HYPERGEOMETRIC CONSTRUCTIONS OF RATIONAL APPROXIMATIONS FOR (MULTIPLE) ZETA VALUES

TANGUY RIVOAL

This survey presents certain results concerning the diophantine nature of zeta values or multiple zeta values that I have obtained over the last few years, with or without coauthors. I did not try to cover all the known results concerning the diophantine theory of the Riemann zeta function and more informations are available in [12] for example. The first part is a presentation of irrationality results for the values of the Riemann zeta function, together with a description of the memoir [17] joint with Christian Krattenthaler on the "Denominators conjecture". The second part describes some of the results in two papers with J. Cresson and S. Fischler [10, 11], both devoted to the construction of linear forms in multiple zeta values, which are generalisations of Riemann zeta function. I warmly thank K. Matsumoto and H. Tsumura, the organisers of the franco-japanese winter school in january 2008 at Miura seaside, for giving me the opportunity to publish this survey in the proceedings of that conference.

In diophantine approximation, proofs of irrationality, linear independence, etc, usually rely on the construction of "auxiliary functions" and this is also the case here. Indeed, the results presented in both parts of the paper are proved by means of explicit auxiliary functions, which turn out to be hypergeometric series in one or several variables. (The underlying aspect "Padé approximants" will not be developped.) Therefore, before going into the subject, it is useful to remind the reader of the definition of those series. These are power series defined by

$$
{}_{q+1}F_q\left[\begin{matrix}\alpha_0, \alpha_1, \ldots, \alpha_q \\ \beta_1, \ldots, \beta_q\end{matrix}; z\right] = \sum_{k=0}^{\infty} \frac{(\alpha_0)_k\,(\alpha_1)_k \cdots (\alpha_q)_k}{k!\,(\beta_1)_k \cdots (\beta_q)_k} z^k,
$$

where $\alpha_j \in \mathbb{C}$, $\beta_j \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ and $(x)_m = x(x+1)\cdots(x+m-1)$ is the Pochhammer symbol. Such series converge for all $z \in \mathbb{C}$ such that $|z| < 1$, and for $z = \pm 1$, provided that $\mathrm{Re}(\beta_1 + \cdots + \beta_q) > \mathrm{Re}(\alpha_0 + \alpha_1 + \cdots + \alpha_q)$. The literature (see [3, 16, 22]) contains various special kind of hypergeometric series whose parameters satisfy particular relations. For example, a hypergeometric series is said to be

Date: February 4, 2009.
1991 *Mathematics Subject Classification.* 11J72 (Primary); 11M41, 33C70 (Secondary).

- *balanced* if $\alpha_0 + \cdots + \alpha_q + 1 = \beta_1 + \cdots + \beta_q$;
- *nearly-poised (of the first kind)* if $\alpha_1 + \beta_1 = \cdots = \alpha_q + \beta_q$;
- *well-poised* if $\alpha_0 + 1 = \alpha_1 + \beta_1 = \cdots = \alpha_q + \beta_q$;
- *very-well-poised* if it is well-poised and $\alpha_1 = \frac{1}{2}\alpha_0 + 1$.

In ths first section, we will show that the very-well-poised case is of special importance. In the second section, we will present multiple series which are non-trivial generalisation of one variable very-well-poised series.

1. Values of the Zeta function and the Denominators Conjecture

In this section, we discuss the appearance of *very-well-poised hypergeometric series* as a tool for studying the diophantine nature of the values of the Riemann zeta function at positive integers. In this context, we give examples of an important and general experimental phenomenon known as the *Denominators Conjecture* which in principle should enable us to obtain better irrationality results. We explain the ideas behind its proof, which was obtained by C. Krattenthaler and the author in [17].

1.1. A general construction.

A simple way of proving irrationality results for zeta values is to start with a rational function of the form

$$R_n(k) = \frac{Q_n(k)}{(k(k+1)\cdots(k+n))^A} = \frac{Q_n(k)}{(k)_{n+1}^A} \in \mathbb{Q}(k)$$

where $n \geq 0$ and $A \geq 1$ are integers, $Q_n(k) \in \mathbb{Q}[k]$, and then consider the series

$$S_n(z) = \sum_{k=1}^{\infty} R_n(k) z^{-k}$$

which we assume to be convergent for $z = 1$, forcing $\deg(Q_n(k)) \leq A(n+1) - 2$. Then, by partial fractions expansion of $R_n(k)$, it is easy to prove that there exist polynomials $(P_{j,n}(z))_{j=0,\ldots,n}$ in $\mathbb{Q}[z]$, of degree at most n such that

$$S_n(z) = P_{0,n}(z) + \sum_{j=1}^{A} P_{j,n}(z) \operatorname{Li}_j(1/z).$$

Here, we have encountered the polylogarithmic functions defined by $\operatorname{Li}_s(z) = \sum_{k=1}^{\infty} z^k/k^s$ for $s \geq 1$, $|z| \leq 1$ and $(s,z) \neq (1,1)$. Note that for $s \geq 2$, $\operatorname{Li}_s(1) = \zeta(s)$ and $\operatorname{Li}_s(-1) = (1 - 2^{1-s})\zeta(s)$. Furthermore, under these conditions, it can be proved that $\mathrm{d}_n^{A-j} P_{j,n}(z) \in \mathbb{Z}[z]$, where $\mathrm{d}_n = \text{l.c.m.}\{1, 2, \ldots, n\}$, and $P_{1,n}(1) = 0$. Consequently, we have that

$$\mathrm{d}_n^A S_n(1) = p_{0,n} + \sum_{j=2}^{A} p_{j,n} \zeta(s)$$

where $p_{j,n} = \mathrm{d}^A P_{j,n}(1) \in \mathbb{Z}$ (and a similar expression for $S_n(-1)$).

We also have at our disposal the "differential" trick, which generalises the previous construction: let $C \geq 0$ be an integer and consider the series

$$S_{n,C}(z) = \sum_{k=1}^{\infty} \frac{1}{C!} \frac{\partial^C R_n(k)}{\partial k^C} \, z^{-k}.$$

Then there exist a polynomial $\tilde{P}_{0,n}(z) \in \mathbb{Q}[z]$, of degree at most n, depending on C such that

$$S_{n,C}(z) = \tilde{P}_{0,n}(z) + (-1)^C \sum_{j=1}^{A} \binom{j+C-1}{j-1} P_{j,n}(z) \, \mathrm{Li}_{j+C}(1/z)$$

where the $P_{j,n}$ are as above and the polylogarithms are shifted by C. It can be proved that $\mathrm{d}_n^{A+C} \tilde{P}_{0,n}(1) \in \mathbb{Z}$, hence there exist integers $\tilde{p}_{j,n}$ such that

$$\mathrm{d}_n^{A+C} S_{n,C}(1) = \tilde{p}_{0,n} + \sum_{j=2}^{A} \tilde{p}_{j,n} \zeta(C+j),$$

and a similar expression holds for $S_{n,C}(-1)$. Given this very general construction, the problem is now to choose suitably A and $Q_n(k)$ in order to apply the following criteria for linear independence, due to Nesterenko [18].

Theorem 1. *Given N real numbers $\theta_1, \theta_2, \ldots, \theta_N$, suppose there exist N sequences $(p_{\ell,n})_{n \geq 0}$ of integers such that (i) $\left|\sum_{\ell=1}^{N} p_{\ell,n}\theta_\ell\right| = \alpha^{n+o(n)}$ and (ii) $\forall \ell = 1, \ldots, N$, $|p_{\ell,n}| \leq \beta^{n+o(n)}$, for some reals $\alpha, \beta > 0$. Then,*

$$\dim_{\mathbb{Q}}(\mathbb{Q}\,\theta_1 + \mathbb{Q}\,\theta_2 + \cdots + \mathbb{Q}\,\theta_N) \geq 1 - \frac{\log(\alpha)}{\log(\beta)} \, .$$

If we are only interested in proving the irrationality of one of the numbers θ_ℓ, then we don't have to check the condition (ii), but only to prove that $\alpha < 1$ to get a dimension > 1. Furthermore, in this case, the proof of Theorem 1 is straightforward. Finally, to get asymptotic values which are as small as possible for $|S_{n,C}(\pm 1)|^{1/n}$, heuristically, the first terms of the sum should be cancelled, *i.e.* $Q_n(k)$ should have a factor $((k-1)\ldots(k-m))^B = (k-m)_m^B$, where $B > C$ (to cancel the effect of the C^{th}-derivative on $R_n(k)$). The parameter m will always be of the form rn for a suitable integer $r \geq 1$.

