_n.

It is somewhat surprising that two *a priori* completely unrelated Padé type approximants computations produce the *same* rational approximations of a given number, all the more since the function $\mathrm{Li}_2(z)$ and $R_2(z)$ do not share at first glance much analytical properties, except that $\mathrm{Li}_2(1) = \zeta(2) = R_2(1)$. The aim of this paper is to prove the similar phenomenon for the exponential function: indeed, we show that the complete table of Padé approximants for the function $\exp(z)$ at $z = 0$ essentially coincide with the approximations obtained by the method described above, which we call the *Remainder Padé approximants* for $\exp(z)$. Our results deal more generally with simultaneous type II Padé-approximations of a family of exponential functions.

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2. Statement of the results

We first introduce some notations. We will consider a finite set of exponential functions $(\exp(a_j z))_{j=1,\dots r}$. The variable z is any complex number and the parameters a_j ($j = 1, \dots, r$) are distinct non-zero complex numbers. We will denote by $\underline{a}$ the family

$(a_j)_{j=1,\dots,r}$. The Pochhammer symbol $(\alpha)_m$ is defined as $\alpha(\alpha+1)\cdots(\alpha+m-1)$ if $m \geq 1$ and 1 if $m=0$. From now on, n denotes a positive integer.

We also define the function $\Phi_z(t)$ as the series

$$\Phi_z(t) = \sum_{k=0}^{\infty} \frac{\Gamma(1-1/t)}{\Gamma(k+1-1/t)}\, z^k,$$

where Γ denotes Euler's Gamma function: it is defined at least for any complex t such that $\Re(1/t) < 1$, and in particular for real negative t. Clearly, for any $j = 1, \dots, r$, we have that

$$\exp(a_j\, z) = \sum_{k=0}^{n-1} \frac{(a_j\, z)^k}{k!} + \frac{(a_j\, z)^n}{n!}\, \Phi_{a_j z}(-1/n).$$

We will show (see Lemma 1 in section 3) that $\Phi_z(t)$ admits an explicit Taylor expansion $\widehat{\Phi}_z(t)$ at $t=0$ of radius of convergence zero, which we therefore view as a formal power series

$$\widehat{\Phi}_z(t) = \sum_{k=0}^{\infty} \varphi_k(-z) t^k.$$

Here, the $\varphi_k(z)$ are exactly *Touchard exponential polynomials of degree* k [14], defined by the generating function

$$e^{z(e^X-1)} = \sum_{k=0}^{\infty} \varphi_k(z)\, \frac{X^k}{k!}.$$

Let us consider a family $\big(F_j(X)\big)_{j=1,\dots r}$ of complex formal power series at 0 and D and p some integers satisfying $D \geq (r-1)p$. We recall that the type II Padé approximants of parameter (D,p) of $\big(F_j(X)\big)_{j=1,\dots r}$ are polynomials $P_1(X), \dots, P_r(X)$ and $Q(X)$ in $\mathbb{C}[X]$ such that

$$\deg(P_j) \leq D, \quad \deg(Q) \leq rp \quad \text{and} \quad Q(X)F_j(X) - P_j(X) = \mathcal{O}\left(X^{D+p+1}\right), \quad 1 \leq j \leq r.$$

When $r=1$, we obtain the usual Padé approximants $P_1(X)/Q(X) = [D/p]_{F_1}(X)$ of the series $F_1(X)$.

Theorem 1. *For fixed z, the type II Padé approximants at $t=0$ of parameter $(rp-1,p)$ for the formal power series $\left(\widehat{\Phi}_{a_j z}(t)\right)_{j=1,\dots,r}$ are given by the following formulae: for $j = 1, \dots, r$,*

$$P_{j,\underline{a},p}(t,z) = (-t)^{rp-1} \sum_{k_1,\dots,k_r=0}^{p} \left(\prod_{i=1}^{r} \binom{p}{k_i} (-a_i z)^{p-k_i} \right) (-1/t)_{k_1+\cdots+k_r} \sum_{i=1}^{k_1+\cdots+k_r} \frac{(a_j z)^{i-1}}{(-1/t)_i},$$

$$Q_{\underline{a},p}(t,z) = (-t)^{rp} \sum_{k_1,\dots,k_r=0}^{p} \left(\prod_{i=1}^{r} \binom{p}{k_i} (-a_i\, z)^{p-k_i} \right) (-1/t)_{k_1+\cdots+k_r}.$$

Remarks. 1) We set the variable z at the same level as t since it will be the main variable below.

2) The polynomial $Q_{\underline{a},p}$ satisfies $Q_{\underline{a},p}(0,z)=1$.

