MULTIVARIATE p-ADIC FORMAL CONGRUENCES AND INTEGRALITY OF TAYLOR COEFFICIENTS OF MIRROR MAPS

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ABSTRACT. We generalise Dwork's theory of p-adic formal congruences from the uni-variate to a multi-variate setting. We apply our results to prove integrality assertions on the Taylor coefficients of (multi-variable) mirror maps. More precisely, with $\mathbf{z} = (z_1, z_2, \ldots, z_d)$, we show that the Taylor coefficients of the multi-variable series $q(\mathbf{z}) = z_i \exp(G(\mathbf{z})/F(\mathbf{z}))$ are integers, where $F(\mathbf{z})$ and $G(\mathbf{z}) + \log(z_i)F(\mathbf{z})$, $i = 1, 2, \ldots, d$, are specific solutions of certain GKZ systems. This result implies the integrality of the Taylor coefficients of numerous families of multi-variable mirror maps of Calabi–Yau complete intersections in weighted projective spaces, as well as of many one-variable mirror maps in the "*Tables of Calabi–Yau equations*" [arχiv:math/0507430] of Almkvist, van Enckevort, van Straten and Zudilin. In particular, our results prove a conjecture of Batyrev and van Straten in [*Comm. Math. Phys.* **168** (1995), 493–533] on the integrality of the Taylor coefficients of canonical coordinates for a large family of such coordinates in several variables.

1. Introduction and statement of the results

In [7, 8, 9, 10, 11], Dwork developed a sophisticated theory for proving analytic and arithmetic properties of solutions to (p-adic) differential equations. In [7, 11], he focussed on the case of hypergeometric differential equations. In particular, the article [11] contains a "formal congruence" criterion that enabled him to address the analytic continuation of quotients of certain solutions and to establish arithmetic properties satisfied by exponentials of such quotients. These exponentials of ratios of solutions to hypergeometric differential equations (in fact, of Picard–Fuchs equations) have recently received great attention in mathematical physics and algebraic geometry under the name of *canonical coordinates*. Their compositional inverses, known as *mirror maps*, are an important ingredient in the computation of the Yukawa coupling in the theory of mirror symmetry. It is conjectured that the coefficients in the Lambert series expansion of the Yukawa coupling produce Gromov–Witten invariants of classes of rational curves.

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It is only relatively recent, that Dwork's theory has been systematically applied to obtain general arithmetic results on the Taylor coefficients of mirror maps. Partial results in this direction were found by Lian and Yau [18, 19], by Zudilin [22], and by Kontsevich, Schwarz and Vologodsky [13, 21]. The (so far) strongest and most general results are contained in [6, 14, 15], where, in particular, numerous integrality results for the Taylor coefficients of univariate mirror maps of Calabi–Yau complete intersections in weighted projective spaces are proven, improving and refining the afore-mentioned results by Lian and Yau, and by Zudilin. However, all these results do not touch the case of *multi-variable* mirror maps, upon which they are not able to say anything. The goal of this paper is to set the basis of a theory which is capable to address questions of integrality of Taylor coefficients of multi-variable mirror maps, and to apply this theory systematically to large classes of such mirror maps.

1.1. **Multivariate theory of formal congruences.** The proof strategy in [6, 14, 15, 18, 19, 22] for obtaining integrality assertions on the Taylor coefficients of one-variable mirror maps is crucially based on a series of reductions and results, of which the corner stones are:

(D1) the conversion of the integrality problem to a p-adic problem;
(D2) a lemma due to Dieudonné and Dwork (cf. [17, Ch. 14, p. 76]) providing a criterion for deciding whether a power series with coefficients over $\mathbb{Q}_p$ has coefficients in $\mathbb{Z}_p$;
(D3) a reduction lemma for harmonic numbers due to the authors (cf. [14, Lemma 1, respectively Lemma 5] and [15, Lemma 3]);
(D4) a combinatorial lemma due to Dwork [11, Lemma 4.2] for rearranging sums that appear in this context in a way tailor-made for p-adic analysis;
(D5) Dwork's theorem on formal congruences (cf. [11, Theorem 1.1]).

We point out that Lian and Yau, and Zudilin do not need item (D3) due to the nature of the special families of mirror maps that they were considering. Indeed, item (D3) is the decisive novelty which enabled the authors to arrive at their general sets of results in [14, 15]. On the side, we remark that Zudilin also condenses (D4) and (D5) into one step in the proof of his main result in [22]. However, in order to arrive at the general results in [14, 15], it turned out to be necessary to follow the full path outlined by (D1)–(D5) above, as attempts to lift Zudilin's variation to this generality failed.

With the exception of (D1), which trivially extends to the multi-variable case, for none of the above items there exist multi-variate extensions in the current literature. In particular, no approach for attacking integrality questions for multi-variable mirror maps has been available so far.

In this paper, we present multi-variate versions for all of (D2)–(D5); all of them seem to be new. Our multi-variate extension of (D2) is the content of Lemma 1 in Section 2, our multi-variate version of (D3) can be found in Lemma 3 in Section 2, while Lemma 5 in Section 6 provides our multi-variate extension of (D4). On the other hand, we state our multi-variate extension of item (D5) in Theorem 1 below. Since its one-variable special case enabled Dwork to address the question of analytic extension of certain ratios of generalised p-adic hypergeometric series in one variable, we expect our result below to be the appropriate tool for analogous studies of multivariable p-adic hypergeometric series.

For the statement of our multi-variate theorem on formal congruences, we need some standard multi-index notation. Namely, given a positive integer d, a real number λ, and vectors $\mathbf{m} = (m_1, m_2, \dots, m_d)$ and $\mathbf{n} = (n_1, n_2, \dots, n_d)$ in $\mathbb{R}^d$, we write $\mathbf{m} + \mathbf{n}$ for $(m_1 + n_1, m_2 + n_2, \dots, m_d + n_d)$, $\lambda\mathbf{m}$ for $(\lambda m_1, \lambda m_2, \dots, \lambda m_d)$, we write $\mathbf{m} \geq \mathbf{n}$ if and only if $m_i \geq n_i$ for $i = 1, 2, \dots, d$, and we write $\mathbf{0}$ for $(0, 0, \dots, 0) \in \mathbb{Z}^d$ and $\mathbf{1}$ for $(1, 1, \dots, 1) \in \mathbb{Z}^d$.

Theorem 1. *Let $A : \mathbb{Z}^d_{\geq 0} \to \mathbb{Z}_p \setminus \{0\}$ and $g : \mathbb{Z}^d_{\geq 0} \to \mathbb{Z}_p \setminus \{0\}$ be maps satisfying the following three properties:*

(i) $v_p\big(A(\mathbf{0})\big) = 0$;

(ii) $A(\mathbf{n}) \in g(\mathbf{n})\mathbb{Z}_p$;

(iii) *for all non-negative integers s and all integer vectors $\mathbf{v}, \mathbf{u}, \mathbf{n} \in \mathbb{Z}^d$ with $\mathbf{v}, \mathbf{u}, \mathbf{n} \geq \mathbf{0}$ with $0 \leq v_i < p$ and $0 \leq u_i < p^s$, $i = 1, 2, \dots, d$,*

$$\frac{A(\mathbf{v} + p\mathbf{u} + \mathbf{n}p^{s+1})}{A(\mathbf{v} + p\mathbf{u})} - \frac{A(\mathbf{u} + \mathbf{n}p^s)}{A(\mathbf{u})} \in p^{s+1} \frac{g(\mathbf{n})}{g(\mathbf{v} + p\mathbf{u})} \mathbb{Z}_p.$$

Then, for all non-negative integers s and all integer vectors $\mathbf{m}, \mathbf{K}, \mathbf{a} \in \mathbb{Z}^d$ with $\mathbf{m} \geq \mathbf{0}$ and $0 \leq a_i < p$, $i = 1, 2, \dots, d$, we have

$$\sum_{p^s\mathbf{m} \leq \mathbf{k} \leq p^s(\mathbf{m}+\mathbf{1})-\mathbf{1}} \big(A(\mathbf{a} + p\mathbf{k})A(\mathbf{K} - \mathbf{k}) - A(\mathbf{a} + p(\mathbf{K} - \mathbf{k}))A(\mathbf{k})\big) \in p^{s+1}g(\mathbf{m})\mathbb{Z}_p,$$

where we extend A to $\mathbb{Z}^d$ by $A(\mathbf{n}) = 0$ if there is an i such that $n_i < 0$.