1.2. **Irrationality of some values of zeta at the integers.** It is well-known that $\zeta(2) = \pi^2/6$, a result due to Euler, and Legendre proved that π^2 is irrational. Hence, one concludes that $\zeta(2)$ is also irrational. But this proof uses a shortcut and it is an interesting problem to prove the irrationality of $\zeta(2)$ without using it. This problem was first solved by Apéry [2], who showed that there exist two sequences $(\alpha_n)_{n\geq 0}$ and $(\beta_n)_{n\geq 0}$ such that $\alpha_n \in \mathbb{Z}$, $\mathrm{d}_n^2\beta_n \in \mathbb{Z}$, where $\mathrm{d}_n = \text{l.c.m}\{1,2,\ldots,n\} = e^{n+o(n)}$, and

$$\lim_{n\to+\infty} |\mathrm{d}_n^2(\alpha_n\zeta(2)-\beta_n)|^{1/n} = e^2\left(\frac{\sqrt{5}-1}{2}\right)^5 < 1. \tag{1.1}$$

These properties immediately imply that:

Theorem 2. $\zeta(2)$ *is irrational.*

There are many different ways of constructing these sequences and the following "hypergeometric" one is particularly simple:

$$\begin{aligned}\alpha_n\zeta(2)-\beta_n &= (-1)^n\frac{n!^4}{(2n+1)!^2}\,{}_3F_2\left[\begin{matrix}n+1,n+1,n+1\\ 2n+2,2n+2\end{matrix};1\right]\\ &= (-1)^n n!\sum_{k=1}^{\infty}\frac{(k-n)_n}{(k)_{n+1}^2} = (-1)^n\int_0^1\int_0^1 \frac{x^n(1-x)^n y^n(1-y)^n}{(1-(1-x)y)^{n+1}}\,\mathrm{d}x\mathrm{d}y.\end{aligned}$$

This hypergeometric series is nearly poised of the first kind. The series on the second line is suitable for proving, after just expanding in partial fractions the rational function $n!(k-n)_n/(k)_{n+1}^2$, the existence of α_n and β_n, while the integral, due to Beukers [5], immediately gives (1.1). The equality between the series and the integral is a straightforward computation. Furthermore, we obtain

$$\alpha_n = \sum_{j=0}^{n}\binom{n}{j}^2\binom{n+j}{n} = {}_3F_2\left[\begin{matrix}-n,-n,n+1\\ 1,\ 1\end{matrix};1\right].$$

We now move to the case of $\zeta(3)$. Contrary to $\zeta(2)$, there is no known shortcut for proving the irrationality of $\zeta(3)$ and in factc, conjecturally, this number has no algebraic relation with π. Apéry's great achievement was to give the first proof of this fact (see [2]). In fact, he proved that there exist two sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ such that $a_n \in \mathbb{Z}$, $\mathrm{d}_n^3 b_n \in \mathbb{Z}$ and

$$\lim_{n\to+\infty} |\mathrm{d}_n^3(2a_n\zeta(3)-b_n)|^{1/n} = e^3(\sqrt{2}-1)^4 < 1.$$

These properties imply that:

Theorem 3. $\zeta(3)$ *is irrational.*

Once again, there exist many ways of constructing these sequences. Gutnik [14] and Beukers [6] independently essentially proposed the following:

$$\begin{aligned}
2a_n\zeta(3) - b_n &= -\sum_{k=1}^{\infty} \frac{\partial}{\partial k}\left(\frac{(k-n)_n^2}{(k)_{n+1}^2}\right) \\
&= \int_0^1\int_0^1\int_0^1 \frac{x^n(1-x)^n y^n(1-y)^n z^n(1-z)^n}{(1-(1-(1-x)y)z)^{n+1}}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z.
\end{aligned}$$

Strictly speaking, the series is not hypergeometric but is linked to solutions of a certain hypergeometric differential equation. Here, the equality between the series and the triple Beukers' integral [5] is not completely straightforward. Finally, we have

$$a_n = \sum_{j=0}^{n} \binom{n}{j}^2 \binom{n+j}{n}^2 = {}_4F_3\left[\begin{matrix}-n, -n, n+1, n+1\\ 1,\ 1,\ 1\end{matrix}; 1\right].$$

We now try to give a new and more complicated proof of the irrationality of $\zeta(2)$: the reason for this will become clear later. Let's consider the following very-well-poised hypergeometric series

$$\begin{aligned}
\mathbf{S}_n &= n!\sum_{k=1}^{\infty}\left(k+\frac{n}{2}\right)\frac{(k-n)_n(k+n+1)_n}{(k)_{n+1}^3}(-1)^k \qquad (1.2)\\
&= \frac{n!^5(3n+2)!}{2(2n+1)!^4}\,{}_6F_5\left[\begin{matrix}3n+2, \frac{3}{2}n+2, n+1, \dots, n+1\\ \frac{3}{2}n+1, 2n+2, \dots, 2n+2\end{matrix}; -1\right],
\end{aligned}$$

which fits into our general scheme. A priori, $\mathbf{S}_n \in \mathbb{Q} + \mathbb{Q}\zeta(2) + \mathbb{Q}\zeta(3)$ but due to the *very* special form of the numerator (we will explain this later), we have $\mathbf{S}_n = -\mathbf{p}_n\frac{1}{2}\zeta(2) - \mathbf{q}_n$, where $\mathrm{d}_n\mathbf{p}_n$ and $\mathrm{d}_n^3\mathbf{q}_n$ are integers. Unfortunately, such estimates are not enough to give a new proof of the irrationality of $\zeta(2)$, but this would be the case if we could "throw away" one power of d_n for $\mathbf{p}_n$ and $\mathbf{q}_n$. It is possible to give the following expression for $\mathbf{p}_n$:

$$\begin{aligned}
\mathbf{p}_n = (-1)^{n+1}\sum_{j=0}^{n}\left(\frac{n}{2}-j\right)\binom{n}{j}^3\binom{n+j}{n}\binom{2n-j}{n}& \\
\cdot\left(4H_{n-j} - 4H_j + H_{n+j} - H_{2n-j} - \frac{1}{\frac{n}{2}-j}\right)&,
\end{aligned}$$

where $H_m = 1 + 1/2 + \cdots + 1/m$ by definition. Hence, there is no reason to expect anything better than a denominator d_n for $\mathbf{p}_n$. But, surprisingly, numerical computations suggest the following:

Claim 1. *The number $\mathbf{p}_n$ and $\mathrm{d}_n^2\mathbf{q}_n$ appear to be integers and furthermore, $\mathbf{p}_n$ and $\mathbf{q}_n$ are the same as Apéry's α_n and $-\beta_n/2$ for $\zeta(2)$.*

Thus, the proof of this claim would be exactly what we need to give a new proof of $\zeta(2) \notin \mathbb{Q}$.