By Theorem 1, the simultaneous Padé approximants at $t=0$, namely the function $P_{j,\underline{a},p}(t,z)/Q_{\underline{a},p}(t,z)$, are not only rational function in t but also in z. Moreover, writing

$$E_{j,\underline{a},p}(t,z)=\widehat{\Phi}_{a_j z}(t)-\frac{P_{j,\underline{a},p}(t,z)}{Q_{\underline{a},p}(t,z)},$$

it turns out that, for fixed t with $\Re(1/t)<1$, the function $E_{j,\underline{a},p}(t,z)$ also has a power series expansion at $z=0$. Using the definition of $\widehat{\Phi}_{a_j z}(t)$, we obtain for fixed values $t=-1/n$ and $j=1,\dots,r$ the relations

$$\exp(a_j z)-\sum_{k=0}^{n-1}\frac{(a_j z)^k}{k!}-\frac{a_j^n z^n}{n!}\frac{P_{j,\underline{a},p}(-1/n,z)}{Q_{\underline{a},p}(-1/n,z)}=\frac{a_j^n z^n}{n!}\,E_{j,\underline{a},p}(-1/n,z). \tag{2.1}$$

This equation defines simultaneous Remainder Padé approximants for $(\exp(a_j z))_{j=1,\dots r}$ (depending on the two parameters n and p) and our next result offers another interpretation of it. We denote by $\mathbf{P}_{1,\underline{a},n,p}(z)$, ..., $\mathbf{P}_{r,\underline{a},n,p}(z)$ and $\mathbf{Q}_{\underline{a},n,p}(z)$ the type II Padé approximants of parameter $(n+r\,p-1,p)$ of $(\exp(a_j z))_{j=1,\dots,r}$.

Theorem 2. *The Remainder Padé approximants* (2.1) *coincide with the type II Padé approximants of parameter* $(n+(r-1)\,p-1,p)$ *of* $(\exp(a_j z))_{j=1,\dots r}$. *More precisely, we have the following formulae: for* $j=1,\dots,r$,

$$\begin{aligned}\mathbf{P}_{j,\underline{a},n,p}(z)&=Q_{\underline{a},p}(-1/n,z)\sum_{k=0}^{n-1}\frac{(a_j z)^k}{k!}+\frac{(a_j\ z)^n}{n!}\,P_{j,\underline{a},p}(-1/n,z), \qquad (2.2)\\ \mathbf{Q}_{\underline{a},n,p}(z)&=Q_{\underline{a},p}(-1/n,z).\end{aligned}$$

Remarks. 1) This coincidence is absolutely non obvious because the degree in the variable z of the right hand side of (2.2) seems to be $n+rp-1$ and not $n+(r-1)p-1$ as it turns out to be.

2) The order at $z=0$ of the functions $\mathbf{R}_{j,\underline{a},n,p}(z)=\mathbf{Q}_{\underline{a},n,p}(z)\exp(a_j z)-\mathbf{P}_{j,\underline{a},n,p}(z)$ is thus at least $n+rp$.

3) When $r=1$, we can summarise this result by the equation

$$\sum_{k=0}^{n-1}\frac{z^k}{k!}+\frac{z^n}{n!}\,[\,p-1/p\,]_{\Phi_z}(-1/n)=[\,n-1/p\,]_{\exp}(z). \tag{2.3}$$

3. Two lemmas

We first provide a proof of a result announced in section 2.

Lemma 1. *The function $\Phi_z(t)$, defined for $t \leq 0$, is C^∞ at $t = 0$. Its Taylor series at $t = 0$ is given by*

$$\widehat{\Phi}_z(t) = \sum_{k=0}^{\infty} \varphi_k(-z) t^k$$

where φ_k are Touchard polynomials.

Proof. We observe that

$$\Phi_z(t) = \sum_{k=0}^{\infty} \frac{\Gamma(1-1/t)}{\Gamma(k+1-1/t)}\, z^k = 1 + z \int_0^1 u^{-1/t} e^{z(1-u)}\, \mathrm{d}u,$$

which is proved using the identities

$$e^{z(1-u)} = \sum_{\ell=0}^{\infty} (1-u)^\ell \frac{z^\ell}{\ell!} \quad \text{and} \quad \int_0^1 u^{-1/t}(1-u)^\ell\, \mathrm{d}u = \frac{\Gamma(1-1/t)\Gamma(1+\ell)}{\Gamma(2+\ell-1/t)}.$$

(The interversion of the series and integral is justified by Fubini's theorem.)

By a change of variables $u = \exp(vt)$ (we use here the fact that $t \leq 0$), we thus obtain that

$$\Phi_z(t) = 1 - zt \int_0^\infty e^{-v(1-t)} e^{(-z)(e^{tv}-1)}\, \mathrm{d}v$$

and, from this, it follows by differentiation under the integral sign that $\Phi_z(t)$ is C^∞ at $t = 0$ and that its Taylor series is as given in the statement of the lemma. □

We also need a property of the sequence of Touchard polynomials (see [14]). To simplify the presentation, we define a linear form $\varphi(z)$ (with z assumed fixed) on the space of formal power series $\mathbb{C}[[u]]$ by $\langle \varphi(z), u^n \rangle = \varphi_n(z)$.

