Addendum to "*Rational approximation to values of G-functions, and their expansions in integer bases*"

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In [1], we proved the following result.

Theorem 1. *Let F be a G-function with rational Taylor coefficients and with $F(z) \notin \mathbb{Q}(z)$, and $t \geq 0$. Then there exist some positive effectively computable constants c_1, c_2, c_3, c_4, depending only on F (and t as well for c_3), such that the following property holds. Let $a \neq 0$ and $b, B \geq 1$ be integers such that*

$$b > (c_1|a|)^{c_2} \text{ and } B \leq b^t. \tag{0.1}$$

Then for any $n \in \mathbb{Z}$ and any $m \geq c_3 \frac{\log(b)}{\log(|a|+1)}$ we have

$$\Big|F\Big(\frac{a}{b}\Big) - \frac{n}{B \cdot b^m}\Big| \geq \frac{1}{B \cdot b^m \cdot (|a|+1)^{c_4 m}}. \tag{0.2}$$

From Theorem 1, we deduced

Corollary 1. *Let F be a G-function with rational Taylor coefficients and with $F(z) \notin \mathbb{Q}(z)$, $\varepsilon > 0$, $t \geq 0$ and $a \in \mathbb{Z}$, $a \neq 0$. Let b and m be positive integers, sufficiently large in terms of F, ε, a (and t for m). Then for any integers n and B with $1 \leq B \leq b^t$, we have*

$$\Big|F\Big(\frac{a}{b}\Big) - \frac{n}{B \cdot b^m}\Big| \geq \frac{1}{b^{m(1+\varepsilon)}}.$$

When $F(\frac{a}{b})$ is an algebraic irrational, this looks like Ridout's Theorem for algebraic irrational numbers, but this is not really the same. First, if $F(\frac{a}{b})$ is an algebraic irrational and b is fixed, then Corollary 1 applies only if ε is not too small with respect to b, and thus we do not get an effective version of Ridout's theorem "in base b" for this number. Second, we don't know if any algebraic irrational number can be represented as a value $F(\frac{a}{b})$ to which these results apply.

In this note, we deduce from Theorem 1 the following result, which partially solves these problems.

Theorem 2. *Let d be a positive rational number such that $\sqrt{d} \notin \mathbb{Q}$. There exist some constants $\eta_d > 0, \kappa_d > 0$ and N_d such that for any convergent $\frac{\alpha}{\beta}$ of the continued fraction expansion of $\sqrt{d}$ with $\alpha, \beta \geq N_d$, we have*

$$\left|\sqrt{d} - \frac{n}{\alpha^m}\right| \geq \frac{1}{(\eta_d \alpha)^m} \quad \text{and} \quad \left|\sqrt{d} - \frac{n}{\beta^m}\right| \geq \frac{1}{(\kappa_d \beta)^m}$$

for any integer $n \in \mathbb{Z}$ and any m large enough with respect to d, α, β.

In particular, for any $\varepsilon > 0$ we have

$$\left|\sqrt{d} - \frac{n}{\alpha^m}\right| \geq \frac{1}{\alpha^{m(1+\varepsilon)}} \quad \text{and} \quad \left|\sqrt{d} - \frac{n}{\beta^m}\right| \geq \frac{1}{\beta^{m(1+\varepsilon)}}$$

provided α and β are large enough (in terms of d and ε).

Proof. Let α, β be any positive integers such that $|\alpha^2 - d\beta^2| \leq c(d)$ for some given constant $c(d)$. Note that if α/β is a convergent to $\sqrt{d}$, then

$$|\alpha^2 - d\beta^2| \leq \frac{\alpha + \sqrt{d}\beta}{\beta} \leq 2\sqrt{d} + 1$$

so that $c(d) = 2\sqrt{d} + 1$ is an admissible value for all convergents.