In fact, the series (1.2) was found after a similar series was produced by K. Ball for $\zeta(3)$. He constructed the following series

$$\begin{aligned}\mathbf{B}_n &= n!^2 \sum_{k=1}^{\infty} \left(k+\frac{n}{2}\right) \frac{(k-n)_n(k+n+1)_n}{(k)_{n+1}^4} \\ &= \frac{n!^7(3n+2)!}{2(2n+1)!^5}\,{}_7F_6\left[\begin{matrix}3n+2, \frac{3}{2}n+2, n+1, \ldots, n+1\\ \frac{3}{2}n+1, 2n+2, \ldots, 2n+2\end{matrix}; 1\right]\end{aligned} \tag{1.3}$$

in the hope that it would give a completely elementary proof of the irrationality of $\zeta(3)$, in the style of the usual irrationality proof of $\exp(1)$ (see the introduction of [19]). Of course, the similarity of $\mathbf{S}_n$ and $\mathbf{B}_n$ is not an accident, and one has $\mathbf{B}_n = \mathbf{a}_n\zeta(3) - \mathbf{b}_n$ with $\mathrm{d}_n\mathbf{a}_n$ and $\mathrm{d}_n^4\mathbf{b}_n$ integers, whereas one would have expected *a priori* $\mathbf{B}_n \in \mathbb{Q}+\mathbb{Q}\zeta(2)+\mathbb{Q}\zeta(3)+\mathbb{Q}\zeta(4)$. Note that, once again, the given denominators are to large to get a new proof of $\zeta(3) \notin \mathbb{Q}$. The expression for $\mathbf{a}_n$ is

$$\mathbf{a}_n = (-1)^{n+1} \sum_{j=0}^{n} \left(\frac{n}{2}-j\right) \binom{n}{j}^4 \binom{n+j}{n} \binom{2n-j}{n} \cdot \left(5H_{n-j} - 5H_j + H_{n+j} - H_{2n-j} - \frac{1}{\frac{n}{2}-j}\right)$$

and numerical computations suggest the following improvements:

Claim 2. *The numbers $\mathbf{a}_n$ and $\mathrm{d}_n^3\mathbf{b}_n$ appear to be integers and furthermore $\mathbf{a}_n$ and $\mathbf{b}_n$ are the same as Apéry's a_n and $b_n/2$ for $\zeta(3)$.*

This would be enough to give an elementary proof of the irrationality of $\zeta(3)$, since a simple application of Stirling's formula proves that

$$\lim_{n\to+\infty} |\mathbf{B}_n|^{1/n} = (\sqrt{2}-1)^4.$$

To generalise the series (1.2) and (1.3), the very-well-poised hypergeometric series

$$\begin{aligned}\mathbf{S}_{n,A,r} &= n!^{A-2r} \sum_{k=1}^{\infty} \left(k+\frac{n}{2}\right) \frac{(k-rn)_{rn}(k+n+1)_{rn}}{(k)_{n+1}^A}(-1)^{kA} \\ &= n!^{A-2r}\frac{(rn)!^{A+1}((2r+1)n+2)!}{2((r+1)n+1)!^{A+1}} \\ &\quad\times {}_{A+3}F_{A+2}\left[\begin{matrix}(2r+1)n+2, \frac{2r+1}{2}n+2, rn+1, \ldots, rn+1\\ \frac{2r+1}{2}n+1, (r+1)n+2, \ldots, (r+1)n+2\end{matrix}; (-1)^A\right]\end{aligned} \tag{1.4}$$

was introduced in [19] and [4] to prove the following result.

Theorem 4. *For any even $A \geq 4$,*

$$\dim_{\mathbb{Q}}(\mathbb{Q} + \mathbb{Q}\zeta(3) + \mathbb{Q}\zeta(5) + \cdots + \mathbb{Q}\zeta(A-1)) \geq \frac{1+o(1)}{1+\log(2)}\log(A). \tag{1.5}$$

(In fact, it was a similar series without the factor $k + n/2$, which is useless in this particular case.) The new parameter r is an integer such that $1 \leq r \leq A/2$. We now briefly indicate the steps in the proof of (1.5). The form of the numerator of the summand first implies that

$$\mathbf{S}_{n,A,r} = p_{0,n} + \sum_{\substack{j=2 \\ j\equiv A-1[2]}}^{A} p_{j,n}\zeta(j),$$

which involves only odd zeta values if A is even, and d_n^{A-j} is a denominator of $p_{j,n}$. Using an explicit expression for the $p_{j,n}$'s, we can prove that

$$\limsup_{n\to+\infty} |p_{j,n}|^{1/n} \leq 2^{A-2r}(2r+1)^{2r+1}.$$

Furthermore, we have a Euler type integral representation

$$\mathbf{S}_{n,A,r} = \frac{((2r+1)n+1)!}{n!^{2r+1}} \int_{[0,1]^{A+1}} \frac{\prod_{j=1}^{A+1} x_j^{rn}(1-x_j)^n \mathrm{d}x_j}{(1-x_1\cdots x_{A+1})^{(2r+1)n}} \frac{1+x_1\cdots x_{A+1}}{(1-x_1\cdots x_{A+1})^3}$$

from which we deduce that

$$0 < \lim_{n\to+\infty} |\mathbf{S}_{n,A,r}|^{1/n} \leq 2^{2r+1}r^{2r-A}.$$

It remains to apply Nesterenko's linear independence criteria, with the optimal choice $r = [A/\log^2(A)]$, to get (1.5).

Although this is not needed here, we note that, as the reader might have already suspected, numerical computations suggest that:

Claim 3. d_n^{A-1} *seems to be a denominator for the* $p_{j,n}$.

Theorem 4 implies that infinitely many numbers $\zeta(2n+1)$ are irrational and the next problem is of course to decide which ones. Presumably, the answer is all; see the discussion following the Conjecture 1 in Section 2 for good reasons to believe that. A more modest aim is to prove the irrationality of $\zeta(5)$ or a result in this direction. To do that, let $A \geq 6$ be an even integer and consider the series

$$\tilde{\mathbf{S}}_{A,n} = n!^{A-6} \sum_{k=1}^{\infty} \frac{1}{2}\frac{\partial^2}{\partial k^2}\left(\left(k+\frac{n}{2}\right)\frac{(k-n)_n^3(k+n+1)_n^3}{(k)_{n+1}^A}\right),$$

which is not exactly hypergeometric, but has enough properties in common with very-well-poised series to give that

$$\tilde{\mathbf{S}}_{A,n} = \tilde{p}_{0,n} + \sum_{\substack{j=5 \\ \text{odd} j}}^{A+1} \tilde{p}_{j,n}\zeta(j).$$

Note that differentiating twice together with the numerator of the summand enable us to have a linear form only in odd zeta values from $\zeta(5)$, and not $\zeta(3)$. Furthermore, d_n^{A+2} is a denominator of $\tilde{p}_{j,n}$ and we now seek the smallest possible A such that $\mathrm{d}_n^{A+2}\tilde{\mathbf{S}}_{A,n}$ tends to 0 as n tends to infinity. This can be done by first noting that $\tilde{\mathbf{S}}_{A,n}$ is the real part of the integral

$$\frac{(-1)^n n!^{A-6}}{2i\pi} \int_{c+i\infty}^{c-i\infty} \left(z+\frac{n}{2}\right) \frac{\Gamma(z)^{A+3}\Gamma(n-z+1)^3\Gamma(z+2n+1)^3}{\Gamma(z+n+1)^{A+3}}\, e^{i\pi z}\mathrm{d}z,$$

(where c is any real in $(0,1)$) and then by applying the saddle point method to estimate the asymptotic behavior of this integral as n tends to infinity. One finds that 20 is the smallest such A, yielding the following result proved by the author in [20].

Theorem 5. *At least one of the nine numbers $\zeta(5), \zeta(7), \ldots, \zeta(21)$ is irrational.*

Numerical computations suggest that:

Claim 4. *For any even $A \geq 6$, d_n^{A+1} seems to be a denominator of the $\tilde{p}_{j,n}$.*

Here, the consequences would very important since the same argument shows that $\mathrm{d}_n^{18+1}\tilde{\mathbf{S}}_{18,n}$ tends to 0 (while $\mathrm{d}_n^{18+2}\tilde{\mathbf{S}}_{18,n}$ does not), thus proving, conjecturally, the irrationality of one of the eight numbers $\zeta(5), \zeta(7), \ldots, \zeta(19)$.

