SOME REMARKS ON CLASSICAL MODULAR FORMS AND HYPERGEOMETRIC SERIES

F. PELLARIN (ST-ÉTIENNE), T. RIVOAL (GRENOBLE), J.-A. WEIL (LIMOGES)

1. Diophantine results for modular forms

Let $z, q \in \mathbb{C}$ be such that $\mathrm{Im}(z) > 0$ and $|q| < 1$.

We consider Ramanujan's q-series

$$P(q) = 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}, \quad Q(q) = 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n}, \quad R(q) = 1 - 504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n}$$

and the function

$$J(q) = 1728\frac{Q(q)^3}{Q(q)^3 - R(q)^2}.$$

We also define the Fourier series $E_2(z) = P(e^{2i\pi z})$, $E_4(z) = Q(e^{2i\pi z})$, $E_6(z) = R(e^{2i\pi z})$ and the modular invariant $j(z) = J(e^{2i\pi z})$, where $\mathrm{Im}(z) > 0$: the functions E_4, E_6 and j are modular forms with respect to $SL_2(\mathbb{Z})$, while E_2 is only quasi-modular with respect to $SL_2(\mathbb{Z})$. Sometimes $E_2(q)$, $E_4(q)$ and $E_6(q)$ are just alternative notations for $P(q), Q(q)$ and $R(q)$ (see [4, p. 159]); the choice of the variable z *vs.* q makes this clear in the sequel. These functions satisfy the non-linear differential system (as functions of q)

$$q\frac{dE_2}{dq} = \frac{E_2^2 - E_4}{12}, \quad q\frac{dE_4}{dq} = \frac{E_2E_4 - E_6}{3}, \quad q\frac{dE_6}{dq} = \frac{E_2E_6 - E_4^2}{2}. \tag{1.1}$$

The Diophantine study of the values taken by modular forms and functions has a long history, see [4]. We quote the following results.

Theorem 1 (Bertrand [3])**.** *For any q such that $0 < |q| < 1$, the numbers $E_4(q)$ and $E_6(q)$ can not be simultaneously algebraic. Equivalently, for any q such that $0 < |q| < 1$ et $J(q) \notin \{0, 1728\}$, the numbers $J(q)$ and $qJ'(q)$ can not be simultaneously algebraic.*

This is in fact a consequence of a more general result of Schneider, expressed in a different language. When q is algebraic, Bertrand's theorem is also a consequence of the more recent

Theorem 2 (Théorème stéphanois [2])**.** *For any q such that $0 < |q| < 1$, the numbers q and $J(q)$ can not be simultaneously algebraic.*

Both theorems are now consequences of

Theorem 3 (Nesterenko [6])**.** *For any q such that $0 < |q| < 1$, at least three of the numbers q, $E_2(q)$, $E_4(q)$ and $E_6(q)$ are algebraically independent over $\mathbb{Q}$.*

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André obtained a particular case of Bertrand's theorem in [1], though the result is stated and proven for certain hypergeometric functions. His method of *simultaneous adelic uniformization* is very general in principle but it has been applied so far essentially only to those he delt with. In particular, his hypergeometric functions have a special connection to the modular world, which is not universal amongst hypergeometric functions.

2. Hypergeometric functions and modular forms

The generalized hypergeometric function is defined as

$$ {}_{p+1}F_p \left[\begin{matrix} a_0, & a_1, & \cdots, & a_p \\ & b_1, & \cdots, & b_p \end{matrix}; x\right] = \sum_{n=0}^{\infty} \frac{(a_0)_n (a_1)\cdots(a_p)_n}{(1)_n (b_1)_n \cdots (b_p)_n} x^n. $$

where $|x| < 1$, $(r)_0 = 1$ and $(r)_m = r(r+1)\cdots(r+m-1)$ for $m \geq 1$, and the parameters a_j, b_j are suitable complex numbers.