Let $f(x) = \sqrt{1-x}$. Then $f(\frac{\alpha^2 - d\beta^2}{\alpha^2}) = \frac{\beta}{\alpha}\sqrt{d}$. Let $d = \frac{u}{v}$ with positive integers u and v. We can apply Theorem 1 to $F = f$, $a = v\alpha^2 - u\beta^2$ and $b = v\alpha^2$, provided that $\alpha^2 > c_1^{c_2}|\alpha^2 - d\beta^2|^{c_2}$ where c_1, c_2 depend only on d. This inequality holds a fortiori if we assume that $\alpha \geq (c_1 c(d))^{c_2/2} =: N_d$, which we now do. Then

$$\left|\frac{\beta}{\alpha}\sqrt{d} - \frac{n}{B \cdot (v\alpha^2)^m}\right| = \left|f\left(\frac{\alpha^2 - d\beta^2}{\alpha^2}\right) - \frac{n}{B \cdot (v\alpha^2)^m}\right| \geq \frac{1}{B \cdot (1 + v|\alpha^2 - d\beta^2|)^{c_4 m} \cdot (v\alpha^2)^m}$$

for any $1 \leq B \leq v\alpha^{2t}$, any $n \in \mathbb{Z}$ and any $m \geq c_3 \frac{\log(v\alpha^2)}{\log(1+v|\alpha^2 - d\beta^2|)}$.

Thus

$$\left|\sqrt{d} - \frac{\alpha n}{\beta B \cdot v^m \alpha^{2m}}\right| \geq \frac{\alpha}{\beta \cdot B \cdot (1 + vc(d))^{c_4 m} \cdot (v\alpha^2)^m}.$$

Note that c_3 depends on f and t. We now choose $t = 2$, so that c_3 becomes absolute. With $B = \alpha$ and $n = \beta v^m n'$ (for any $n' \in \mathbb{Z}$), we get

$$\left|\sqrt{d} - \frac{n'}{\alpha^{2m}}\right| \geq \frac{1}{\beta \cdot (1 + vc(d))^{c_4 m} \cdot v^m \cdot \alpha^{2m}}.$$

On the other hand, with $B = \alpha^2$ and $n = \beta v^m n'$ (for any $n' \in \mathbb{Z}$), we obtain

$$\left|\sqrt{d} - \frac{n'}{\alpha^{2m+1}}\right| \geq \frac{1}{\beta \cdot (1 + vc(d))^{c_4 m} \cdot v^m \cdot \alpha^{2m+1}}.$$

Moreover, assuming $m \geq C(d, \alpha, \beta)$ we have

$$\beta \cdot (1 + vc(d))^{c_4 m} v^m \leq \delta^m$$

for some constant δ that depends only on d. Therefore combining the previous inequalities yields

$$\left| \sqrt{d} - \frac{n'}{\alpha^m} \right| \geq \frac{1}{(\eta_d \alpha)^m}$$

for any $n' \in \mathbb{Z}$ and any $m \geq C(d, \alpha, \beta)$, where $\eta_d > 0$ depends only on d.

We now prove the other inequality

$$\left| \sqrt{d} - \frac{n}{\beta^m} \right| \geq \frac{1}{(\kappa_d \beta)^m}.$$

Any convergent of $1/\sqrt{d}$ (except maybe the first ones) is of the form β/α where α/β is a convergent of $\sqrt{d}$. Therefore we may apply the above result with $1/d$ and β/α: we obtain

$$\left| \frac{1}{\sqrt{d}} - \frac{n'}{\beta^m} \right| \geq \frac{1}{(\eta_d \beta)^m}.$$

Since the map $x \mapsto 1/x$ is Lipschitz around $\sqrt{d}$, we deduce the lower bound of Theorem 2 by choosing an appropriate constant κ_d.

□

References

[1] S. Fischler, T. Rivoal, *Rational approximation to values of G-functions, and their expansions in integer bases*, preprint (2015), 18 pages, Manuscripta Mathematica, to appear.

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