1.3. **A very general phenomenon.** In the light of the previous examples, it is time to adopt a general approach to very-well-poised series of hypergeometric kind.

Let z be a complex number such that $|z| \geq 1$, and A, B, C, r be positive integers such that $1 \leq 2Br \leq A$. Consider the series

$$\mathbf{S}_{n,A,B,C,r}(z) = n!^{A-2Br} \sum_{k=1}^{\infty} \frac{1}{C!}\frac{\partial^C}{\partial k^C} \left(\left(k+\frac{n}{2}\right) \frac{(k-rn)_{rn}^B (k+n+1)_{rn}^B}{(k)_{n+1}^A} \right) z^{-k},$$

which is really hypergeometric when $C=0$. According to the general scheme developed in section 1.1, we have that

$$\mathbf{S}_{n,A,B,C,r}(z) = \mathbf{p}_{0,C,n}(z) + (-1)^C \sum_{m=1}^{A} \binom{C+m-1}{m-1} \mathbf{p}_{m,n}(z)\,\mathrm{Li}_{C+m}(1/z),$$

where the polynomials $\mathbf{p}_{m,n}(X)$ also depend on A,B and r, but not C, except $\mathbf{p}_{0,C,n}(X)$. Using the trivial but important relation $(\alpha)_m = (-1)^m(-\alpha-m+1)_m$ (for any $\alpha \in \mathbb{C}$), one immediately proves that the rational summand of $\mathbf{S}_{n,A,B,C,r}(z)$

$$R_n(k) = \left(k + \frac{n}{2}\right) \frac{(k-rn)_{rn}^B (k+n+1)_{rn}^B}{(k)_{n+1}^A}$$

satisfies the symmetry $R_n(-n-k) = (-1)^{A(n+1)+1} R_n(k)$, from which one deduces that $z^n \mathbf{p}_{j,n}(1/z) = (-1)^{A+j+1}\mathbf{p}_{j,n}(z)$ $(j \geq 1)$. Consequently, when A is even, $\mathbf{S}_{n,A,B,C,r}((-1)^A)$ is a rational linear combination of 1, $\zeta(C+3), \zeta(C+5), \ldots, \zeta(C+A-1)$, whereas when A is odd, $\mathbf{S}_{n,A,B,C,r}((-1)^A)$ is a rational linear combination of 1, $\zeta(C+2), \zeta(C+4), \ldots, \zeta(C+A-1)$. The coefficients of these linear forms satisfy $\mathrm{d}_n^{A+C}\mathbf{p}_{0,C,n}(X) \in \mathbb{Z}[X]$ and $\mathrm{d}_n^{A-m}\mathbf{p}_{m,n}(X) \in \mathbb{Z}[X]$, but the evidence above, along with many other numerical computations, led the author to formulate the following conjecture in [21], which contains the previous Claims 1–4.

Denominators Conjecture. *Fix integers $A \geq 2$, $B \geq 0$, $r \geq 0$ and $n \geq 0$ (with $0 \leq 2Br \leq A$). Then the rational numbers $\mathrm{d}_n^{A+C-1}\mathbf{p}_{0,C,n}\left((-1)^A\right)$ and $\mathrm{d}_n^{A-m-1}\mathbf{p}_{m,n}\left((-1)^A\right)$ (for all $m \in \{1, \ldots, A\}$) are integers.*

Nothing similar holds if the factor $k + n/2$ is omitted from the series: it corresponds to the word *very* in very-well-poised and its presence is crucial. After many partial steps towards the proof of the conjecture (as witnessed by the different versions posted in the arXiv), C. Krattenthaler and the author finally proved it completely in [17].

Theorem 6. *The Denominators Conjecture is true.*

It follows that Claims 1 to 4 are all true. We now explain the ideas behind this result, which are based on a refined Denominators Conjecture. We will only consider the case of the "leading" coefficient, that is to say the coefficient $\mathbf{p}_{A-1,n}\left((-1)^A\right)$ of $\zeta(C+A-1)$, which depends on A, B and r but not on C. From now on, we assume that $r=1$ and set $P_n(A,B) = (-1)^{B(n+1)}\mathbf{p}_{A-1,n}\left((-1)^A\right)$. Then,

$$P_n(A,B) = \sum_{j=0}^{n} \left(\frac{n}{2} - j\right) \binom{n}{j}^A \binom{n+j}{n}^B \binom{2n-j}{n}^B \cdot \left((A+B)H_{n-j} - (A+B)H_j + BH_{n+j} - BH_{2n-j} - \frac{1}{\frac{n}{2}-j}\right). \tag{1.6}$$

The general estimates prove that $\mathrm{d}_n P_n(A,B) \in \mathbb{Z}$.

The Denominators Conjecture claims that $P_n(A,B)$ is always an integer, which is not at all obvious, all the more because the summands on the right hand side of (1.6) are almost never integers themselves. In the case of $\zeta(2)$ ($A=3$, $B=1$) and $\zeta(3)$ ($A=4$, $B=1$), we could even identify these coefficients as Apéry's numbers:

$$P_n(3,1)=\alpha_n \quad \text{and} \quad P_n(4,1)=a_n. \tag{1.7}$$

The advantage of (1.7) is that it is much easier to prove an identity and, in fact, (1.7) can be proved with the help of Zeilberger's program Ekhad: both sides are shown to satisfy the same recurrences relations, with the same initial values (see the introduction of [17] for references). But in the general situation, we do not have an identity to prove and the first thing to do is to find one. This can be done as follows. In [25], Vasilyev introduced the following generalisation of Beukers' integrals for $\zeta(2)$ and $\zeta(3)$: let $E\geq 2$ and

$$J_{n,E}=\int_{[0,1]^E}\frac{\prod_{j=1}^{E}x_j^n(1-x_j)^n\,\mathrm{d}x_j}{Q_E(x_1,x_2,\ldots,x_E)^{n+1}}$$

where $Q_E(x_1,x_2,\ldots,x_E)=1-(\cdots(1-(1-x_1)x_2)\cdots)x_E$. He proved that $J_{n,4}\in\mathbb{Q}+\mathbb{Q}\zeta(2)+\mathbb{Q}\zeta(4)$ and $J_{n,5}\in\mathbb{Q}+\mathbb{Q}\zeta(3)+\mathbb{Q}\zeta(5)$, while Beukers showed that $J_{n,2}\in\mathbb{Q}+\mathbb{Q}\zeta(2)$ and $J_{n,3}\in\mathbb{Q}+\mathbb{Q}\zeta(3)$. Vasilyev conjectured that this dichotomy is valid for all E, depending on the parity of E and this was obtained by Zudilin in [30], who proved the following result.

Theorem 7. *For all $E\geq 2$, we have that*

$$J_{n,E}=\frac{n!^{2E+1}(3n+2)!}{2(2n+1)!^{E+2}}\;{}_{E+4}F_{E+3}\left[\begin{matrix}3n+2,\frac{3}{2}n+2,n+1,\ldots,n+1\\ \frac{3}{2}n+1,2n+2,\ldots,2n+2\end{matrix};(-1)^{E+1}\right]. \tag{1.8}$$

Indeed, the hypergeometric series (1.8) is a special case of the very-well-poised series (1.4) and consequently it can be represented as a linear form in odd/even zeta values, where the coefficient of $\zeta(E)$ is exactly $(-1)^{n+1}P_n(E+1,1)$. But it is also possible, though quite difficult, to expand the integral on the left hand side of (1.8) as a linear combinations of zeta values (and also some multiple zeta values), and then to isolate the coefficient of $\zeta(E)$. Assuming the reasonable, but still conjectural, fact that the values $\zeta(n)$, $n\geq 2$, are $\mathbb{Q}$–linearly independent, the comparison of both sides of (1.8) led to guess the following

Refined Denominators Conjecture in the case $B=1$. *For $A=2m+1\geq 3$ odd, set*

$$p_n(A,1)=\sum_{0\leq i_1\leq i_2\leq\cdots\leq i_m\leq n}\binom{n}{i_m}^2\binom{n+i_m}{n}\prod_{k=1}^{m-1}\binom{n}{i_k}^2\binom{n+i_{k+1}-i_k}{n},$$

and for $A=2m\geq 2$ even, set

$$p_n(A,1)=\sum_{0\leq i_1\leq i_2\leq\cdots\leq i_m\leq n}(-1)^{i_m}\binom{n}{i_m}\binom{n+i_m}{n}\prod_{k=1}^{m-1}\binom{n}{i_k}^2\binom{n+i_{k+1}-i_k}{n},$$

Then, for all integers $A \geq 2$, $n \geq 0$, we have that $P_n(A,1) = (-1)^{An+1} p_n(A,1)$.