While the proofs of Lemmas 1 and 5 (corresponding to items (D2) and (D4)) are relatively straightforward extensions of the one-variable proofs given in [17, Ch. 14, p. 76] and [11, proof of Lemma 4.2], respectively, the proofs of Lemma 3 and Theorem 1 (corresponding to items (D3) and (D5)) need new ideas. The proof of Lemmas 1 is given in Section 3. Section 5 is devoted to the proof of Lemma 3. This proof was kindly provided by an anonymous referee. (For our original proof, see [16, Sec. 5].) Even in the one-dimensional case, this proof (as well as the one in [16]) is new, as it simplifies the earlier proofs [14, proofs of Lemma 1, respectively Lemma 5] and [15, proof of Lemma 3]. In fact, it turned out, that these earlier proofs could not be extended to the multi-variate case. The proof of Lemma 5 can be found in Section 6. Finally, in Section 7 we prove Theorem 1.

The main application of our multi-variate theory of formal congruences that we present in this paper concerns the proof that, for a large class of multi-variable mirror maps, their Taylor coefficients are integers. We state the corresponding general theorem in the next subsection. The subsequent subsection collects some particularly interesting special cases and consequences.

1.2. **A family of GKZ functions and their associated mirror maps.** In order to state the results in this section conveniently, we need to further enlarge our set of multi-index notations given before Theorem 1. Given vectors $\mathbf{m} = (m_1, m_2, \dots, m_d)$ and $\mathbf{n} =$

$(n_1, n_2, \dots, n_d)$ in $\mathbb{R}^d$, we write $\mathbf{m}\cdot\mathbf{n}$ for the scalar product $m_1n_1+m_2n_2+\cdots+m_dn_d$, and we write $|\mathbf{m}|$ for $m_1+m_2+\cdots+m_d$. Furthermore, given a vector $\mathbf{z}=(z_1,z_2,\dots,z_d)$ of variables and $\mathbf{n}=(n_1,n_2,\dots,n_d)\in\mathbb{Z}^d$, we write $\mathbf{z}^{\mathbf{n}}$ for the *product* $z_1^{n_1}z_2^{n_2}\cdots z_d^{n_d}$. On the other hand, if n is an integer, we write $\mathbf{z}^n$ for the *vector* $(z_1^n, z_2^n, \dots, z_d^n)$.

Given k vectors $\mathbf{N}^{(j)}=(N_1^{(j)}, N_2^{(j)}, \dots, N_d^{(j)})\in\mathbb{Z}^d$, $j=1,\dots,k$, with $\mathbf{N}^{(j)}\geq\mathbf{0}$, let us define the series

$$F_{\mathbf{N}}(\mathbf{z})=\sum_{\mathbf{m}\geq\mathbf{0}}\mathbf{z}^{\mathbf{m}}\prod_{j=1}^{k}\frac{(\mathbf{N}^{(j)}\cdot\mathbf{m})!}{\prod_{i=1}^{d}m_i!^{N_i^{(j)}}}=\sum_{\mathbf{m}\geq\mathbf{0}}\mathbf{z}^{\mathbf{m}}\prod_{j=1}^{k}\frac{\left(\sum_{i=1}^{d}N_i^{(j)}m_i\right)!}{\prod_{i=1}^{d}m_i!^{N_i^{(j)}}}.$$

Since the Taylor coefficients of $F_{\mathbf{N}}(\mathbf{z})$ are products of multinomial coefficients, it follows that $F_{\mathbf{N}}(\mathbf{z})\in 1+\sum_{i=1}^{d}z_i\mathbb{Z}[[\mathbf{z}]]$, where $\mathbb{Z}[[\mathbf{z}]]$ denotes the set of all (formal) power series in the variables $z_1, z_2, \dots, z_d$ with integer coefficients.

This series is a GKZ hypergeometric function ([1]) and it is known to "come from geometry," i.e., it can be viewed as the period of certain multi-parameter families of algebraic varieties in a product of weighted projective spaces (see [12] for details). It satisfies a linear differential system $\{\mathcal{L}_{i,\mathbf{N}}(F_{\mathbf{N}})=0 : i=1,\dots,d\}$ defined by the operators

$$\mathcal{L}_{i,\mathbf{N}}=\theta_i^{N_i^{(1)}+\cdots+N_i^{(k)}}-z_i\prod_{j=1}^{k}\prod_{r_j=1}^{N_i^{(j)}}\Big(\sum_{\ell=1}^{d}N_\ell^{(j)}\theta_\ell+r_j\Big),\quad i=1,\dots,d,$$

where $\theta_i=z_i\frac{\partial}{\partial z_i}$. Amongst the other solutions of this system, we find the d functions $\log(z_i)F_{\mathbf{N}}(\mathbf{z})+G_{i,\mathbf{N}}(\mathbf{z})$, $i=1,\dots,d$, defined by

$$G_{i,\mathbf{N}}(\mathbf{z})=\sum_{\mathbf{m}\geq\mathbf{0}}\mathbf{z}^{\mathbf{m}}\Big(\sum_{j=1}^{k}N_i^{(j)}H_{\mathbf{N}^{(j)}\cdot\mathbf{m}}-H_{m_i}\sum_{j=1}^{k}N_i^{(j)}\Big)\prod_{j=1}^{k}\frac{(\mathbf{N}^{(j)}\cdot\mathbf{m})!}{\prod_{i=1}^{d}m_i!^{N_i^{(j)}}}.$$

Here and in the rest of the article, $H_m=\sum_{j=1}^{m}1/j$ denotes the m-th harmonic number, with the convention $H_0=0$.

This set of solutions enables us to define d *canonical coordinates* $q_{i,\mathbf{N}}(\mathbf{z})$ by

$$q_{i,\mathbf{N}}(\mathbf{z})=z_i\exp\big(G_{i,\mathbf{N}}(\mathbf{z})/F_{\mathbf{N}}(\mathbf{z})\big),$$

which are objects with many fundamental properties for the "mirror symmetry" study of the underlying multi-parameter families of varieties. The compositional inverse of the map

$$\mathbf{z}\mapsto(q_{1,\mathbf{N}}(\mathbf{z}), q_{2,\mathbf{N}}(\mathbf{z}),\dots,q_{d,\mathbf{N}}(\mathbf{z}))$$

defines the vector $(z_{1,\mathbf{N}}(\mathbf{q}), z_{2,\mathbf{N}}(\mathbf{q}),\dots,z_{d,\mathbf{N}}(\mathbf{q}))$ of *mirror maps*. In this paper, by abuse of terminology, we will also use the term "mirror map" for any canonical coordinate. ([2])

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[1]See [20] for an introduction to these functions, which are a far-reaching generalisation of the classical hypergeometric functions to several variables.

[2]Canonical coordinates and mirror maps have distinct geometric meanings. However, in the number-theoretic study undertaken in the present paper, they play strictly the same role, because $q_{i,\mathbf{N}}(\mathbf{z})\in z_i\mathbb{Z}[[\mathbf{z}]]$, $i=1,2,\dots,d$, implies that $z_{i,\mathbf{N}}(\mathbf{q})\in q_i\mathbb{Z}[[\mathbf{q}]]$, $i=1,2,\dots,d$, and conversely.

Let us define the series

$$G_{\mathbf{L},\mathbf{N}}(\mathbf{z}) = \sum_{\mathbf{m}\geq\mathbf{0}} \mathbf{z}^{\mathbf{m}} H_{\mathbf{L}\cdot\mathbf{m}} \prod_{j=1}^{k} \frac{(\mathbf{N}^{(j)}\cdot\mathbf{m})!}{\prod_{i=1}^{d} m_i!^{N_i^{(j)}}} \in \sum_{i=1}^{d} z_i\mathbb{Q}[[\mathbf{z}]],$$

where $\mathbf{L}\in\mathbb{Z}^d$ is $\geq \mathbf{0}$. For any $i=1,\dots,d$, the function $G_{i,\mathbf{N}}(\mathbf{z})$ is a finite linear combination with integer coefficients in the functions $G_{\mathbf{L},\mathbf{N}}(\mathbf{z})$, where the summation runs over various vectors $\mathbf{L}$, each one with the property that $\mathbf{0}\leq\mathbf{L}\leq\mathbf{N}^{(j(\mathbf{L}))}$ for some $j(\mathbf{L})\in\{1,\dots,k\}$. Therefore, the following theorem concerns as well our mirror maps $q_{i,\mathbf{N}}(\mathbf{z})\in z_i\mathbb{Z}[[\mathbf{z}]]$ (see Corollary 1).