Lemma 2. *For all integers $n \geq 0$, we have that $\left\langle \varphi(z), \binom{u}{n} \right\rangle = z^n/n!$.*

Proof. We first observe that, for all integers $n \geq 0$,

$$\varphi_n(z) = e^{-z} \sum_{k=0}^{\infty} \frac{k^n}{k!}\, z^k,$$

which is a consequence of the fact that (see [14])

$$e^{z(e^u-1)} = e^{-z} \sum_{k=0}^{\infty} \frac{z^k}{k!}\, e^{ku} = e^{-z} \sum_{k=0}^{\infty} \frac{z^k}{k!} \left(\sum_{n=0}^{\infty} \frac{(ku)^n}{n!} \right) = e^{-z} \sum_{n=0}^{\infty} \frac{u^n}{n!} \left(\sum_{k=0}^{\infty} \frac{k^n}{k!}\, z^k \right).$$

Let us expand the binomial $\binom{u}{n} = \sum_{i=0}^{n} c_i u^i$, where the c_i's are suitable coefficients. Hence, by definition of the linear form $\varphi(z)$, we have

$$\left\langle \varphi(z), \binom{u}{n} \right\rangle = \sum_{i=0}^{n} c_i \left\langle \varphi(z), u^i \right\rangle = \sum_{i=0}^{n} c_i \varphi_i(z) = e^{-z} \sum_{k=0}^{\infty} \frac{z^k}{k!} \sum_{i=0}^{n} c_i k^i = e^{-z} \sum_{k=0}^{\infty} \frac{z^k}{k!} \binom{k}{n}$$

$$= e^{-z} \sum_{k=n}^{\infty} \frac{z^k}{k!} \binom{k}{n} = \frac{z^n}{n!}\, e^{-z} \sum_{k=n}^{\infty} \frac{z^{k-n}}{(k-n)!} = \frac{z^n}{n!}$$

and the proof is complete. $\square$

4. Proof of Theorem 1

The main property of the exponential polynomials $\varphi_n(z)$ used here is the fact that they are the moments of a discrete measure. More precisely, let the *Poisson distribution* $\mathbf{P}_z$ be the measure on $\mathbb{N}$ defined by $\mathbf{P}_z(u) = e^{-z}z^u/u!$ for $u \in \mathbb{N}$ (z is a parameter in this situation). Then, we have

$$\varphi_n(z) = \int_{\mathbb{N}} u^n \,\mathrm{d}\mathbf{P}_z(u) = e^{-z}\sum_{u=0}^{\infty}\frac{u^n z^u}{u!},$$

which is also equal to $\langle \varphi(z), u^n\rangle$. (This equation is shown during the proof of Lemma 2.) With this interpretation, finding simultaneous Padé approximants for the functions $\widehat{\Phi}_{a_j z}(t)$ is reduced to finding a sequence of polynomials $C_{-\underline{a}z,\underline{p}}(X)$ simultaneously orthogonal for the measures $\mathrm{d}\mathbf{P}_{-a_j z}$ ($j = 1, \dots, r$): see [9, 10, 12] for detailed expositions of this well-known connection based upon the orthogonality property

$$\int_{\mathbb{N}} u^k C_{-\underline{a}z,\underline{p}}(u)\,\mathrm{d}\mathbf{P}_{-a_j z}(u) = 0 \quad \text{for} \quad k = 0, \dots, p_j - 1, \quad j = 1, \dots, r,$$

where $-\underline{a}z = (-a_1 z, \dots, -a_r z)$ and $\underline{p} = (p_1, \dots, p_r)$.

Fortunately, an explicit form for these polynomials already exists in the literature, under the name of *multiple Charlier polynomials* (see [2, 3] but with a different normalization):

$$C_{-\underline{a}z,\underline{p}}(X) = \sum_{k_1}^{p_1}\cdots\sum_{k_r}^{p_r}\left(\prod_{i=1}^{r}\binom{p_i}{k_i}(-a_i\, z)^{p_i-k_i}\right)(-X)_{k_1+\cdots+k_r}, \tag{4.1}$$

(Although this is not apparent here, the construction in [3] uses the fact that $a_1, \dots, a_r$ are all distinct.)

If $p_1 = p_2 = \cdots = p_r = p \in \mathbb{N}$, then $C_{-\underline{a}z,\underline{p}}$ will be denoted by $C_{-\underline{a}z,p}$.

From this, we can obtain an explicit expression for our type II Padé approximants problem (up to a multiplicative constant chosen here equal to $(-1)^{rp}$): we have

$$Q_{\underline{a},p}(t,z) = (-1)^{r\,p}\, t^{r\,p}\, C_{-\underline{a}z,p}(1/t),$$

which is the required formula, and also

$$P_{j,\underline{a},p}(t,z) = (-1)^{r\,p}\, t^{r\,p-1}\, D_{j,-\underline{a}z,p}(1/t) \quad \text{for} \quad j = 1, \dots, r,$$

where $D_{j,-\underline{a}z,p}(X)$ are the associated polynomial of $C_{-\underline{a}z,p}(X)$ given by

$$D_{j,-\underline{a}z,p}(X) = \int_{\mathbb{N}} \frac{C_{-\underline{a}z,p}(X) - C_{-\underline{a}z,p}(u)}{X-u}\,\mathrm{d}\mathbf{P}_{-a_j z}(u) = \left\langle \varphi(-a_j z), \frac{C_{-\underline{a}z,p}(X) - C_{-\underline{a}z,p}(u)}{X-u}\right\rangle.$$