It is now quite clear why $P_n(A,1)$ should be integers, since these refined conjectures express them as multiple sums of products of binomial coefficients. C. Krattenthaler has produced an extremely useful electronic version, which he called HYP, of Gasper & Rahman's book [16] that lists the almost infinitely many identities between (q–)hypergeometric series: HYP is not only an electronic library but, more importantly, a "tool box", that is to say one can feed it a hypergeometric series, ask it to perform a certain transform and output the result. Using this software, it becomes easier (but still very difficult in our situation) to handle hypergeometric sums and prove the "refined conjectural identities" above, which are in fact special cases of the following key identity:

$$
\begin{aligned}
&{}_{2s+5}F_{2s+4}\left[\begin{matrix} a, \frac{a}{2}+1, b_1, c_1, \dots, b_{s+1}, c_{s+1}, -N \\ \frac{a}{2}, 1+a-b_1, 1+a-c_1, \dots, 1+a-b_{s+1}, 1+a-c_{s+1}, 1+a+N \end{matrix}; 1\right] \\
&\quad = \frac{(1+a)_N\,(1+a-b_{s+1}-c_{s+1})_N}{(1+a-b_{s+1})_N\,(1+a-c_{s+1})_N} \sum_{k_1,k_2,\dots,k_s \geq 0} \frac{(-N)_{k_1+\cdots+k_s}}{(b_{s+1}+c_{s+1}-a-N)_{k_1+\cdots+k_s}} \\
&\qquad\qquad \cdot \prod_{j=1}^{s} \frac{(1+a-b_j-c_j)_{k_j}\,(b_{j+1})_{k_1+\cdots+k_j}\,(c_{j+1})_{k_1+\cdots+k_j}}{k_j!\,(1+a-b_j)_{k_1+\cdots+k_j}\,(1+a-c_j)_{k_1+\cdots+k_j}}. \qquad (1.9)
\end{aligned}
$$

Note that the left hand side is a very-well-poised hypergeometric series. It turned out that this identity had already been proved by Andrews [1] in the seventies, but without HYP. From the key identity, it is not only possible to prove the refined Denominators Conjecture for $P_n(A,0)$ mentioned above, but also to handle the case of $B=0$ and $B \geq 2$. Thus the truth of the Denominators Conjecture for the "leading" coefficients follows more or less from Andrews' identity (1.9). More "human" work is required to extract from the key identity the conjectured denominators of the other coefficients: the interested reader is refered to [17] for the details, which are far from trivial. For completeness, the conjecture is proved for $2\mathbf{p}_{0,C,n}(\pm 1)$ rather than for $\mathbf{p}_{0,C,n}(\pm 1)$: this is the most difficult case of the conjecture, and also the most important since its denominator is always the one of the linear forms in zeta values.

As we have already mentioned in Section 1.2, the Denominators Conjecture enables us to give new proofs that $\zeta(2)$, $\zeta(3)$ are irrational and, more importantly, that at least one of the eight numbers $\zeta(5), \zeta(7), \dots, \zeta(19)$ is irrational. This is proved via the series (corresponding to $A=18$, $B=3$, $C=2$ and $r=1$)

$$
n!^{12} \sum_{k=1}^{\infty} \frac{1}{2} \frac{\partial^2}{\partial k^2} \left(\left(k + \frac{n}{2}\right) \frac{(k-n)_n^3 (k+n+1)_n^3}{(k)_{n+1}^{18}} \right) = \tilde{p}_{0,n} + \sum_{\substack{j=5 \\ j \text{ odd}}}^{19} \tilde{p}_{j,n} \zeta(j),
$$

where the a priori denominator d_n^{20} of $\tilde{p}_{j,n}$ can now be replaced by d_n^{19}. But this refinement is useless since W. Zudilin proved the following much better result in [31].

Theorem 8. *At least one of the four numbers $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational.*

To prove this, he uses more complicated very-well-poised series (in a sense, those considered in this text are the simplest of this kind) and he also formulates a "super" Denominators Conjecture for his linear forms, which is still open. Hence there might be room to prove that at least one of three numbers $\zeta(5), \zeta(7), \zeta(9)$ is irrational.

2. Linear forms in Multiple Zeta Values

We now turn our attention to a generalisation of the Riemann zeta function $\zeta(s)$, given by the multiple zeta values (abreviated as MZVs; note that in french, the word *polyzêtas* is now often used for these series). These are multiple series defined for all integers $p \geq 1$ and all p-tuples $\underline{s} = (s_1, s_2, \dots, s_p)$ of integers ≥ 1, with $s_1 \geq 2$, by

$$\zeta(s_1, s_2, \dots, s_p) = \sum_{k_1 > k_2 > \dots > k_p \geq 1} \frac{1}{k_1^{s_1} k_2^{s_2} \dots k_p^{s_p}}.$$

The integers p and $s_1 + s_2 + \dots + s_p$ are the depth and the weight of $\zeta(s_1, s_2, \dots, s_p)$ respectively.

2.1. Goncharov-Zagier's conjecture.

MZVs naturally appear when, for example, one considers products of values of the zeta function, e.g $\zeta(n)\zeta(m) = \zeta(n+m) + \zeta(n,m) + \zeta(m,n)$. In a certain sense, this enables us to "linearise" these products. Except a few identities such as $\zeta(2,1) = \zeta(3)$ (due to Euler), the arithmetical nature of MZVs is no better understood than that of $\zeta(s)$. However, the set of MZVs has a very rich structure which is well understood, at least conjecturally. (See [26]). For example, let us consider the $\mathbb{Q}$-vector spaces $\mathcal{Z}_p$ of $\mathbb{R}$ which are spanned by the 2^{p-2} MZVs of weight $p \geq 2$: $\mathcal{Z}_2 = \mathbb{Q}\zeta(2)$, $\mathcal{Z}_3 = \mathbb{Q}\zeta(3) + \mathbb{Q}\zeta(2,1)$, $\mathcal{Z}_4 = \mathbb{Q}\zeta(4) + \mathbb{Q}\zeta(3,1) + \mathbb{Q}\zeta(2,2) + \mathbb{Q}\zeta(2,1,1)$, etc. Set $v_p = \dim_{\mathbb{Q}}(\mathcal{Z}_p)$. We have the following conjecture, whose (i) is due to Zagier and (ii) to Goncharov.

Conjecture 1. (i) *For any integer $p \geq 2$, we have $v_p = c_p$, where c_p is defined by the linear recursion $c_{p+3} = c_{p+1} + c_p$, where $c_0 = 1$, $c_1 = 0$ and $c_2 = 1$.*

(ii) *The $\mathbb{Q}$-vector spaces $\mathbb{Q}$ and $\mathcal{Z}_p$ $(p \geq 2)$ are in direct sum.*

Hence, the sequence $(v_p)_{p\geq 2}$ should grow like α^p (where $\alpha \approx 1,3247$ is a root of the polynomial $X^3 - X - 1$), which is much less than 2^{p-2}. Thus, conjecturally, there exist many linear relations between MZVs of the same weight and none between those of different weight: in this direction, the theorem of Goncharov [13] and Terasoma [24] claims that $v_p \le c_p$ for all integers $p \ge 2$. It remains to prove the opposite inequality to show (i), but no non-trivial lower bound for v_p is yet known: even if classical relations give $v_2 = v_3 = v_4 = 1$, we do not know how to prove that $v_5 = 2$, which is equivalent to the irrationality of $\zeta(5)/(\zeta(3)\zeta(2))$. Conjecture 1 is also interesting because it implies the following one.