Theorem 2. *Let d and k be positive integers. For all vectors $\mathbf{L}=(L_1,L_2,\dots,L_d)\in\mathbb{Z}^d$ and $\mathbf{N}^{(j)}=(N_1^{(j)},N_2^{(j)},\dots,N_d^{(j)})\in\mathbb{Z}^d$, $j=1,2,\dots,k$, with $\mathbf{0}\leq\mathbf{L}\leq\mathbf{N}^{(1)}$, $\mathbf{N}^{(2)}\geq\mathbf{0},\dots,\mathbf{N}^{(k)}\geq\mathbf{0}$, we have*

$$q_{\mathbf{L},\mathbf{N}}(\mathbf{z}) := \exp\big(G_{\mathbf{L},\mathbf{N}}(\mathbf{z})/F_{\mathbf{N}}(\mathbf{z})\big)\in\mathbb{Z}[[\mathbf{z}]].$$

Remarks 1. (a) Given the fact that the canonical coordinates $q_{i,\mathbf{N}}(\mathbf{z})$ (which, in their turn, define the mirror maps $z_{i,\mathbf{N}}(\mathbf{q})$) can be expressed as products of several series of the form $q_{\mathbf{L},\mathbf{N}}(\mathbf{z})$ (with varying $\mathbf{L}$), we call $q_{\mathbf{L},\mathbf{N}}(\mathbf{z})$ a *mirror-type map.*

(b) By carefully going through our arguments, one sees that minor modifications lead to the slightly stronger statement that, under the assumptions of Theorem 2, we have

$$\exp\big(G_{\mathbf{L},\mathbf{N}}(\mathbf{z})/F_{\mathbf{N}}(\mathbf{z})\big)\in\prod_{j=2}^{k}\big(\min\{N_1^{(j)},N_2^{(j)},\dots,N_d^{(j)}\}\big)!\,\mathbb{Z}[[\mathbf{z}]].$$

The statement of the theorem might suggest that $\mathbf{N}^{(1)}$ plays a special role amongst the vectors $\mathbf{N}^{(1)},\mathbf{N}^{(2)},\dots,\mathbf{N}^{(k)}$. Of course, this is not the case: by symmetry, given any $j\in\{1,\dots,k\}$, a similar result holds for any $\mathbf{L}$ such that $\mathbf{0}\leq\mathbf{L}\leq\mathbf{N}^{(j)}$. This remark implies the following result for the mirror maps $q_{i,\mathbf{N}}(\mathbf{z})\in z_i\mathbb{Z}[[\mathbf{z}]]$, proving a conjecture of Batyrev and van Straten [3, Conjecture 7.3.4] for a large family of canonical coordinates in several variables.

Corollary 1. *Let d,k be positive integers. For all vectors $\mathbf{N}^{(j)}=(N_1^{(j)},N_2^{(j)},\dots,N_d^{(j)})\in\mathbb{Z}^d$, $j=1,2,\dots,k$, with $\mathbf{N}^{(1)}\geq\mathbf{0}$, $\mathbf{N}^{(2)}\geq\mathbf{0},\dots,\mathbf{N}^{(k)}\geq\mathbf{0}$, we have $q_{i,\mathbf{N}}(\mathbf{z})\in\mathbb{Z}[[\mathbf{z}]]$, $i=1,2,\dots,d$.*

We outline the proof of Theorem 2 in Section 2, thereby showing how the various pieces of our multi-variate theory of formal congruences fit together in order to prove integrality assertions for multi-variable mirror(-type) maps. The details are deferred to Sections 3–9.

1.3. **Consequences of Theorem 2.** In order to illustrate the range of applicability of Theorem 2, we collect in this subsection some examples and applications that are of particular interest to multi-variable *and* one-variable mirror-type maps.

(1) A classical multi-variate example, studied in detail in [3, Sec. 7] and [20, Sec. 8.4], is the case of the two parameters (w and z say) family of hypersurfaces V of degree $(3,3)$ in

$\mathbb{P}^2(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C})$, which is a family of Calabi–Yau threefolds. The periods of the associated mirror family of Calabi–Yau hypersurfaces can be expressed in term of the double series

$$F(w,z) = \sum_{m\geq 0}\sum_{n\geq 0} \frac{(3m+3n)!}{m!^3 n!^3}\, w^m z^n, \tag{1.1}$$

which is symmetric and holomorphic in $\{(w,z) \in \mathbb{C}^2 : |w|^{1/3} + |z|^{1/3} < \frac{1}{3}\}$. It is a solution of the linear differential system $\{\mathcal{L}_1(F) = 0, \mathcal{L}_2(F) = 0\}$ defined by the operators

$$\begin{cases} \mathcal{L}_1 = \theta_1^3 - w(3\theta_1 + 3\theta_2 + 1)(3\theta_1 + 3\theta_2 + 2)(3\theta_1 + 3\theta_2 + 3), \\ \mathcal{L}_2 = \theta_2^3 - z(3\theta_1 + 3\theta_2 + 1)(3\theta_1 + 3\theta_2 + 2)(3\theta_1 + 3\theta_2 + 3), \end{cases}$$

where $\theta_1 = w\frac{\partial}{\partial w}$ and $\theta_2 = z\frac{\partial}{\partial z}$.

Two solutions of this system are of the form $\log(w)F(w,z)+G_1(w,z)$ and $\log(z)F(w,z)+G_2(w,z)$ where $G_1(w,z)$ and $G_2(w,z)$ are holomorphic in $\{(w,z) \in \mathbb{C}^2 : |w|^{1/3}+|z|^{1/3} < \frac{1}{3}\}$, and are given explicitly by

$$G_1(w,z) = \sum_{m\geq 0}\sum_{n\geq 0} \big(3H_{3m+3n} - 3H_m\big) \frac{(3m+3n)!}{m!^3 n!^3}\, w^m z^n,$$

$$G_2(w,z) = \sum_{m\geq 0}\sum_{n\geq 0} \big(3H_{3m+3n} - 3H_n\big) \frac{(3m+3n)!}{m!^3 n!^3}\, w^m z^n.$$

Let us now define the two variable mirror maps $q_1(w,z) = w\exp\big(G_1(w,z)/F(w,z)\big)$ and $q_2(w,z) = z\exp\big(G_2(w,z)/F(w,z)\big)$. Here, $q_1(w,z) = q_2(z,w)$, but this is not the case in general. It was observed in the early developments of mirror symmetry theory that $q_1(w,z)$ and $q_2(w,z)$ seem to have integral Taylor coefficients (see the end of Section 7.1 in [3] for example). Corollary 1 with $d = 2$, $k = 1$, $\mathbf{N}^{(1)} = (3,3)$ now provides a proof for this observation.

(2) Interesting consequences result also by considering the series expansion $q_{\mathbf{L},\mathbf{N}}(\mathbf{z})$ for cases where some or all of the variables z_i are equal to each other. The obtained series is obviously still a formal power series. Furthermore, since the initial power series has integer coefficients, any such specialisation leads again to a series with integer coefficients. In this way, we can construct many new mirror-type maps, and, for several of them, this leads to proofs of conjectures in the literature on the integrality of their Taylor coefficients.

Here, we provide details for a corresponding example derived from the mirror-type map of item (1). Subsequently, Item (3) will address another family of one-variable examples derived from two-variable series, which, for example, includes the series whose coefficients form the famous sequences that appear in Apéry's proof of the irrationality of $\zeta(2)$ and $\zeta(3)$. Finally, in Item (4), we mention briefly certain cases studied in [1, 2, 3].