We proceed to get a more explicit formula for $D_{j,-\underline{a}z,p}(X)$ and from now on, we set $K_r = k_1 + \cdots + k_r$. We first observe that

$$(-X)_{K_r} = (-1)^{K_r} K_r!\binom{X}{K_r},$$

hence, by linearity of $\langle\varphi(z),\cdot\rangle$, we have that

$$D_{j,-\underline{a}z,p}(X) = \sum_{k_1,\dots,k_r=0}^{p} \left(\prod_{i=1}^{r} \binom{p}{k_i}(-a_i\, z)^{p-k_i}\right)(-1)^{K_r} K_r! \left\langle \varphi(-a_j z), \frac{\binom{X}{K_r}-\binom{u}{K_r}}{X-u} \right\rangle.$$

We remark that lemma 2 only gives the expression of the modified moments of $\varphi(z)$ on the basis $\binom{u}{n}, n \in \mathbb{N}$. Thus, we now proceed to obtain the expression of the polynomial $\frac{\binom{X}{K_r}-\binom{u}{K_r}}{X-u}$ on the basis $\binom{u}{k}, k = 0,\dots,K_r-1$. Let us define a polynomial P (of degree K_r-1) by

$$P(u) = \frac{\binom{X}{K_r}-\binom{u}{K_r}}{X-u}.$$

Like any polynomial, P can be expressed on the Newton basis $(\nu_0,\nu_1,\dots,\nu_{K_r-1})$ (with $\nu_k = (u-k+1)_k$) using integers as interpolation points:

$$P(u) = \sum_{i=0}^{\deg(\mathrm{P})} \alpha_i\, \nu_i(u).$$

The α's are related to the divided differences of P: indeed, we have $\alpha_i = \Delta^i P(0)/i!$ where the operator Δ is defined recursively by $\Delta^0 P(t) = 1$ and $\Delta^{i+1}P(t) = \Delta^i P(t+1) - \Delta^i P(t)$.

In our situation, we have $P(k) = \frac{\binom{X}{K_r}}{X-k}$ for any integer k such that $0 \le k \le K_r - 1$. Thus $\Delta^j P(0) = (-1)^j \binom{X}{K_r} \Delta^j \left(\frac{1}{X-j}\right)$, where Δ is applied to the variable X. By induction, one proves that $\Delta^j\left(\frac{1}{X-j}\right) = (-1)^j \frac{j!}{(X-j)_{j+1}}$. Finally, we obtain

$$P(u) = \sum_{i=0}^{K_r-1} \frac{1}{i!}\frac{i!}{(X-i)_{i+1}}\binom{X}{K_r}\nu_i(u) = \binom{X}{K_r}\sum_{i=1}^{K_r}\frac{1}{i\binom{X}{i}}\binom{u}{i-1}$$

and

$$\begin{aligned}
&D_{j,-\underline{a}z,p}(X)\\
&= \sum_{k_1,\dots,k_r=0}^{p}\left(\prod_{i=1}^{r}\binom{p}{k_i}(-a_i\, z)^{p-k_i}\right)(-1)^{K_r}K_r!\binom{X}{K_r}\sum_{i=1}^{K_r}\frac{1}{i\binom{X}{i}}\left\langle\varphi(-a_j z),\binom{u}{i-1}\right\rangle\\
&= \sum_{k_1,\dots,k_r=0}^{p}(-X)_{K_r}\left(\prod_{i=1}^{r}\binom{p}{k_i}(-a_i z)^{p-k_i}\right)\sum_{i=1}^{K_r}\frac{(-a_j z)^{i-1}}{i!\binom{X}{i}}\\
&= \sum_{k_1,\dots,k_r=0}^{p}(-X)_{K_r}\left(\prod_{i=1}^{r}\binom{p}{k_i}(-a_i z)^{p-k_i}\right)\sum_{i=1}^{K_r}\left(-\frac{(a_j z)^{i-1}}{(-X)_i}\right),
\end{aligned}$$

thanks to Lemma 2. With $X = 1/t$, we obtain the formula for $P_{j,\underline{a},p}(t,z)$ stated in theorem 1.

Remark. If needed, one can also give a precise estimation of the remainder term $E_{j,\underline{a},p}(t,z) = \Phi_{a_j z}(t) - P_{j,\underline{a},p}(t,z)/Q_{\underline{a},p}(t,z)$, $j = 1,\dots,r$.