Conjecture 2. *The numbers* $\pi, \zeta(3), \zeta(5), \zeta(7), \zeta(9)$, *etc, are algebraically independent over* $\mathbb{Q}$.

This conjecture seems completely out of reach. As mentioned in Section 1, a number of diophantine results have been proved in weight 1, i.e, in the case of the Riemann zeta function and we saw that these results can all be proved by the study of certain series of the form

$$\sum_{k=1}^{\infty} \frac{P(k)}{(k)_{n+1}^{A}} \tag{2.1}$$

where $P(X) \in \mathbb{Q}[X]$, $n \ge 0$, $A \ge 1$. The above series can be written as a linear combination over $\mathbb{Q}$ of 1 and the values of zeta at integers. The crucial point is that we can find special polynomials P such that in these combinations only certain value of zeta occur: $\zeta(3)$ in case (i), values $\zeta(s)$ with s odd in cases (ii) and (iii). This comes from a symmetry linked to the very-well-poised properties of the series (2.1), which we summarize by the following result.

Proposition 1. *Let* $P \in \mathbb{Q}[X]$ *of degree at most* $A(n+1)-2$, *such that*

$$P(-n-X) = (-1)^{A(n+1)+1} P(X).$$

Then, the series (2.1) *is a linear combination, with rational coefficients, of* 1 *and* $\zeta(s)$ *with* s *an odd integer between* 3 *and* A.

2.2. **Well-poised-symmetry in several dimensions.** Our aim is to present two generalisations, in arbitrary depth, of the symmetry phenomenon stated in Proposition 1, and whose proofs are given in [11]. Even though no new diophantine results (like those presented in Section 1) have been obtained so far for the underlying MZVs possible, we hope that such generalisations will provide new ideas towards such theorems.

Our first result deals with "uncoupled" series, i.e, series over all p-tuples $(k_1, \ldots, k_p) \in \mathbb{N}^{*p}$:

Theorem 9. *Consider integers $p \geq 1$, $n \geq 0$ and $A \geq 1$. Let $P \in \mathbb{Q}[X_1, \dots, X_p]$ be a polynomial of degree $\leq A(n+1)-2$ with respect to each of the variables, such that*

$$P(X_1, \dots, X_{j-1}, -X_j - n, X_{j+1}, \dots, X_p) \\ = (-1)^{A(n+1)+1} P(X_1, \dots, X_{j-1}, X_j, X_{j+1}, \dots, X_p)$$

for any $j \in \{1, \dots, p\}$. Then, the multiple series

$$\sum_{k_1, \dots, k_p \geq 1} \frac{P(k_1, \dots, k_p)}{(k_1)_{n+1}^A \dots (k_p)_{n+1}^A} \tag{2.2}$$

is a polynomial with rational coefficients, of degree at most p, in the $\zeta(s)$, for s an odd integer between 3 and A.

For example, when $A = 3$ or $A = 4$, this series is a polynomial in $\zeta(3)$. When $p = 1$, we exactly obtain Proposition 1 (for all A).

From the point of view of diophantine applications, the main drawback of Theorem 9 is that the summation of $k_1, \dots, k_p$ is uncoupled. We now describe three disadvantages of uncoupled series.

First of all, uncoupled series always give polynomials in values of ζ at integers, even if we omit the symmetry condition in Theorem 9. This remark shows that MZVs cannot really appear in this setup. Secondly, let us consider again Ball's series

$$\mathbf{B}_n = n!^2 \sum_{k=1}^{\infty} (k + \frac{n}{2}) \frac{(k-n)_n (k+n+1)_n}{(k)_{n+1}^4}.$$

introduced in Section 1. For all integer n, $\mathbf{B}_n$ is a linear form in 1 and $\zeta(3)$; this follows from Proposition 1. For all integers $p \geq 1$, the series $\mathbf{B}_n^p$ is obviously an uncoupled series of the the form considered in Theorem 9 with

$$P(X_1, \dots, X_p) \\ = n!^{2p} (X_1 + \frac{n}{2}) \dots (X_p + \frac{n}{2}) (X_1 - n)_n \dots (X_p - n)_n (X_1 + n + 1)_n \dots (X_p + n + 1)_n$$

and $A = 4$. Therefore, $\mathbf{B}_n^p$ is a polynomial in $\zeta(3)$ of degree (at most) p, from which we could hope to deduce the transcendence of $\zeta(3)$. However, $\mathbf{B}_n^p$ does not contain anymore diophantine information than $\mathbf{B}_n$ and it can only gives the irrationality of $\zeta(3)$. Finally, the multiple series which appear in irrationality proofs are generally of the form

$$\sum_{k_1 \geq \dots \geq k_p \geq 1} \frac{P(k_1, \dots, k_p)}{(k_1)_{n+1}^A \dots (k_p)_{n+1}^A}, \tag{2.3}$$

i.e, the summation is over ordered indices; it is to this kind of series that one can apply the algorithm decribed in [10]. For example, when $p = 2$, $A = 2$ and

$$P(X_1, X_2) = n!(X_1 - X_2 + 1)_n (X_2 - n)_n (X_2)_{n+1},$$

Sorokin [23] shows that the sum (2.3) is exactly the linear form in 1 and $\zeta(3)$ used by Apéry. More generaly, a conjecture of Vasilyev [25] claimed that a certain multiple integral, equals to

$$n!^{p-\varepsilon} \sum_{k_1 \geq \cdots \geq k_p \geq 1} \frac{(k_1 - k_2 + 1)_n \ldots (k_{p-1} - k_p + 1)_n (k_p - n)_n}{(k_1)_{n+1}^2 \ldots (k_{p-1})_{n+1}^2 (k_p)_{n+1}^{2-\varepsilon}}, \tag{2.4}$$

is a rational linear form in zeta values at integers ≥ 2 of the same parity as $\varepsilon \in \{0, 1\}$. The integral formulation of this conjecture was proved in [30] and a refined version was proved in [17]: the method is to prove that the series (2.4) is also equal to a simple series to which Theorem 1 applies. Zlobin [28] recently obtained a completely different proof by a direct study of the series (2.4), in the spirit of the combinatorial methods developped in [10, 11]. It is then possible to prove results of essentially the same nature as those of [4, 19]: this confirms our feeling that multiple series with ordered indices are the interesting ones.

We showed in [10] that any convergent series of the form (2.3) can be written as a rational linear form in MZVs of weight at most pA and of depth at most p (this result was also obtained independently by Zlobin [27]). Furthermore, we produced an algorithm, implemented [9] in Pari, to explicitly compute such a linear combination. This enabled us to discover the symmetry property that we now describe in the special case of depth 2 for the reader's convenience.

Theorem 10. *Consider integers $n \geq 0$ et $A \geq 1$, with n even. Let $P \in \mathbb{Q}[X_1, X_2]$ be a polynomial in two variables, of degree $\leq A(n+1) - 2$ in each one, such that*

$$\begin{cases} P(X_1, X_2) = -P(X_2, X_1) \\ P(-n - X_1, X_2) = (-1)^{A(n+1)+1} P(X_1, X_2) \\ P(X_1, -n - X_2) = (-1)^{A(n+1)+1} P(X_1, X_2) \end{cases} \tag{2.5}$$

Then, the double series (2.3) *is a linear combination, with rational coefficients,*

- *of 1,*
- *of the values $\zeta(s)$ with s an odd integer such that $3 \leq s \leq 2A$,*
- *of the differences $\zeta(s, s') - \zeta(s', s)$ with s, s' odd integers such that $3 \leq s < s' \leq A$.*

(Let us note here that in the series (2.3), the variables $k_1, \ldots, k_p$ are linked by non-strict inequalities, as in [10], but contrary to the definition of MZVs. This does not cause any problems, since it is easy to go from statements with non-strict inequalities to statements with strict inequalities, and vice-versa.)