We put $w = z$ in the example (1.1) considered in Item (1) above and get

$$f(z) = \sum_{m\geq 0}\sum_{n\geq 0} z^{m+n} \frac{(3m+3n)!}{m!^3 n!^3} = \sum_{k=0}^{\infty} z^k \frac{(3k)!}{k!^3} \sum_{j=0}^{k} \binom{k}{j}^3$$

after rearrangement. This map is studied in [3, Sec. 7.3], where it is shown to be of significance in the theory of mirror symmetry. The function f satisfies a Fuchsian differential equation of order 4 with maximal unipotent monodromy at the origin: it is annihilated by the minimal operator

$$\theta^4 - 3z(7\theta^2 + 7\theta + 2)(3\theta + 1)(3\theta + 2) - 72z^2(3\theta + 5)(3\theta + 4)(3\theta + 2)(3\theta + 1).$$

Another solution is $g(z) + \log(z)f(z)$, where $g(z)$ is given by

$$g(z) = \sum_{k=0}^{\infty} z^k \frac{(3k)!}{k!^3} \sum_{j=0}^{k} \binom{k}{j}^3 (3H_{3k} - 3H_{k-j}).$$

The function $g(z)$ is a linear combination with integer coefficients of the functions

$$g_{\mathbf{L}}(z) = \sum_{k=0}^{\infty} z^k \frac{(3k)!}{k!^3} \sum_{j=0}^{k} \binom{k}{j}^3 H_{L_1 j + L_2(k-j)},$$

where $\mathbf{L} = (L_1, L_2) \in \mathbb{Z}^2$ is such that $0 \le L_1, L_2 \le 3$. For these $\mathbf{L}$, equating the variables in Theorem 2 leads to

$$\exp\big(g_{\mathbf{L}}(z)/f(z)\big) \in \mathbb{Z}[[z]],$$

which, in particular, implies the new result that $z\exp\big(g(z)/f(z)\big) \in z\mathbb{Z}[[z]]$.

(3) For any integers α, β such that $0 \le \beta \le \alpha$, we consider the function

$$\sum_{m\ge 0}\sum_{n\ge 0} \left(\frac{(m+n)!}{m!\,n!}\right)^{\alpha-\beta} \left(\frac{(2m+n)!}{m!^2 n!}\right)^{\beta} w^m z^n. \tag{1.2}$$

The specialisation $w = z$ produces the function

$$\mathcal{A}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \binom{k}{j}^{\alpha} \binom{k+j}{j}^{\beta}\right) z^k,$$

to which we associate the function $\mathcal{B}_{\alpha,\beta}(z) + \log(z)\mathcal{A}_{\alpha,\beta}(z)$ defined by

$$\mathcal{B}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \binom{k}{j}^{\alpha} \binom{k+j}{j}^{\beta} \big((\alpha-\beta)H_k - \alpha H_{k-j} + \beta H_{k+j}\big)\right) z^k.$$

Let $\mathcal{L}_{\alpha,\beta}$ denote the minimal Fuchsian differential operator that annihilates $\mathcal{A}_{\alpha,\beta}(z)$: it does not always have maximal unipotent monodromy at $z = 0$, as the case $(\alpha, \beta) = (6, 0)$ shows (cf. [1, Sec. 10]). The operator $\mathcal{L}_{\alpha,\beta}$ also annihilates $\mathcal{B}_{\alpha,\beta}(z) + \log(z)\mathcal{A}_{\alpha,\beta}(z)$ and we define the mirror map $z\exp\big(\mathcal{B}_{\alpha,\beta}(z)/\mathcal{A}_{\alpha,\beta}(z)\big)$. We observe that $\mathcal{B}_{\alpha,\beta}(z)$ is a linear combination with integer coefficients in the functions

$$\mathcal{B}_{\mathbf{L},\alpha,\beta}(z) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \binom{k}{j}^{\alpha} \binom{k+j}{j}^{\beta} H_{L_1 j + L_2(k-j)}\right) z^k.$$

Here, $\mathbf{L} = (L_1, L_2) \in \mathbb{Z}^2$ is such that $0 \le L_1 \le 2$ and $0 \le L_2 \le 1$. For these $\mathbf{L}$, equating the variables in Theorem 2 leads to

$$\exp\big(\mathcal{B}_{\mathbf{L},\alpha,\beta}(z)/\mathcal{A}_{\alpha,\beta}(z)\big) \in \mathbb{Z}[[z]],$$

and this implies that $z\exp\big(\mathcal{B}_{\alpha,\beta}(z)/\mathcal{A}_{\alpha,\beta}(z)\big) \in z\mathbb{Z}[[z]]$. This example is particularly interesting because it proves that maximal unipotent monodromy at the origin is not a necessary condition to obtain mirror-type maps with integer Taylor coefficients.

It is interesting to note that the Taylor coefficients of $\mathcal{A}_{2,1}(z)$ and $\mathcal{A}_{2,2}(z)$ form the sequences appearing in Apéry's proof of the irrationality of $\zeta(2)$ and $\zeta(3)$, respectively. Beukers [4] showed that $\mathcal{A}_{2,1}(z)$ and $\mathcal{A}_{2,2}(z)$ are strongly related to modular forms, a fact which also explains the integrality properties of the associated mirror-type maps. (For p-adic properties of $\mathcal{A}_{2,1}(z)$, we refer the reader to [5].)

(4) Equating variables in Theorem 2 can explain the integrality properties of many of the mirror-type maps in [1], many of which have been incorporated in the table [2] of "Calabi–Yau differential equations". This table contains a list of more than 300 Fuchsian differential equations of order 4 with certain analytic properties, amongst which are maximal unipotent monodromy at the origin and conjectural integrality of the instanton-type numbers. Only the first 29 items are currently known to have a geometric origin, meaning that they have an interpretation in mirror symmetry; for example, the instanton-type numbers in these cases are really instanton numbers. In particular, the table contains the mirror-type maps of geometric origin considered in Sections 8.1, 8.2, 8.3 and 8.4 of [3], which all come from equating variables in series covered by Theorem 2.

Although this is not mentioned explicitly in [2], it is plausible that the mirror-type maps associated to each example of the table have integer Taylor coefficients. In this direction, we have checked that the functions whose Taylor coefficients are given in items 15 to 23, 25, 34, 39, 45, 58, 60, 72, 76, 78, 79, 81, 91, 93, 96, 97, 127, 130, 188, 190 and 191, are specialisations of multi-variable series that can be treated with Theorem 2. Hence the mirror-type maps associated to these items have integer Taylor coefficients. Incidentally, items 1 to 14 are all covered by the results in [14, 18, 22] and therefore, amongst the "geometric" items 1 to 29, there remains to understand only items $24, 26, 27, 28, 29$.

We could use many other ways of specialisation in conjunction with Theorem 2, for example "weighted" equating such as $z_1 = Mz_2^N$ for some integer parameters $M \neq 0$ and $N \ge 1$.

2. Outline of the proof of Theorem 2

In this section, we present a decomposition of the proof of Theorem 2 into various assertions, which form our multi-variate theory of formal congruences described in Subsection 1.1. The individual assertions will be proved in the later sections.

The starting point (listed as (D1) in Subsection 1.1) is the observation that, given a power series $S(\mathbf{z}) = S(z_1, z_2, \dots, z_d)$ in $\mathbb{Q}[[\mathbf{z}]]$, the series $S(\mathbf{z})$ is an element of $\mathbb{Z}[[\mathbf{z}]]$ if and only if, for all primes p, it is an element of $\mathbb{Z}_p[[\mathbf{z}]]$.

Next, we want to get rid of the exponential function in the definition of the mirror-type map $q_{\mathbf{L},\mathbf{N}}(\mathbf{z})$. To achieve this, we use a generalisation of a lemma attributed to Dieudonné and Dwork in [17, Ch. 14, p. 76] to several variables, the latter being the univariate case of the following lemma (corresponding to (D2) in Subsection 1.1).

Lemma 1. *For $S(\mathbf{z}) \in 1 + \sum_{i=1}^{d} z_i \mathbb{Q}_p[[\mathbf{z}]]$, we have*

$$S(\mathbf{z}) \in 1 + \sum_{i=1}^{d} z_i \mathbb{Z}_p[[\mathbf{z}]] \quad \textit{if and only if} \quad \frac{S(\mathbf{z}^p)}{S(\mathbf{z})^p} \in 1 + p \sum_{i=1}^{d} z_i \mathbb{Z}_p[[\mathbf{z}]].$$

This lemma enables us to prove the following reduction of our problem.