For example, when $r = 1$ and $a_1 = 1$, the remainder behaves like $g_p(t,z)p!/(-1/t)_p^2$ for $t < 0$ (at least), with $g_p(t,z)$ of geometric growth of p at most. Without going into details (see [10, 12] for similar but complete computations), let us just mention that the proof uses the fact that

$$\Phi_z(t) - \frac{P_p(t,z)}{Q_p(t,z)} = \frac{1}{C_{z,p}(1/t)^2}\int_{\mathbb{N}} \frac{C_{z,p}(u)^2}{1-zu}\,\mathrm{d}\mathbf{P}_z u = \frac{1}{C_{z,p}(1/t)^2}\left\langle \varphi(z), \frac{C_{z,p}(u)^2}{1-zu}\right\rangle$$

and that

$$\big\langle \varphi(z), C_{z,p}(u)^2\big\rangle = \big\langle \varphi(z), u^p C_{z,p}(u)\big\rangle = (-z)^n\, n!.$$

Although this implies that the Padé approximants in question tend to $\Phi_z(t)$ and thus "sum" the divergent power series $\widehat{\Phi}_z(t)$, this does not seem to be enough to deduce any diophantine result for the values of $\Phi_z(t)$, because $Q_p(t,z)$ grows like $t^p(-1/t)_p$.

5. Proof of Theorem 2

First, we provide another expression for polynomials occuring in the simultaneous Remainder-Padé approximants in (2.1) : the denominator is

$$Q_{\underline{a},p}(-1/n,z) = n^{-rp}\sum_{k_1,\dots,k_r=0}^{p}\left(\prod_{i=1}^{r}\binom{p}{k_i}(-a_i\, z)^{p-k_i}\right)(n)_{K_r}$$

and the associated polynomial on the numerator can be trivially expressed as

$$P_{j,\underline{a},p}(-1/n,z) = \frac{(n-1)!}{(a_j z)^n} n^{-r\,p+1}\sum_{k_1,\dots,k_r=0}^{p}\left(\prod_{i=1}^{r}\binom{p}{k_i}(-a_i\, z)^{p-k_i}\right)(n)_{K_r}\big(S_{n+K_r-1}(a_j z) - S_{n-1}(a_j z)\big)$$

where S_n is the partial sum of the exponential series $S_n(t) = \sum_{k=0}^{n} t^k/k!$.

Hence, after some simplifications, the Remainder Padé approximant is

$$\sum_{k=0}^{n-1}\frac{(a_j z)^k}{k!} + \frac{a_j^n z^n}{n!}\frac{P_{j,\underline{a},p}(-1/n,z)}{Q_{\underline{a},p}(-1/n,z)} = \frac{\displaystyle\sum_{k_1,\dots,k_r=0}^{p}\left(\prod_{i=1}^{r}\binom{p}{k_i}(-a_i\, z)^{p-k_i}\right)(n)_{K_r}S_{n+K_r-1}(a_j z)}{\displaystyle\sum_{k_1,\dots,k_r=0}^{p}\left(\prod_{i=1}^{r}\binom{p}{k_i}(-a_i\, z)^{p-k_i}\right)(n)_{K_r}}. \tag{5.1}$$

At first glance, this rational fraction has degree $(n+rp-1/rp)$ in z but this is not optimal: we prove below that its degree is $(n+(r-1)p-1/rp)$.

Indeed, with $K'_{j,r} = k_1 + \cdots + k_{j-1} + k_{j+1} + \cdots + k_r$ and $\underline{k}'_{j,r} = (k_1, \ldots, k_{j-1}, k_{j+1}, \ldots, k_r)$, the numerator of (5.1) can be expressed as

$$\sum_{k_1,\ldots,k_{j-1},k_{j+1},\ldots,k_r=0}^{p} \left(\prod_{i=1, i\neq j}^{r} \binom{p}{k_i} (-a_i\ z)^{p-k_i} \right) (n)_{K'_{j,r}} A_{j,\underline{k}'_{j,r}}(z)$$

with

$$A_{j,\underline{k}'_{j,r}}(z) = \sum_{k_j=0}^{p} \binom{p}{k_j} (-a_j\ z)^{p-k_j} (n + K'_{j,r})_{k_j} S_{n+K'_{j,r}-1}(a_j z). \tag{5.2}$$

We claim that the degree in z of the polynomial $A_{j,\underline{k}'_{j,r}}(z)$ is at most $n + K'_{j,r} - 1$. ([1]) To prove this observation, which is non trivial, we first observe that, by Leibniz's rule, we have the identity

$$\frac{\mathrm{d}^p}{\mathrm{d}t^p} \left(t^{-q} S_m(zt) \right)$$
$$= \sum_{k=0}^{p} \binom{p}{k} \frac{\mathrm{d}^k}{\mathrm{d}t^k}(t^{-q}) \frac{\mathrm{d}^{p-k}}{\mathrm{d}t^{p-k}} \left(S_m(zt) \right) = \sum_{k=0}^{p} \binom{p}{k} \frac{(-1)^k (q)_k}{t^{k+q}} z^{p-k} S_{m-p+k}(zt).$$

By comparison with (5.2), we thus obtain the expression

$$A_{j,\underline{k}'_{j,r}}(z) = (-1)^p a_j^{n+p+K'_{j,r}} \frac{\mathrm{d}^p}{\mathrm{d}a_j^p} \left(a_j^{-n-K'_{j,r}} S_{n+K'_{j,r}+p-1}(a_j z) \right). \tag{5.3}$$

A simple computation based on (5.3) then proves that the degree in z of $A_{j,\underline{k}'_{j,r}}(z)$ is at most $n + K'_{j,r} - 1$. It follows that the degree in z of the polynomial in (5) is at most $n + (r-1)p - 1$.