Of course, in (2.5), the third condition is a consequence of the first two. If $A = 4$, this theorem shows that the double series

$$\sum_{k_1 \geq k_2 \geq 1} \frac{P(k_1, k_2)}{(k_1)_{n+1}^4 (k_2)_{n+1}^4}$$

is a linear form in 1, $\zeta(3)$, $\zeta(5)$ and $\zeta(7)$ (which was far from obvious a priori since this a double series). For $A = 3$, we get a linear form in 1, $\zeta(3)$, $\zeta(5)$. Finally, for $A = 2$, we get a linear form in 1 and $\zeta(3)$.

To state our main result in arbitrary depth, we need the following notation. For integers $p \geq 0$ and $s_1, \ldots, s_p \geq 2$, we set

$$\zeta^{\mathrm{as}}(s_1, \ldots, s_p) = \sum_{\sigma \in \mathfrak{S}_p} \varepsilon_\sigma \zeta(s_{\sigma(1)}, \ldots, s_{\sigma(p)}),$$

where ε_σ is the signature of the permutation σ. We call such a linear combination of MZVs an *antisymmetric MZV* (even if, for $p \geq 2$, it is not an MZV in general). These are convergent series since each s_i is supposed ≥ 2. For $p = 1$, we have $\zeta^{\mathrm{as}}(s) = \zeta(s)$. The natural convention is to set $\zeta^{\mathrm{as}}(s_1, \ldots, s_p) = 1$ when $p = 0$ because there exists one unique bijection of the empty set onto itself. For $p = 2$, we have $\zeta^{\mathrm{as}}(s_1, s_2) = \zeta(s_1, s_2) - \zeta(s_2, s_1)$ and, when $p = 3$,

$$\begin{aligned} &\zeta^{\mathrm{as}}(s_1, s_2, s_3) \\ &\quad = \zeta(s_1, s_2, s_3) + \zeta(s_2, s_3, s_1) + \zeta(s_3, s_1, s_2) - \zeta(s_2, s_1, s_3) - \zeta(s_1, s_3, s_2) - \zeta(s_3, s_2, s_1). \end{aligned}$$

By definition, for all $\sigma \in \mathfrak{S}_p$, we have

$$\zeta^{\mathrm{as}}(s_{\sigma(1)}, \ldots, s_{\sigma(p)}) = \varepsilon_\sigma \zeta^{\mathrm{as}}(s_1, \ldots, s_p),$$

and $\zeta^{\mathrm{as}}(s_1, \ldots, s_p) = 0$ once two of the s_i's are equal. It seems reasonable to us that in general an antisymmetric MZV is not a polynomial in values of the Riemann zeta function. However, any "symmetric" MZV (defined as $\zeta^{\mathrm{as}}(s_1, \ldots, s_p)$ but omiting the signature ε_σ) is a polynomial in $\zeta(s)$ (by [15], Theorem 2.2).

Let $\mathscr{A}_p$ denotes the set of polynomials $P(X_1, \ldots, X_p) \in \mathbb{Q}[X_1, \ldots, X_p]$ such that:

$$\begin{cases} \text{For all } \sigma \in \mathfrak{S}_p, \text{ we have} \\ \qquad\qquad P(X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(p)}) = \varepsilon_\sigma P(X_1, X_2, \ldots, X_p). \\ \\ \text{For all } j \in \{1, \ldots, p\}, \text{ we have} \\ \qquad\qquad P(X_1, \ldots, X_{j-1}, -X_j - n, X_{j+1}, \ldots, X_p) \\ \qquad\qquad\qquad = (-1)^{A(n+1)+1} P(X_1, \ldots, X_{j-1}, X_j, X_{j+1}, \ldots, X_p). \end{cases}$$

There are redondances in these conditions. If the first one is satisfied, then it is enough to check the second one for one single value of j. For example, $\mathscr{A}_2$ is exactly the set of polynomials P satisfying the conditions (2.5). Moreover, if $P \in \mathscr{A}_p$ then P has the same degree in each variable $X_1, \ldots, X_p$. Clearly, the definition of $\mathscr{A}_p$ also depends on the parity of $A(n+1)$. We can now state our main result.

Theorem 11. *Consider integers $n \geq 0$ and $A, p \geq 1$, with n even. Let $P \in \mathscr{A}_p$ be of degree $\leq A(n+1) - 2$ in each of the variables. Then, the series*

$$\sum_{k_1 \geq \ldots \geq k_p \geq 1} \frac{P(k_1, \ldots, k_p)}{(k_1)_{n+1}^A \cdots (k_p)_{n+1}^A} \tag{2.6}$$

is a rational linear combination of products of the form

$$\zeta(s_1) \ldots \zeta(s_q) \zeta^{\mathrm{as}}(s'_1, \ldots, s'_{q'}),$$

where

$$\begin{cases} q, q' \geq 0 \text{ integers such that } 2q + q' \leq p, \\ s_1, \ldots, s_q, s'_1, \ldots, s'_{q'} \text{ odd integers } \geq 3, \\ s_i \leq 2A - 1 \text{ for all } i \in \{1, \ldots, q\}, \\ s'_i \leq A \text{ for all } i \in \{1, \ldots, q'\}. \end{cases} \tag{2.7}$$

When $q' = 0$, the antisymmetric MZV $\zeta^{\mathrm{as}}(s'_1, \ldots, s'_{q'})$ is equal to 1 and we obtain a product of values of ζ at odd integers. When $q = q' = 0$, this produit is empty and we obtain 1.

If $p = 1$, Theorem 11 states that (2.6) is a linear combination of 1 and the $\zeta(s)$ with odd s such that $3 \leq s \leq A$: this is just Theorem 1.

If $p = 2$, we obtain exactly Theorem 10.

If $p = 3$, the theorem states that the series is a linear combination of

- products of at most two values of ζ at odd integers ≥ 3,
- antisymmetric MZVs $\zeta^{\mathrm{as}}(s_1, s_2)$ with $s_1, s_2 \geq 3$ odd,
- antisymmetric MZVs $\zeta^{\mathrm{as}}(s_1, s_2, s_3)$ with $s_1, s_2, s_3 \geq 3$ odd.

In depth $p \geq 4$, terms such as $q \geq 1$ and $q' \geq 2$ can appear: it seems that the series is not always the sum of a polynomial in values of $\zeta(s)$ (with s odd) and of a linear combination of antisymmetric MZVs $\zeta^{\mathrm{as}}(s_1, \dots, s_q)$ with $s_1, \dots, s_q$ odd.

2.3. **Corollaries and examples.** When $A \leq 2$, we necessarily have $q' = 0$ in all the products, which implies the following corollary.

Corollary 1. *Under the hypotheses of Theorem 11, if $A \leq 2$, then the series (2.6) is a polynomial in $\zeta(3)$ with rationals coefficients.*

Theorem 11 also contains, for example, the following special case.

Corollary 2. *Consider integers $n, r, t, \varepsilon \geq 0$ and $A, p \geq 1$, with n even, such that*

$$\varepsilon \equiv (A+1)(n+1) + 1 \bmod 2$$

and

$$\varepsilon + (4r+2)p + 2t \leq (A-1)(n+1) + 4r.$$

Then, the convergent series

$$\sum_{k_1 \geq \ldots \geq k_p \geq 1} \Big[\prod_{i=1}^{p} (k_i + \frac{n}{2})\Big]^{\varepsilon} \frac{\Big[\prod_{1 \leq i < j \leq p} (k_i - k_j - r)_{2r+1} (k_i + k_j + n - r)_{2r+1}\Big] \Big[\prod_{i=1}^{p} (k_i - t)_{2t+n+1}\Big]}{(k_1)_{n+1}^{A} \dots (k_p)_{n+1}^{A}}$$

is a linear combination as described in Theorem 11.