Lemma 2. *Given two formal series $F(\mathbf{z}) \in 1 + \sum_{i=1}^{d} z_i \mathbb{Z}[[\mathbf{z}]]$ and $G(\mathbf{z}) \in \sum_{i=1}^{d} z_i \mathbb{Q}[[\mathbf{z}]]$, let $q(\mathbf{z}) := \exp(G(\mathbf{z})/F(\mathbf{z}))$. Then we have $q(\mathbf{z}) \in 1 + \sum_{i=1}^{d} z_i \mathbb{Z}_p[[\mathbf{z}]]$ if and only if*

$$F(\mathbf{z})G(\mathbf{z}^p) - pF(\mathbf{z}^p)G(\mathbf{z}) \in p \sum_{i=1}^{d} z_i \mathbb{Z}_p[[\mathbf{z}]].$$

These two lemmas are proved in Sections 3 and 4, respectively.

We write $B_{\mathbf{N}}(\mathbf{m}) = \prod_{j=1}^{k} B(\mathbf{N}^{(j)}, \mathbf{m})$, where

$$B(\mathbf{P}, \mathbf{m}) = \frac{\left(\sum_{i=1}^{d} P_i m_i\right)!}{\prod_{i=1}^{d} m_i!^{P_i}} \tag{2.1}$$

for all vectors $\mathbf{P}, \mathbf{m} \in \mathbb{Z}^d$ with $\mathbf{P} \geq \mathbf{0}$ and $\mathbf{m} \geq \mathbf{0}$, while we define $B(\mathbf{P}, \mathbf{m}) = 0$ for vectors $\mathbf{m}$ for which $m_i < 0$ for some i. (If we interpret factorials $n!$ as $\Gamma(n+1)$, where Γ stands for the gamma function, then this convention is in accordance with the behaviour of the gamma function.) Note that, using this notation, we have $F_{\mathbf{N}}(\mathbf{z}) = \sum_{\mathbf{m} \geq \mathbf{0}} \mathbf{z}^{\mathbf{m}} B_{\mathbf{N}}(\mathbf{m})$ and $G_{\mathbf{L},\mathbf{N}}(\mathbf{z}) = \sum_{\mathbf{m} \geq \mathbf{0}} \mathbf{z}^{\mathbf{m}} H_{\mathbf{L}\cdot\mathbf{m}} B_{\mathbf{N}}(\mathbf{m})$.

As already mentioned, we have $F_{\mathbf{N}}(z) \in 1 + \sum_{i=1}^{d} z_i \mathbb{Z}[[\mathbf{z}]]$ and thus we can use Lemma 2 with $F(\mathbf{z}) = F_{\mathbf{N}}(\mathbf{z})$ and $G(\mathbf{z}) = G_{\mathbf{L},\mathbf{N}}(\mathbf{z})$. The coefficient of $\mathbf{z}^{\mathbf{a}+p\mathbf{K}}$ (with $0 \leq a_i < p$ for all i) in the Taylor expansion of the formal power series $F_{\mathbf{N}}(\mathbf{z})G_{\mathbf{L},\mathbf{N}}(\mathbf{z}^p) - pF_{\mathbf{N}}(\mathbf{z}^p)G_{\mathbf{L},\mathbf{N}}(\mathbf{z})$ can be written in the form

$$C(\mathbf{a} + p\mathbf{K}) = \sum_{\mathbf{0} \leq \mathbf{k} \leq \mathbf{K}} B_{\mathbf{N}}(\mathbf{a} + p\mathbf{k}) B_{\mathbf{N}}(\mathbf{K} - \mathbf{k}) \Big(H_{\mathbf{L}\cdot(\mathbf{K}-\mathbf{k})} - pH_{(\mathbf{L}\cdot\mathbf{a} + p\mathbf{L}\cdot\mathbf{k})} \Big).$$

Lemma 2 tells us that we have to show that $C(\mathbf{a} + p\mathbf{K})$ is in $p\mathbb{Z}_p$.

To prove this, we will proceed step by step. First, because of the congruence ([3])

$$pH_{(\mathbf{L}\cdot\mathbf{a} + p\mathbf{L}\cdot\mathbf{k})} \equiv H_{\lfloor \frac{1}{p}\mathbf{L}\cdot\mathbf{a} \rfloor + \mathbf{L}\cdot\mathbf{k}} \mod p\mathbb{Z}_p,$$

[3] This is an immediate consequence of the identity $H_J = \sum_{j=1}^{\lfloor J/p \rfloor} \frac{1}{pj} + \sum_{j=1, p\nmid j}^{J} \frac{1}{j}$.

we obtain

$$C(\mathbf{a}+p\mathbf{K}) \equiv \sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{K}} B_{\mathbf{N}}(\mathbf{a}+p\mathbf{k})B_{\mathbf{N}}(\mathbf{K}-\mathbf{k})\Big(H_{\mathbf{L}\cdot(\mathbf{K}-\mathbf{k})} - H_{\lfloor\frac{1}{p}\mathbf{L}\cdot\mathbf{a}\rfloor+\mathbf{L}\cdot\mathbf{k}}\Big) \mod p\mathbb{Z}_p.$$

Then, the following lemma (corresponding to (D3) in Subsection 1.1) is proved in Section 5.

Lemma 3. *For any prime p, vectors $\mathbf{a},\mathbf{k},\mathbf{L},\mathbf{N}^{(1)} \in \mathbb{Z}^d$ with $\mathbf{k}\ge\mathbf{0}$, $\mathbf{0}\le\mathbf{L}\le\mathbf{N}^{(1)}$, and $0\le a_i<p$ for $i=1,2,\dots,d$, we have*

$$B(\mathbf{N}^{(1)},\mathbf{a}+p\mathbf{k})\Big(H_{\lfloor\frac{1}{p}\mathbf{L}\cdot\mathbf{a}\rfloor+\mathbf{L}\cdot\mathbf{k}} - H_{\mathbf{L}\cdot\mathbf{k}}\Big) \in p\mathbb{Z}_p, \tag{2.2}$$

where $B(\mathbf{N}^{(1)},\mathbf{a}+p\mathbf{k})$ is defined in (2.1).

Since $B(\mathbf{N}^{(1)},\mathbf{a}+p\mathbf{k})$ is a factor of $B_{\mathbf{N}}(\mathbf{a}+p\mathbf{k})$, it follows that

$$C(\mathbf{a}+p\mathbf{K}) \equiv \sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{K}} B_{\mathbf{N}}(\mathbf{a}+p\mathbf{k})B_{\mathbf{N}}(\mathbf{K}-\mathbf{k})\big(H_{\mathbf{L}\cdot(\mathbf{K}-\mathbf{k})} - H_{\mathbf{L}\cdot\mathbf{k}}\big) \mod p\mathbb{Z}_p.$$

For the right-hand side, we obviously have

$$\begin{aligned}\sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{K}} & B_{\mathbf{N}}(\mathbf{a}+p\mathbf{k})B_{\mathbf{N}}(\mathbf{K}-\mathbf{k})\big(H_{\mathbf{L}\cdot(\mathbf{K}-\mathbf{k})} - H_{\mathbf{L}\cdot\mathbf{k}}\big)\\ &= -\sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{K}} H_{\mathbf{L}\cdot\mathbf{k}}\big(B_{\mathbf{N}}(\mathbf{a}+p\mathbf{k})B_{\mathbf{N}}(\mathbf{K}-\mathbf{k}) - B_{\mathbf{N}}(\mathbf{a}+p(\mathbf{K}-\mathbf{k}))B_{\mathbf{N}}(\mathbf{k})\big).\end{aligned} \tag{2.3}$$