In the sequel, we will be consider the function

$$ F(t) = {}_2F_1 \left[\begin{matrix} \frac{1}{12}, & \frac{5}{12} \\ & 1 \end{matrix}; t\right]^2 = {}_3F_2 \left[\begin{matrix} \frac{1}{2}, & \frac{1}{6}, & \frac{5}{6} \\ & 1, & 1 \end{matrix}; t\right]. \tag{2.1} $$

The second equality is a particular case of Clausen's identity [7, p. 75, eq. (2.5.7)].

There exists a well-known connection between modular forms and hypergeometric series, going back to Klein at least. More recently, Stiller obtained a very elegant formulation. Let us define $t(q) = \frac{1728}{J(q)} \in \mathbb{Q}[[q]]$. It is an holomorphic function at $q = 0$, and $|t(q)| < 1$ iff $|J(q)| > 1728$ (i.e, if $|q|$ is sufficiently close to 0).

Theorem 4 (Stiller [8])**.** *For any $q \in \mathbb{C}$ such that $|q| < 1$ and $|J(q)| > 1728$, we have*

$$ E_4(q) = F\big(t(q)\big)^2 \quad \text{and} \quad E_6(q) = \big(1 - t(q)\big)^{1/2} F\big(t(q)\big)^3, \tag{2.2} $$

where the principal branch of the logarithm is chosen to define the square root.

It is natural question to wonder if $E_2(q)$ can also be expressed with similar hypergeometric functions. We prove the following result.

Theorem 5. *For any $q \in \mathbb{C}$ such that $|q| < 1$ and $|J(q)| > 1728$, we have*

$$ E_2(q) = \big(1 - t(q)\big)^{1/2} \Big(F\big(t(q)\big) + 6t(q)F'\big(t(q)\big)\Big) \tag{2.3} $$

$$ = {}_2F_1 \left[\begin{matrix} \frac{1}{12}, & \frac{5}{12} \\ & 1 \end{matrix}; t(q)\right] \cdot {}_2F_1 \left[\begin{matrix} -\frac{1}{12}, & \frac{7}{12} \\ & 1 \end{matrix}; t(q)\right]. \tag{2.4} $$

Proof. The hypergeometric function $F(t)$ is solution of the differential equation

$$ \Big[\theta^3 - t\big(\theta + \frac{1}{2}\big)\big(\theta + \frac{1}{6}\big)\big(\theta + \frac{5}{6}\big)\Big] y(t) = 0, \quad \theta = t\frac{d}{dt}. \tag{2.5} $$

Moreover,

$$ J'(q) = -\frac{E_6(q)}{qE_4(q)} J(q) $$

from which we deduce that

$$t'(q) = \frac{E_6(q)}{qE_4(q)}t(q). \tag{2.6}$$

Let $D(q)$ denote the function on the right-hand side of (2.3), which is holomorphic at $q = 0$. Using (2.2), (2.5) and (2.6), some tedious computations show that $D(q)$, $E_4(q)$ and $E_6(q)$ satisfy

$$q\frac{dD}{dq} = \frac{D^2 - E_4}{12}, \quad q\frac{dE_4}{dq} = \frac{DE_4 - E_6}{3}, \quad q\frac{dE_6}{dq} = \frac{DE_6 - E_4^2}{2}. \tag{2.7}$$

This system is formally the same as (1.1) with $E_2(q)$ replaced by $D(q)$, and we shall now prove that $D(q) = E_2(q)$.

For this, let us consider the differential equation (extracted from (2.7)):

$$12q\frac{dY}{dq} = Y^2 - E_4 \tag{2.8}$$

where $Y(q) = \sum_{n=0}^{\infty} y_n q^n \in \mathbb{C}[[q]$ is an unknown function. For simplicity, we set $E_4(q) = \sum_{n=0}^{\infty} e_n q^n$. Eq. (2.8) implies that

$$12ny_n = -e_n + \sum_{j=0}^{n} y_j y_{n-j}, \quad n \geq 0.$$

For $n = 0$, we deduce that $y_0^2 = 1$. Moreover, for $n \geq 1$, we have

$$(12n - 2y_0)y_n = -e_n + \sum_{j=1}^{n-1} y_j y_{n-j}$$

which determines each y_n uniquely once the value of y_0 is fixed. Hence, (2.8) has exactly two solutions $Y(q) \in \mathbb{C}[[q]]$, one for $y_0 = 1$ and the other one for $y_0 = -1$. Now, $D(q)$ and $E_2(q)$ are both solutions of (2.8) and $D(0) = E_2(0) = 1$. Hence $D(q) = E_2(q)$ as expected.

