LAGRANGIAN INTERPOLATION AND THE RIEMANN ZETA FUNCTION

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Interpolation series theory (i.e., expansion of entire functions in series of polynomials where the roots of the polynomials belong to a fixed set of $\mathbb{C}$) played an important rôle in diophantine approximation at the beginning of the twentieth century. In particular, it was used by Pólya [6] when he proved that the function 2^z is the entire function of smallest growth which sends $\mathbb{N}$ in $\mathbb{Z}$. The transcendence of e^α for any algebraic number $\alpha \neq 0$ was also obtained by Siegel [8] by expanding $\exp(z)$ at suitable interpolation points.

Interpolation methods were crucial in Gel'fond's proof the transcendence of e^π (see [3]): this was a first step towards the proof of Hilbert's 7th problem that α^β is transcendental when α, β are algebraic numbers, with $\alpha \neq 0, 1$ and β irrational. He first expanded the entire function $\exp(\pi z)$ in interpolation series at interpolation points $(\alpha_n)_{n\geq 0}$ given by the gaußian integers ordered by increasing modulus and argument, without multiplicity: we have $e^{\pi z} = \sum_{n=0}^\infty A_n \, z(z-\alpha_1)\cdots(z-\alpha_{n-1})$ for all $z \in \mathbb{C}$, where the coefficients A_n are given by a certain complex integral. By the residue theorem, we obtain

$$A_n = \sum_{k=0}^{n} \frac{e^{\pi\alpha_k}}{\prod_{\substack{0\leq j\leq n\\ j\neq k}}(\alpha_k - \alpha_j)} = \sum_{k=0}^{n} \frac{e^{\pi\alpha_k}}{\omega_{n,k}} = P_n(e^\pi),$$

where $P_n(X) \in \mathbb{Q}(i)[X, 1/X]$ is of degree $\sqrt{n/\pi} + o(\sqrt{n})$ in X and $1/X$. Gel'fond then proved the following results:

1) The number $P_n(e^\pi)$ is non zero for infinitely many n because $\exp(\pi z)$ is not a polynomial.

2) There exists $\Omega_n \in \mathbb{Q}(i)$ such that $\Omega_n P_n(e^\pi) \in \mathbb{Z}[i][e^\pi, e^{-\pi}]$ and the height H_n of the Laurent polynomial $\Omega_n P_n(X)$ is bounded by $e^{\mathcal{O}(n)}$.

3) We have $\Omega_n P_n(e^\pi) \ll \exp(-n\log(n) + \mathcal{O}(n))$.

The conclusion follows by standard arguments. Despite some works by Boehle [2], Kuzmin [4] and Siegel [8] for example, interpolation methods were replaced by more powerful (and less explicit) methods based on auxiliary functions contructed thanks to Siegel's lemma.

The aim of my talk during the Oberwolfach meeting was to report on my recent work [7], in which I show how another kind of interpolation process can be used in irrationality theory. More precisely, I show that the irrationality of $\log(2), \zeta(2)$ and $\zeta(3)$ (Apéry's theorem [1]) can be obtained by expanding the Hurwitz zeta function $\zeta(s,z) = \sum_{k=1}^\infty 1/(k+z)^s$ or related functions in interpolation series of rational functions (not only polynomials). Such an interpolation process was first studied by René Lagrange [5] in 1935 when the degree of the numerators and denominators of the rational summands are essentially equal. For example, using certain of his formulae, I proved the following:

Theorem 1 (Rivoal, 2006). *For all $z \in \mathbb{C} \setminus \{-1, -2, \ldots\}$, we have that*

$$\zeta(2,z) = \sum_{n=0}^{\infty} A_{2n} \frac{(z-n+1)_n^2}{(z+1)_n^2} + \sum_{n=0}^{\infty} A_{2n+1} \frac{(z-n+1)_n^2}{(z+1)_n^2} \frac{z-n}{z+n+1},$$

where $A_0 = \zeta(2)$ and, for all $n \geq 0$,

$$A_{2n+1} = \frac{2n+1}{2\pi i} \int_{\mathscr{C}_n} \frac{(x+1)_n^2}{(x-n)_{n+1}^2} \zeta(2,x)\, \mathrm{d}x \in \mathbb{Q}\zeta(3) + \mathbb{Q}$$

and

$$A_{2n+2} = \frac{2n+2}{2\pi i} \int_{\mathscr{C}_n} \frac{(x+1)_n^2}{(x-n)_{n+1}^2} \frac{x+n+1}{x-n-1} \zeta(2,x)\, \mathrm{d}x \in \mathbb{Q}\zeta(3) + \mathbb{Q}.$$