We now use the multi-variable extension of the combinatorial lemma of Dwork (corresponding to (D4) in Subsection 1.1; stated here as Lemma 5 in Section 6, with proof in the same section) in order to decompose the sum over $\mathbf{k}$. Namely, if in Lemma 5 we let $Z(\mathbf{k}) = H_{\mathbf{L}\cdot\mathbf{k}}$,

$$W(\mathbf{k}) = B_{\mathbf{N}}(\mathbf{a}+p\mathbf{k})B_{\mathbf{N}}(\mathbf{K}-\mathbf{k}) - B_{\mathbf{N}}(\mathbf{a}+p(\mathbf{K}-\mathbf{k}))B_{\mathbf{N}}(\mathbf{k}),$$

and choose an integer r that satisfies $p^{r-1} > \max\{K_1,K_2,\dots,K_d\}$, then

$$\begin{aligned}C(\mathbf{a}+p\mathbf{K}) \equiv -\sum_{s=0}^{r-1} \sum_{\mathbf{0}\le\mathbf{m}\le(p^{r-s}-1)\mathbf{1}} & \big(H_{\sum_{i=1}^d L_i m_i p^s} - H_{\sum_{i=1}^d L_i \lfloor\frac{m_i}{p}\rfloor p^{s+1}}\big)\\ \cdot \sum_{p^s\mathbf{m}\le\mathbf{k}\le p^s(\mathbf{m}+\mathbf{1})-\mathbf{1}} & \big(B_{\mathbf{N}}(\mathbf{a}+p\mathbf{k})B_{\mathbf{N}}(\mathbf{K}-\mathbf{k}) - B_{\mathbf{N}}(\mathbf{a}+p(\mathbf{K}-\mathbf{k}))B_{\mathbf{N}}(\mathbf{k})\big)\\ & \mod p\mathbb{Z}_p.\end{aligned} \tag{2.4}$$

(Since for the first term appearing on the right-hand side of (6.1) we have $Z(\mathbf{0})\overline{W}_r(\mathbf{0}) = H_0\overline{W}_r(\mathbf{0}) = 0$, the right-hand sides of (2.3) and (2.4) are in fact equal.)

To deal with the sum over $\mathbf{k}$ in (2.4), we invoke Theorem 1 (corresponding to (D5) in Subsection 1.1). (Its proof is given in Section 7). We show in Section 8 that Theorem 1 can be applied with $A = g = B_{\mathbf{N}}$. Using this, we obtain

$$\sum_{p^s\mathbf{m}\le\mathbf{k}\le p^s(\mathbf{m}+\mathbf{1})-\mathbf{1}} \big(B_{\mathbf{N}}(\mathbf{a}+p\mathbf{k})B_{\mathbf{N}}(\mathbf{K}-\mathbf{k})-B_{\mathbf{N}}(\mathbf{a}+p(\mathbf{K}-\mathbf{k}))B_{\mathbf{N}}(\mathbf{k})\big) \in p^{s+1}B_{\mathbf{N}}(\mathbf{m})\mathbb{Z}_p. \quad (2.5)$$

We now have to deal with the harmonic sums

$$H_{\sum_{i=1}^d L_i m_i p^s} - H_{\sum_{i=1}^d L_i \lfloor \frac{m_i}{p} \rfloor p^{s+1}}$$

occurring on the right-hand side of (2.4). In this regard, we prove the following lemma in Section 9. (As we show there, it can be reduced to Lemma 3.)

Lemma 4. *For all primes p, vectors $\mathbf{m}, \mathbf{L}, \mathbf{N}^{(1)}, \mathbf{N}^{(2)}, \dots, \mathbf{N}^{(d)} \in \mathbb{Z}^d$ with $\mathbf{m}, \mathbf{L}, \mathbf{N}^{(1)}, \mathbf{N}^{(2)}, \dots, \mathbf{N}^{(d)} \ge \mathbf{0}$, we have*

$$B_{\mathbf{N}}(\mathbf{m})\big(H_{\sum_{i=1}^d L_i m_i p^s} - H_{\sum_{i=1}^d L_i \lfloor \frac{m_i}{p} \rfloor p^{s+1}}\big) \in \frac{1}{p^s}\,\mathbb{Z}_p. \quad (2.6)$$

Consequently, putting the congruences (2.5) and (2.6) together, it follows from (2.4) that $C(\mathbf{a}+p\mathbf{k})$ is congruent mod $p\mathbb{Z}_p$ to a multiple sum (over s and $\mathbf{m}$) whose terms are all in $p\mathbb{Z}_p$. Hence, we have established that

$$C(\mathbf{a}+p\mathbf{k}) \in p\mathbb{Z}_p.$$

This concludes our outline of the proof Theorem 2.

3. Proof of Lemma 1

Proof of the "only if" part. We have to show that if $S(\mathbf{z}) \in 1 + \sum_{i=1}^d z_i\mathbb{Z}_p[[\mathbf{z}]]$, then

$$\frac{S(\mathbf{z}^p)}{S(\mathbf{z})^p} \in 1 + p\sum_{i=1}^d z_i\mathbb{Z}_p[[\mathbf{z}]].$$

To do this, we set $S(\mathbf{z}) = \sum_{\mathbf{i}\ge\mathbf{0}} a_{\mathbf{i}}\mathbf{z}^{\mathbf{i}}$. The congruence $(u+v)^p \equiv u^p + v^p \mod p\mathbb{Z}_p$ and Fermat's Little Theorem imply that

$$S(\mathbf{z})^p = \Big(\sum_{\mathbf{i}\ge\mathbf{0}} a_{\mathbf{i}}\mathbf{z}^{\mathbf{i}}\Big)^p \equiv \sum_{\mathbf{i}\ge\mathbf{0}} a_{\mathbf{i}}^p\mathbf{z}^{p\mathbf{i}} \mod p\sum_{i=1}^d z_i\mathbb{Z}_p[[\mathbf{z}]]$$

$$\equiv \sum_{\mathbf{i}\ge\mathbf{0}} a_{\mathbf{i}}\mathbf{z}^{p\mathbf{i}} \mod p\sum_{i=1}^d z_i\mathbb{Z}_p[[\mathbf{z}]].$$

This means that $S(\mathbf{z})^p = S(\mathbf{z}^p) + pH(\mathbf{z})$ with $H(\mathbf{z}) \in \sum_{i=1}^d z_i\mathbb{Z}_p[[\mathbf{z}]]$. Hence,

$$\frac{S(\mathbf{z}^p)}{S(\mathbf{z})^p} = 1 - p\,\frac{H(\mathbf{z})}{S(\mathbf{z})^p} \in 1 + p\sum_{i=1}^d z_i\mathbb{Z}_p[[\mathbf{z}]],$$

because the formal series $S(\mathbf{z}) \in 1 + \sum_{i=1}^d z_i\mathbb{Z}_p[[\mathbf{z}]]$ is invertible in $\mathbb{Z}_p[[\mathbf{z}]]$.

Proof of the "if" part. Suppose that $S(\mathbf{z}^p) = S(\mathbf{z})^p R(\mathbf{z})$ with $R(\mathbf{z}) = 1 + p\sum_{|\mathbf{i}|\geq \mathbf{1}} b_{\mathbf{i}}\mathbf{z}^{\mathbf{i}} \in 1 + p\sum_{i=1}^{d} z_i\mathbb{Z}_p[[\mathbf{z}]]$ and $S(\mathbf{0}) = 1$. Set $S(\mathbf{z}) = \sum_{\mathbf{i}\geq 0} a_{\mathbf{i}}\mathbf{z}^{\mathbf{i}}$. We have $a_{\mathbf{0}} = 1$, and we proceed by induction on $|\mathbf{i}|$ to show that $a_{\mathbf{i}} \in \mathbb{Z}_p$.