We are now in position to prove that the rational fraction (5.1) is exactly the type II Padé approximant of parameter $(\,n + (r-1)p - 1,\ p\,)$ of $(\exp(a_j z))_{j=1,\ldots,r}$. The expression of this II Padé approximants is well-known and goes back to Hermite [7], who used them for proving the transcendency of e. For example in [9], one finds the following integral expressions of Hermite:

$$\begin{aligned} \mathbf{Q}_{\underline{a},n,p}(z) &= \frac{z^{n+rp}}{(n+rp-1)!} \int_0^\infty T(x)\, e^{-zx}\, \mathrm{d}x \\ \mathbf{P}_{j,\underline{a},n,p}(z) &= \frac{e^{a_j z} z^{n+rp}}{(n+rp-1)!} \int_{a_j}^\infty T(x)\, e^{-zx}\, \mathrm{d}x, \quad j = 1, \ldots, r \\ \mathbf{R}_{j,\underline{a},n,p}(z) &= \frac{e^{a_j z} z^{n+rp}}{(n+rp-1)!} \int_0^{a_j} T(x)\, e^{-zx}\, \mathrm{d}x, \quad j = 1, \ldots, r \end{aligned} \tag{5.4}$$

where $T(x) = x^{n-1} \prod_{j=1}^{r} (x - a_j)^p$. The polynomials $\mathbf{Q}_{\underline{a},n,p}(z)$ and $\mathbf{P}_{j,\underline{a},n,p}(z)$ are of degree respectively less than rp and $n + (r-1)p - 1$; the factor $(n + rp - 1)!$ ensures that

[1]This is a very special property: if in (5.2), one replaces $S_{n+K'_{j,r}+p-1}$ by any polynomial of degree $n + K'_{j,r} + p - 1$, then the degree of the corresponding $A_{j,\underline{k}'_{j,r}}(z)$ is usually $n + K'_{j,r} + p - 1$

$\mathbf{Q}_{\underline{a},n,p}(1) = 1$. Finally, we clearly have

$$\mathbf{R}_{j,\underline{a},n,p}(z) = \mathbf{Q}_{\underline{a},n,p}(z)e^{a_j z} - \mathbf{P}_{j,\underline{a},n,p}(z) = \mathcal{O}\left(z^{n+rp}\right), \quad j = 1, \ldots, r.$$

It remains to prove that the quotients $\dfrac{\mathbf{P}_{j,\underline{a},n,p}(z)}{\mathbf{Q}_{\underline{a},n,p}(z)}$, $j = 1, \ldots, r$, are equal to the Remainder Padé approximant given by (5.1). But from the expansion

$$T(x) = x^{n-1} \sum_{k_1,\ldots,k_r=0}^{p} \left(\prod_{i=1}^{r} \binom{p}{k_i} (-a_i)^{p-k_i} \right) x^{K_r}$$

and the formula

$$S_n(az) = \sum_{k=0}^{n} \frac{(az)^k}{k!} = \frac{z^{n+1}e^{az}}{n!} \int_a^\infty x^n \, e^{-zx} \, \mathrm{d}x,$$

we readily find that

$$\begin{aligned} \mathbf{Q}_{\underline{a},n,p}(z) &= \frac{1}{(n)_{rp}} \sum_{k_1,\ldots,k_r=0}^{p} \left(\prod_{i=1}^{r} \binom{p}{k_i} (-a_i \; z)^{p-k_i} \right) (n)_{K_r} \qquad (5.5) \\ \mathbf{P}_{j,\underline{a},n,p}(z) &= \frac{1}{(n)_{rp}} \sum_{k_1,\ldots,k_r=0}^{p} \left(\prod_{i=1}^{r} \binom{p}{k_i} (-a_i \; z)^{p-k_i} \right) (n)_{K_r} S_{n+K_r-1}(a_j z). \end{aligned}$$

These expressions are found to match exactly with their respective counter-parts in the right hand side of (5.1) for $j = 1, \ldots, r$. The proof of theorem 2 is therefore complete.

We conclude this section with a marginal but interesting observation. The proofs of the previous theorems use the link between multiple Charlier polynomial (4.1) for $p_1 = p_2 = \cdots = p_r$ and the denominator (5.4) of the simultaneous Padé approximations of exponential function:

$$\begin{aligned} C_{-\underline{a}z,p}(-n) &= (n)_{rp} \;\; \mathbf{Q}_{\underline{a},n,p}(z) \\ &= \frac{z^{n+rp}}{(n-1)!} \int_0^\infty T(x) \, e^{-zx} \, \mathrm{d}x \\ &= \frac{1}{(n-1)!} \int_0^\infty s^{n-1} \prod_{j=1}^{r} (s - a_j \; z)^p \;\; e^{-s} \, \mathrm{d}s \end{aligned}$$

The formulas in (5.4) and (5.5) can be re-written in the general case $\underline{p} = (p_1, \ldots, p_r)$, providing the following corollary which complements previous formulas in [3].