Eq. (2.4) follows from (2.1) and (2.3) by Euler's hypergeometric identity [7, p. 10, eq. (1.3.15)]. □

3. Further remarks

Let us consider the three series in $\mathbb{Q}[[t]]$, of hypergeometric type, defined by

$$A(t) := (1-t)^{1/2}\big(F(t) + 6tF'(t)\big), \quad B(t) := F(t)^2, \quad C(t) := (1-t)^{1/2}F(t)^3.$$

Using Ramanujan's system for $E_2(q), E_4(q), E_6(q)$ and Eq. (2.6), we get

Proposition 1. *We have*

$$t\frac{A(t)}{dt} = \frac{B(t)}{12C(t)}\big(A(t)^2 - B(t)\big)$$
$$t\frac{B(t)}{dt} = \frac{B(t)}{3C(t)}\big(A(t)B(t) - C(t)\big)$$
$$t\frac{C(t)}{dt} = \frac{B(t)}{2C(t)}\big(A(t)C(t) - B(t)^2\big).$$

Up to the multiplicative factor $B(t)/C(t)$, this is the same differential system as the one satisfied by $E_2(q), E_4(q), E_6(q)$. Hence, the ring $\mathbb{C}[A(t), B(t), C(t)]$ is left stable by the differential operator $\frac{tC(t)}{B(t)}\frac{d}{dt}$.

As we now explain, the functions A, B, C are "universal" in some sense. Let $T(q)$ be an hauptmodul holomorphic at 0 and vanishing at $q = 0$. We can use $T(q)$ instead of $t(q) = \frac{1728}{J(q)}$. Since $J(q)$ generates over $\mathbb{C}$ the field of modular functions (over $SL_2(\mathbb{Z})$), there exists $\varphi(X) \in \mathbb{C}(X)$ such that $T(q) = \varphi(t(q))$.

Given such a T, consider $a(t), b(t), c(t)$ the functions (holomorphic at $t = 0$) such that

$$E_2(q) = a(T(q)), \quad E_4(q) = b(T(q)), \quad E_6(q) = c(T(q)).$$

We thus have

$$A(t(q)) = a(T(q)),\ B(t(q)) = b(T(q)),\ C(t(q)) = c(T(q))$$

in a neighborhood of $q = 0$, so that

$$A(t) = a(\varphi(t)),\ B(t) = B(\varphi(t)),\ c(t) = C(\varphi(t)),$$

in a neighborhood of $t = 0$.

Thus, we see that a, b, c are not really different from A, B, C as they can expressed in terms of the hypergeometric function F and the reciprocal of φ. We remark that the functions in the sets $\{a(t), b(t), c(t)\}$, $\{a(t), a'(t), a''(t)\}$, $\{b(t), b'(t), b''(t)\}$ et $\{c(t), c'(t), c''(t)\}$ are algebraically dependent over $\mathbb{C}(t)$. It is thus not clear if one can get Nesterenko's theorem with such functions by means of [1, p. 119, Theorem 4], where algebraic independence is an hypothesis.

André did not use Stiller's hypergeometric $F(t)$, but instead ${}_2F_1[\frac{1}{2}, \frac{1}{2}; 1; t]$, which satisfies

$${}_2F_1\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2} \\ & 1\end{matrix}; \Big(\frac{\theta_2(q)}{\theta_3(q)}\Big)^4\right]^4 = \theta_3(q)^8$$

where

$$\theta_2(q) = 2\sum_{n=1}^{\infty} q^{(n-1/2)^2}, \quad \theta_3(q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2}.$$

Note that $\big(\frac{\theta_2(q)}{\theta_3(q)}\big)^4$ is a modular function and that

$$E_4(q^2) = \theta_2(q)^8 + \theta_3(q)^8 - \theta_2(q)^4\theta_3(q)^4.$$

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