So, let us assume that $a_{\mathbf{i}} \in \mathbb{Z}_p$ for all vectors $\mathbf{i} \in \mathbb{Z}^d$ with $|\mathbf{i}| \leq r - 1$. Let $\mathbf{n} \in \mathbb{Z}^d$ be a vector with $|\mathbf{n}| = r$. The Taylor coefficient $C_{\mathbf{n}}$ of $\mathbf{z}^{\mathbf{n}}$ in $S(\mathbf{z}^p)$ is

$$\begin{cases} a_{\frac{1}{p}\mathbf{n}} & \text{if } p \mid n_1,\ p \mid n_2, \dots,\ p \mid n_d\,; \\ 0 & \text{otherwise.} \end{cases}$$

The Taylor coefficient $C_{\mathbf{n}}$ is at the same time also equal to the coefficient of $\mathbf{z}^{\mathbf{n}}$ in the expansion of the series

$$\Bigg(\sum_{\mathbf{i}\geq \mathbf{0}} a_{\mathbf{i}}\mathbf{z}^{\mathbf{i}}\Bigg)^p \Bigg(1 + p \sum_{\mathbf{i}\geq \mathbf{0},\, |\mathbf{i}|\geq 1} b_{\mathbf{i}}\mathbf{z}^{\mathbf{i}}\Bigg).$$

The coefficient of $\mathbf{z}^{\mathbf{n}}$ in this series is thus $C_{\mathbf{n}} = B_{\mathbf{n}} + pD_{\mathbf{n}}$, where

$$B_{\mathbf{n}} = \sum_{\mathbf{i}^{(1)}+\cdots+\mathbf{i}^{(p)}=\mathbf{n}} a_{\mathbf{i}^{(1)}} \cdots a_{\mathbf{i}^{(p)}} \tag{3.1}$$

and

$$D_{\mathbf{n}} = \sum_{\substack{\mathbf{i}^{(1)}+\cdots+\mathbf{i}^{(p+1)}=\mathbf{n} \\ \left|\mathbf{i}^{(p+1)}\right|>0}} a_{\mathbf{i}^{(1)}} \cdots a_{\mathbf{i}^{(p)}} b_{\mathbf{i}^{(p+1)}} \in \mathbb{Z}_p. \tag{3.2}$$

Case 1. If $p \mid n_1, \dots, p \mid n_d$, in the multiple sum $B_{\mathbf{n}}$ a term $\prod_{\ell=1}^{m} a_{\mathbf{i}_\ell}^{e_\ell}$ with $a_{\mathbf{i}_{\ell_1}} \neq a_{\mathbf{i}_{\ell_2}}$ occurs

$$\frac{(e_1 + \cdots + e_m)!}{e_1! \cdots e_m!} = \frac{p!}{e_1! \cdots e_m!} \tag{3.3}$$

times. The multinomial coefficient (3.3) is an integer divisible by p, except if $m = 1$ and $e_1 = p$; that is, if we are looking at the term $a_{\frac{1}{p}\mathbf{n}}^p$, which occurs with coefficient 1 in $B_{\mathbf{n}}$. The term $a_{\mathbf{n}}$ appears in the form $pa_{\mathbf{n}}a_{\mathbf{0}}^{p-1} = pa_{\mathbf{n}}$ in the expression (3.1) for $B_{\mathbf{n}}$. For all other terms in the sum on the right-hand side of (3.1), we have $\left|\mathbf{i}^{(\ell)}\right| < |\mathbf{n}|$ for $\ell = 1, 2, \dots, p$. Hence, the induction hypothesis applies to all the factors in the corresponding terms $a_{\mathbf{i}^{(1)}} \cdots a_{\mathbf{i}^{(p)}}$, whence $B_{\mathbf{n}} = pa_{\mathbf{n}} + a_{\frac{1}{p}\mathbf{n}}^p \mod p\mathbb{Z}_p$.

In the multiple sum (3.2) for $D_{\mathbf{n}}$, the condition $\left|\mathbf{i}^{(p+1)}\right| > 0$ guarantees that $\left|\mathbf{i}^{(\ell)}\right| < |\mathbf{n}|$ for $\ell = 1, \dots, p$, and therefore we can apply the induction hypothesis to each factor $a_{\mathbf{i}^{(\ell)}}$. This shows that $D_{\mathbf{n}} \in \mathbb{Z}_p$.

We therefore have

$$a_{\frac{1}{p}\mathbf{n}} = C_{\mathbf{n}} \equiv pa_{\mathbf{n}} + a_{\frac{1}{p}\mathbf{n}}^p \mod p\mathbb{Z}_p,$$

whence,

$$pa_{\mathbf{n}} \equiv a_{\frac{1}{p}\mathbf{n}} - a_{\frac{1}{p}\mathbf{n}}^p \mod p\mathbb{Z}_p.$$

This shows that $a_{\mathbf{n}} \in \mathbb{Z}_p$ since $a_{\frac{1}{p}\mathbf{n}} - a_{\frac{1}{p}\mathbf{n}}^p \in p\mathbb{Z}_p$ by Fermat's Little Theorem.

Case 2. If $p \nmid n_i$ for some i between 1 and d, the only change compared to the preceding case is that the term $a^p_{\frac{1}{p}\mathbf{n}}$ does not occur. Therefore, in this case we have

$$0 = C_{\mathbf{n}} \equiv pa_{\mathbf{n}} \mod p\mathbb{Z}_p.$$

Hence,

$$pa_{\mathbf{n}} \equiv 0 \mod p\mathbb{Z}_p,$$

which shows again that $a_{\mathbf{n}} \in \mathbb{Z}_p$.

This completes the proof of the lemma. □

4. Proof of Lemma 2

We begin by showing that, if $S(\mathbf{z}) \in \sum_{i=1}^d z_i\mathbb{Q}_p[[\mathbf{z}]]$, then

$$\exp(S(\mathbf{z})) \in 1 + \sum_{i=1}^d z_i\mathbb{Z}_p[[\mathbf{z}]] \quad \text{if and only if} \quad S(\mathbf{z}^p) - pS(\mathbf{z}) \in p\sum_{i=1}^d z_i\mathbb{Z}_p[[\mathbf{z}]].$$

The formal power series $\exp(X)$ and $\log(1+X)$ are defined by their usual expansions.

Proof of the "if" part. By Lemma 1 with $S(\mathbf{z})$ replaced by $\exp(S(\mathbf{z}))$, we have

$$\exp\big(S(\mathbf{z}^p) - pS(\mathbf{z})\big) \in 1 + p\sum_{i=1}^d z_i\mathbb{Z}_p[[\mathbf{z}]].$$

Therefore, we have $S(\mathbf{z}^p) - pS(\mathbf{z}) = \log(1+pH(\mathbf{z}))$ with $H(\mathbf{z}) \in \sum_{i=1}^d z_i\mathbb{Z}_p[[\mathbf{z}]]$. This yields

$$S(\mathbf{z}^p) - pS(\mathbf{z}) = -\sum_{n=1}^{\infty} \frac{p^n}{n}(-H(\mathbf{z}))^n \in p\sum_{i=1}^d z_i\mathbb{Z}_p[[\mathbf{z}]]$$

since $v_p(p^n/n) \ge 1$ for all integers $n \ge 1$.

Proof of the "only if" part. We have $S(\mathbf{z}^p) - pS(\mathbf{z}) = pJ(\mathbf{z})$ with $J(\mathbf{z}) \in \sum_{i=1}^d z_i\mathbb{Z}_p[[\mathbf{z}]]$. Therefore, we have

$$\exp\big(S(\mathbf{z}^p) - pS(\mathbf{z})\big) = 1 + \sum_{n=1}^{\infty} \frac{p^n}{n!}J(\mathbf{z})^n \in 1 + p\sum_{i=1}^d z_i\mathbb{Z}_p[[\mathbf{z}]],$$

since

$$v_p\left(\frac{p^n}{n!}\right) = n - \sum_{k=1}^{\infty}\left\lfloor \frac{n}{p^k}\right\rfloor > n - \sum_{k=1}^{\infty}\frac{n}{p^k} = \frac{p-2}{p-1}n \ge 0.$$

By Lemma 1 with $S(\mathbf{z})$ replaced by $\exp\big(S(\mathbf{z})\big)$, it follows that

$$\exp\big(S(\mathbf{z})\big) \in 1 + \sum_{i=1}^d z_i\mathbb{Z}_p[[\mathbf{z}]].$$

In order to finish the proof of the lemma, we observe that for $S = G/F$ with $F(\mathbf{z}) \in 1 + \sum_{i=1}^{d} z_i \mathbb{Z}_p[[\mathbf{z}]]$, we have the equivalence

$$S(\mathbf{z}^p) - pS(\mathbf{z}) \in p \sum_{i=1}^{d} z_i \mathbb{Z}_p[[\mathbf{z}]] \quad \text{if and only if} \quad F(\mathbf{z})G(\mathbf{z}^p) - pF(\mathbf{z}^p)G(\mathbf{z}) \in p \sum_{i=1}^{d} z_i \mathbb{Z}_p[[\mathbf{z}]],$$

since $F(\mathbf{z})$ is invertible in $\mathbb{Z}_p[[\mathbf{z}]]$. □

5. Proof of Lemma 3

The following proof was kindly provided by an anonymous referee. It differs from the proof in Section 6 of [14] even in the case $d = 1$, and thus provides an alternative argument.