Corollary 1. *The multiple Charlier polynomials*

$$C_{\underline{c},\,\underline{p}}(X) = \sum_{k_1=0}^{p_1} \cdots \sum_{k_r=0}^{p_r} \left(\prod_{i=1}^{r} \binom{p_i}{k_i} (c_i)^{p_i-k_i} \right) (-X)_{k_1+\cdots+k_r}$$

satisfy

$$C_{\underline{c}\,,\underline{p}}(X) = \frac{1}{\Gamma(-X)} \int_0^\infty s^{-X-1} \prod_{j=1}^{r} (s+c_j)^{p_j}\, e^{-s} \mathrm{d}s, \tag{5.6}$$

where $\underline{c} = (c_1, \ldots, c_r)$ *and* $\underline{p} = (p_1, \ldots, p_r) \in \mathbb{N}^r$.

Proof. We expand the integrand in (5.6) as:

$$\prod_{j=1}^{r} (s+c_j)^{p_j} = \sum_{k_1=0}^{p_1} \cdots \sum_{k_r=0}^{p_r} \prod_{j=1}^{r} \binom{p_j}{k_j} s^{k_j}\, c_j^{p_j-k_j}$$

form which we deduce another expression of the integral

$$\begin{aligned}\int_0^\infty s^{-X-1} \prod_{j=1}^{r} (s+c_j)^{p_j} e^{-s} \mathrm{d}s &= \sum_{k_1=0}^{p_1} \cdots \sum_{k_r=0}^{p_r} \int_0^\infty \left(\prod_{j=1}^{r} \binom{p_j}{k_j} c_j^{p_j-k_j} \right) s^{-X-1+K_r} e^{-s} \mathrm{d}s \\ &= \sum_{k_1=0}^{p_1} \cdots \sum_{k_r=0}^{p_r} \prod_{j=1}^{r} \binom{p_j}{k_j} c_j^{p_j-k_j} \Gamma(K_r - X),\end{aligned}$$

where $K_r = k_1 + \cdots + k_r$. The simplification $\frac{\Gamma(K_r-X)}{\Gamma(-X)} = (-X)_{K_r}$ completes the proof. □

6. The case of logarithms and some questions

Two natural questions one may ask about the Remainder Padé phenomenon are "Why does it occur?" and "Does it occur often?". Concerning the first question, we must admit that the coincidence between two different Padé approximants construction is a mystery, for which we can't offer an explanation but rather a verification. Formally, the problem is the following: assume that we are given a power series $F(z) = \sum_{k=0}^{\infty} f_k\, z^k$ such that there exist a function $G_z(t)$, C^∞ at 0 with a (possibly formal) Taylor series $\widehat{G}_z(t) = \sum_{k=0}^{\infty} g_k(z) t^k$, and a suitable normalizing sequence $a_n(z)$ such that

$$F(z) = \sum_{k=0}^{n-1} f_k\, z^k + a_n(z)\, G_z(1/n).$$

(For the exponential function, we have $a_n(z) = f_n z^n$.)

Under what conditions on F does there exist a link between the Padé approximants to $F(z)$ and those to $\widehat{G}_z(t)$? Concerning the second question, we are aware of only one other example (except those mentioned in the introduction, which are more related to numbers than to power series). Indeed, in [10], the first author also discovered a Remainder Padé phenomenon for the function $L(z) = -\log(1-z)$: he wrote the remainder as

$$L(z) = \sum_{k=1}^{n-1} \frac{z^k}{k} + \frac{z^n}{n}\, \vartheta_z(1/n)$$

and noted that the function $\vartheta_z(t)$ admits a Taylor expansion (with zero radius of convergence) $\widehat{\vartheta}_z(t) = \sum_{k=0}^{\infty} \frac{R_k(-1/z)}{1-z}(-t)^k$ where the R's are the eulerian numbers defined by

$$\frac{1+z}{\exp(x)+z} = \sum_{k=0}^{\infty} R_k(z)\frac{x^k}{k!}.$$

A classical family of orthogonal polynomials of Carlitz [5] enabled him to prove the following result, reminiscent of (2.3): for $n \geq p$,

$$\sum_{k=1}^{n-1} \frac{z^k}{k} + \frac{z^n}{n}\,[p-1/p]_{\widehat{\vartheta}_z}(1/n) = [n-1/p]_L(z). \tag{6.1}$$

In the following, we display a sketch of the proof of (6.1), slightly modified to be in harmony with the similar results proved in Theorem 2 for Padé Hermite approximants of exponential functions.