The curve $\mathscr{C}_n$ encloses the points $0, 1, \ldots, n$ but none of the poles of $\zeta(2,z)$.

(By definition, $(u)_0 = 1$ and $(u)_m = u(u+1)\cdots(u+m-1)$ for $m \geq 1$.) The irrationality of $\zeta(3)$ is a corollary of this theorem. Indeed, by the residue theorem, it is easy to compute explicitely the coefficient A_n and to deduce that

$$d_n^3 A_n = u_n \zeta(3) - v_n \in \mathbb{Z}\zeta(3) + \mathbb{Z}$$

where $d_n = \mathrm{lcm}(1, 2, \ldots, n)$. Furthermore, from the integral representation of A_n, we obtain that

$$\limsup_{n \to +\infty} (d_n^3 A_n)^{1/n} \leq e^3(\sqrt{2}-1)^4 < 1.$$

Since $\zeta(2,z)$ is not a rational function of z, we necessarily have $A_n \neq 0$ for infinitely many n and the irrationality of $\zeta(3)$ is proved.

Similarly, the irrationality of $\log(2)$ can be deduced from the following result. Let

$$\widetilde{\zeta}(1,z) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+z}.$$

Theorem 2 (Rivoal, 2006). *For all $z \in \mathbb{C} \setminus \{-1, -2, \ldots\}$, we have*

$$\widetilde{\zeta}(1,z) = \sum_{n=1}^{\infty} A_n \frac{(z-n+2)_{n-1}}{(z+1)_n} \tag{1}$$

where, for all $n \geq 0$,

$$A_{n+1} = \frac{2n+1}{2\pi i} \int_{\mathscr{C}_n} \frac{(x+1)_n}{(x-n)_{n+1}} \widetilde{\zeta}(1,x)\, \mathrm{d}x \in \mathbb{Q}\log(2) + \mathbb{Q}.$$

The curve $\mathscr{C}_n$ encloses the points $0, 1, \ldots, n$ but none of the poles of $\widetilde{\zeta}(1,z)$.

I don't know if it is possible to obtain the irrationality of $\zeta(2)$ by means of R. Lagrange's interpolation. Instead, I found new interpolation formulae which enabled me to use rational functions with unequal degrees for the numerators and denominators. The irrationality of $\zeta(2)$ is then a consequence of the following theorem. By a slight abuse of notations, let

$$\zeta(1,z)=\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+z}\right).$$

Theorem 3 (Rivoal, 2006)**.** *For all $z\in\mathbb{C}\setminus\{-1,-2,\ldots\}$, we have*

$$\zeta(1,z)=\sum_{n=0}^{\infty}A_n\,\frac{(z-n+1)_n^2}{(z+1)_n}+\sum_{n=0}^{\infty}B_n\,\frac{(z-n+1)_n^2}{(z+1)_n}\frac{z-n}{z+n+1}$$

where $A_0=B_0=0$ and, for all $n\geq 1$,

$$A_n=\frac{1}{2\pi i}\int_{\mathscr{C}_n}\frac{(x+1)_n(x-n)}{(x-n)_{n+1}^2}\,\zeta(1,x)\,\mathrm{d}x\in\mathbb{Q}\zeta(2)+\mathbb{Q}$$

and

$$B_n=\frac{2n}{2\pi i}\int_{\mathscr{C}_n}\frac{(x+1)_n}{(x-n)_{n+1}^2}\,\zeta(1,x)\,\mathrm{d}x\in\mathbb{Q}\zeta(2)+\mathbb{Q}.$$

The curve $\mathscr{C}_n$ encloses the points $0,1,\ldots,n$ but none of the poles of $\zeta(1,z)$.

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