An example of application of this corollary is the following series (in which we take $t = 0$ and the Pochhammer symbols $(k_i)_{n+1}$ at the numerator cancel out with those at the denominator):

$$\sum_{k_1 \geq k_2 \geq k_3 \geq 1} \big(k_1 + \frac{1}{2}\big)\big(k_2 + \frac{1}{2}\big)\big(k_3 + \frac{1}{2}\big) \times \frac{(k_1 - k_2)(k_2 - k_3)(k_1 - k_3)(k_1 + k_2 + 1)(k_1 + k_3 + 1)(k_2 + k_3 + 1)}{(k_1)_2^4 \, (k_2)_2^4 \, (k_3)_2^4}$$
$$= -\frac{1}{4} - \zeta(3) + \frac{1}{4}\zeta(5) + \zeta(3)^2 - \frac{1}{4}\zeta(7).$$

$$\sum_{k_1 \geq k_2 \geq 1} \big(k_1 + \frac{1}{2}\big)\big(k_2 + \frac{1}{2}\big) \frac{(k_1 - k_2 - 1)_3 (k_1 + k_2)_3 (k_1 - 1)_4 (k_2 - 1)_4}{(k_1)_2^7 \, (k_2)_2^7}$$
$$= -1156 + 891\,\zeta(3) + \frac{189}{2}\zeta(5) + 78\big(\zeta(5,3) - \zeta(3,5)\big).$$

Finally, let us mention that the series described in the above theorems are related to the multiple hypergeometric series that can be associated to root systems: see [7, 8] for example as well as the discussion in [11].

References

[1] G. E. Andrews, *Problems and prospects for basic hypergeometric functions*, Theory and application of special functions, R. A. Askey, ed., Math. Res. Center, Univ. Wisconsin, Publ. No. 35, Academic Press, New York, pp. 191–224, 1975.

[2] R. Apéry, *Irrationalité de* $\zeta(2)$ *et* $\zeta(3)$, Journées Arithmétiques (Luminy, 1978), Astérisque, no. **61**, 1979, p. 11–13.

[3] W. N. Bailey, *Generalized hypergeometric series*, Cambridge University Press, Cambridge, 1935.

[4] K. Ball and T. Rivoal, *Irrationalité d'une infinité de valeurs de la fonction zêta aux entiers impairs*, Invent. Math. **146** (2001), no. 1, p. 193–207.

[5] F. Beukers, *A note on the irrationality of* $\zeta(2)$ *and* $\zeta(3)$, Bull. London Math. Soc. **11** (1979), 268–272.

[6] F. Beukers, *Padé approximations in Number Theory* in "Padé approximation and its applications", Amsterdam 1980, LNM 888, 90–99, Springer (1981).

[7] G. Bhatnagar and M. Schlosser, C_n *and* D_n *very well-poised* ${}_{10}\phi_9$ *transformations*, Constr. Approx. **14** (1998), p. 531–567.

[8] H. Coksun, *An Elliptic* BC_n *Bailey Lemma, Multiple Rogers–Ramanujan Identities and Euler's Pentagonal Number Theorems*, Trans. Am. Math. Soc. **360**.10 (2008), 5397–5433.

[9] J. Cresson, S. Fischler and T. Rivoal, Algorithm available at `http//www.math.u-psud.fr/~fischler/algo.html`.

[10] J. Cresson, S. Fischler and T. Rivoal *Séries hypergéométriques multiples et polyzêtas*, Bull. SMF **136**.1 (2008), 97–145.

[11] J. Cresson, S. Fischler and T. Rivoal, *Phénomènes de symétrie dans des formes linéaires en polyzêtas*, J. reine angew. Math. **617** (2008), 109–152.

[12] S. Fischler, *Irrationalité de valeurs de zêta (d'après Apéry, Rivoal, ...)*, Sém. Bourbaki 2002/03, Astérisque **294**, 2004, exp. no. 910, p. 27–62.

[13] A. Goncharov, *Multiple polylogarithms and mixed Tate motives*, preprint available at `http//front.math.ucdavis.edu/math.AG/0103059`, 2001.

[14] L.A. Gutnik, *The irrationality of certain quantities involving* $\zeta(3)$, Russ. Math. Surv. **34**.3 (1979), 200. In russian in Acta Arith. **42**.3 (1983), 255–264.

[15] M. Hoffman, *Multiple harmonic series*, Pacific J. of Math. **152** (1992), p. 275–290.

[16] G. Gasper et M. Rahman, *Basic hypergeometric series*, Encyclopedia of Mathematics And Its Applications 35, Cambridge University Press, Cambridge, 1990.

[17] C. Krattenthaler and T. Rivoal, *Hypergéométrie et fonction zêta de Riemann*, Memoirs of the AMS **186** (2007), 93 pages.

[18] Yu.V. Nesterenko, *On the linear independence of numbers*, Mosc. Univ. Math. Bull.**40**.1 (1985), 69–74, translation of Vest. Mosk. Univ., Ser. I (1985), no. 1, 46–54.

[19] T. Rivoal, *La fonction zêta de Riemann prend une infinité de valeurs irrationnelles aux entiers impairs*, C. R. Acad. Sci. Paris, Ser. I **331** (2000), no. 4, p. 267–270.

[20] T. Rivoal, *Irrationalité d'au moins un des neuf nombres* $\zeta(5)$, $\zeta(7), \ldots, \zeta(21)$, Acta Arith. **103**.2 (2002), 157–167.

[21] T. Rivoal, *Séries hypergéométriques et irrationalité des valeurs de la fonction zêta de Riemann*, J. Théor. Nombres Bordeaux **15**.1 (2003), 351–365.

[22] L. J. Slater, *Generalized hypergeometric functions*, Cambridge University Press, Cambridge, 1966.

[23] V. Sorokin, *Apéry's theorem*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. [Moscow Univ. Math. Bull.] **53** (1998), no. 3, p. 48–53 [48–52].

[24] T. Terasoma, *Mixed Tate motives and multiple zeta values*, Invent. Math. **149** (2002), no. 2, p. 339–369.

[25] D. Vasilyev, *Approximations of zero by linear forms in values of the Riemann zeta-function*, Doklady Nats. Akad. Nauk Belarusi **45** (2001), no. 5, p. 36–40, in russian ; extended version in english anglais : *On small linear forms for the values of the Riemann zeta-function at odd points*, preprint no.1 (558), Nat. Acad. Sci. Belarus, Institute Math., Minsk (2001), 14 pages.

[26] M. Waldschmidt, *Valeurs zêta multiples : une introduction*, J. Théor. Nombres Bordeaux **12** (2000), no. 2, p. 581–595.

[27] S. Zlobin, *Expansion of multiple integrals in linear forms*, Mat. Zametki [Math. Notes] **77** (2005), no. 5, 683–706 [630–652].

[28] S. Zlobin, *Properties of coefficients of certain linear forms in generalized polylogarithms*, Fundamentalnaya i Prikladnaya Matematika [Fundamental and Applied Mathemetics] **11** (2005), no. 6, p. 41–58,

[29] W. Zudilin, *One of the numbers* $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ *is irrational*, Uspekhi Mat. Nauk [Russian Math. Surveys] **56** (2001), no. 4, p. 149–150 [774–776].

[30] W. Zudilin, *Well-poised hypergeometric service for diophantine problems of zeta values*, J. Théor. Nombres Bordeaux **15** (2003), no. 2, p. 593–626.

[31] W. Zudilin, *Arithmetic of linear forms involving odd zeta values*, à paraître au J. Théor. Nombres Bordeaux. `http://arXiv.org/abs/math.NT/0206176`.

T. Rivoal, Institut Fourier, CNRS et Université Grenoble 1, 100 rue des Maths, BP 74, 38402 Saint-Martin d'Hères cedex, France.