Let Θ be defined as the linear functional acting on the space of polynomials by $\left\langle \Theta, x^k \right\rangle = R_k(-1/z)/(1-z)$, where z is considered as a parameter. The moments of Θ can be seen as a sum of Dirac distribution on $\mathbb{N}$ (see [5]). Hence, the orthogonal polynomial with respect to Θ are linked with Meixner I polynomials and given by

$$t^p \sum_{k=0}^{p} z^{-k}(z-1)^k \binom{p}{k}\binom{-1/t}{k}.$$

Then, the Remainder Padé approximant of L (i.e., the left hand side of (6.1)) is

$$\sum_{k=1}^{n-1} \frac{z^k}{k} - \frac{\displaystyle\sum_{k=0}^{p} z^{n-k}(z-1)^k \binom{p}{k}\binom{-n}{k} \sum_{i=1}^{k} \frac{1}{i\binom{-n}{i}} \frac{z^{i-1}}{(1-z)^i}}{\displaystyle\sum_{k=0}^{p} z^{-k}(z-1)^k \binom{p}{k}\binom{-n}{k}}. \tag{6.2}$$

The proof is based on a simple expression for the modified moments

$$\left\langle \Theta, \binom{-x}{k} \right\rangle = \frac{z^k}{(1-z)^{k+1}},$$

whose proof is similar to the one of lemma 2. The denominator of the Remainder Padé approximant is related to Legendre polynomial

$$\sum_{k=0}^{p} z^{-k}(z-1)^k \binom{p}{k}\binom{-n}{k} = P_p^{(n)}\left(\frac{2}{z}-1\right)$$

where $P_p^{(n)}$ is the Legendre polynomial of degree p, orthogonal with respect to the weight x^n on $[0,1]$. Another expression of the numerator of the RPA

$$-\sum_{k=0}^{p} z^{-k}(z-1)^k \binom{p}{k}\binom{-n}{k}\left(\sum_{k=1}^{n-1}\frac{z^k}{k}-\sum_{i=1}^{k}\frac{1}{i\binom{-n}{i}}\frac{z^{n+i-1}}{(1-z)^i}\right),$$

implies that the difference between $L(z)$ and the expression in (6.2) can be written as

$$\begin{aligned}
-\frac{\displaystyle\sum_{k=0}^{p} z^{-k}(z-1)^k \binom{p}{k}\binom{-n}{k}\left(L(z)-\int_0^z \frac{t^{n+k-1}}{(1-t)^{k+1}\binom{-n}{k}}\,\mathrm{d}t\right)}{P_p^{(n)}(2/z-1)}
&= L(z)-\frac{1}{P_p^{(n)}(2/z-1)}\int_0^z \frac{(z-t)^p}{(1-t)^{p+1}}t^{n-1}z^{-p}\,\mathrm{d}t\\
&= L(z)-\frac{1}{z^pP_p^{(n)}(2/z-1)}\int_0^1 \frac{(1-u)^p}{(1-z\,u)^{p+1}}u^{n-1}z^{-n+p}\,\mathrm{d}u.
\end{aligned}$$

We remark that this is exactly the error for the Padé approximant $[n-1/p]$ of L, which was the desired result.

Surprisingly, this example does not seem to extend to the family $\big(-\log(1-a_jz)\big)_{j=1,\dots,r}$ for which certain type II Padé approximants are known: see [11]. Indeed, one has

$$-\ln(1-a_jz)=\sum_{k=1}^{n-1}\frac{(a_jz)^k}{k}+\frac{(a_jz)^n}{n}\,\vartheta_{a_jz}(1/n)$$

with the above notation. For $n \geq rp$, we obtain a simultaneous Remainder Padé approximants for $-\log(1-a_jz)$, $j=1,\dots,r$, of the form

$$\sum_{k=1}^{n-1}\frac{(a_jz)^k}{k}-(a_jz)^n\frac{\displaystyle\sum_{k_1,\cdots,k_r=0}^{p}\left(\prod_{j=1}^{r}(1-1/a_jz)^{k_j}\binom{p}{k_j}\right)\binom{-n}{K_r}\sum_{i=1}^{K_r}\frac{1}{i\binom{-n}{i}}\frac{(a_jz)^{i-1}}{(1-a_jz)^i}}{\displaystyle\sum_{k_1,\cdots,k_r=0}^{p}\left(\prod_{j=1}^{r}(1-1/a_jz)^{k_j}\binom{p}{k_j}\right)\binom{-n}{K_r}} \tag{6.3}$$

with $K_r=\sum_{j=1}^{r}k_j$. The proof follows the same lines as for Theorem 1, using the fact that the numbers $\vartheta(a_jz)$ are the moments of the distribution on $\mathbb{N}$ defined by $(a_jz)^\ell$ for $\ell\in\mathbf{N}$.

The simultaneous Remainder Padé approximants for the L's are naturally related to the multiple Meixner I polynomials, also introduced in [2] but, when $r\geq 2$, we don't recover any type II Padé approximants for $\big(\log(1-a_jz)\big)_{j=1,\dots,r}$: indeed, the remainder terms provided by the approximations in (6.3) don't satisfy the property of matching the terms of the initial series as far as the type II Padé approximants do. The case $r=1$ is thus an exception. In any case, it would be very interesting to find others examples in order to understand better the Remainder Padé phenomenon.

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