For convenience, we shall drop the upper index in $N_i^{(1)}$ in this section, that is, we write

$$B(\mathbf{N}^{(1)}, \mathbf{m}) = \frac{(\mathbf{N}^{(1)} \cdot \mathbf{m})!}{\prod_{i=1}^{d} m_i!^{N_i^{(1)}}} = \frac{(\mathbf{N} \cdot \mathbf{m})!}{\prod_{i=1}^{d} m_i!^{N_i}}.$$

We claim that one may restrict oneself to the case $\mathbf{L} = \mathbf{N}$. Indeed, we have the factorisation

$$\frac{(\mathbf{N} \cdot \mathbf{m})!}{\prod_{i=1}^{d} m_i!^{N_i}} = \frac{(\mathbf{L} \cdot \mathbf{m})!}{\prod_{i=1}^{d} m_i!^{L_i}} \times \frac{(\mathbf{N} \cdot \mathbf{m})!}{(\mathbf{L} \cdot \mathbf{m})! \prod_{i=1}^{d} m_i!^{N_i - L_i}},$$

where the second factor is obviously an integer. If this is inserted in (2.2), with $m_i = a_i + pk_i$ for all i, then the claim becomes evident.

We continue to use the notation $m_i = a_i + pk_i$ for all i, so that we have to prove

$$\frac{(\mathbf{N} \cdot \mathbf{m})!}{\prod_{i=1}^{d} m_i!^{N_i}} \left(H_{\lfloor \frac{1}{p} \mathbf{N} \cdot \mathbf{m} \rfloor} - H_{\mathbf{N} \cdot \lfloor \frac{1}{p} \mathbf{m} \rfloor} \right) \in p\mathbb{Z}_p, \tag{5.1}$$

where $\lfloor \frac{1}{p} \mathbf{m} \rfloor = (\lfloor \frac{1}{p} m_1 \rfloor, \lfloor \frac{1}{p} m_2 \rfloor, \dots, \lfloor \frac{1}{p} m_d \rfloor)$. If $\lfloor \frac{1}{p} \mathbf{N} \cdot \mathbf{m} \rfloor = \mathbf{N} \cdot \lfloor \frac{1}{p} \mathbf{m} \rfloor$, then the difference of harmonic numbers in (5.1) is zero, whence Lemma 3 is trivially true in this case. On the other hand, if $\lfloor \frac{1}{p} \mathbf{N} \cdot \mathbf{m} \rfloor > \mathbf{N} \cdot \lfloor \frac{1}{p} \mathbf{m} \rfloor$, then we claim that the first factor in the expression in (5.1) is divisible by

$$p\big(\mathbf{N} \cdot \lfloor \tfrac{1}{p} \mathbf{m} \rfloor + 1\big)\big(\mathbf{N} \cdot \lfloor \tfrac{1}{p} \mathbf{m} \rfloor + 2\big) \cdots \big(\lfloor \mathbf{N} \cdot \tfrac{1}{p} \mathbf{m} \rfloor\big).$$

Clearly, this would immediately imply Lemma 3 in this case also.

In order to establish the claim, we observe that

$$v_p(n!) = \lfloor n/p \rfloor + v_p(\lfloor n/p \rfloor !),$$

and hence

$$v_p\left(\frac{(\mathbf{N} \cdot \mathbf{m})!}{\prod_{i=1}^{d} m_i!^{N_i}} \right) = \lfloor \tfrac{1}{p} \mathbf{N} \cdot \mathbf{m} \rfloor - \mathbf{N} \cdot \lfloor \tfrac{1}{p} \mathbf{m} \rfloor + v_p\left(\frac{\lfloor \frac{1}{p} \mathbf{N} \cdot \mathbf{m} \rfloor !}{\prod_{i=1}^{d} \lfloor \frac{1}{p} m_i \rfloor !^{N_i}} \right). \tag{5.2}$$

We have the factorisation

$$\frac{\lfloor \frac{1}{p} \mathbf{N} \cdot \mathbf{m} \rfloor !}{\prod_{i=1}^{d} \lfloor \frac{1}{p} m_i \rfloor !^{N_i}} = \big(\lfloor \mathbf{N} \cdot \tfrac{1}{p} \mathbf{m} \rfloor\big) \cdots \big(\mathbf{N} \cdot \lfloor \tfrac{1}{p} \mathbf{m} \rfloor + 2\big)\big(\mathbf{N} \cdot \lfloor \tfrac{1}{p} \mathbf{m} \rfloor + 1\big) \frac{(\mathbf{N} \cdot \lfloor \frac{1}{p} \mathbf{m} \rfloor)!}{\prod_{i=1}^{d} \lfloor \frac{1}{p} m_i \rfloor !^{N_i}},$$

where the last factor on the right-hand side is an integer. If this is inserted in (5.2), then together with our assumption that $\left\lfloor \frac{1}{p}\mathbf{N}\cdot\mathbf{m}\right\rfloor > \mathbf{N}\cdot\left\lfloor \frac{1}{p}\mathbf{m}\right\rfloor$ and the fact that (5.2) holds for all primes p, the claim follows. This finishes the proof of the lemma. □

6. A combinatorial lemma

In this section, we generalise a combinatorial lemma due to Dwork (see [11, Lemma 4.2]) to several variables.

Lemma 5. *Let r be a non-negative integer, let Z and W be maps from $\mathbb{Z}^d$ to a ring R, and let*

$$\overline{W}_r(\mathbf{m}) = \sum_{p^r\mathbf{m}\le\mathbf{k}\le p^r(\mathbf{m}+\mathbf{1})-\mathbf{1}} W(\mathbf{k}).$$

Then

$$\sum_{\mathbf{0}\le\mathbf{k}\le(p^r-1)\mathbf{1}} Z(\mathbf{k})W(\mathbf{k}) = Z(\mathbf{0})\overline{W}_r(\mathbf{0})$$
$$+\sum_{s=0}^{r-1}\left(\sum_{\mathbf{0}\le\mathbf{m}\le(p^{r-s}-1)\mathbf{1}}\left(Z(m_1p^s,\dots,m_dp^s) - Z\left(\left\lfloor\tfrac{m_1}{p}\right\rfloor p^{s+1},\dots,\left\lfloor\tfrac{m_d}{p}\right\rfloor p^{s+1}\right)\right)\overline{W}_s(\mathbf{m})\right). \tag{6.1}$$

We give two proofs, both of which have their merits. The first one shows that behind the formula there is a combinatorial decomposition of the summation range, see (6.2). The second one, which was kindly supplied to us by an anonymous referee, provides a recursive "construction" of the summation formula.

First proof of Lemma 5. Let

$$X_s = \sum_{\mathbf{0}\le\mathbf{m}\le(p^{r-s}-1)\mathbf{1}} Z(m_1p^s,\dots,m_dp^s)\overline{W}_s(\mathbf{m})$$

and

$$Y_s = \sum_{\mathbf{0}\le\mathbf{m}\le(p^{r-s}-1)\mathbf{1}} Z\left(\left\lfloor\tfrac{m_1}{p}\right\rfloor p^{s+1},\dots,\left\lfloor\tfrac{m_d}{p}\right\rfloor p^{s+1}\right)\overline{W}_s(\mathbf{m}).$$

By definition, we have

$$X_s = \sum_{\mathbf{0}\le\mathbf{m}\le(p^{r-s}-1)\mathbf{1}}\left(\sum_{p^s\mathbf{m}\le\mathbf{k}\le p^s(\mathbf{m}+\mathbf{1})-\mathbf{1}} Z(m_1p^s,\dots,m_dp^s)W(k_1,\dots,k_d)\right).$$

For $k_j\in\left\{m_jp^s,\dots,(m_j+1)p^s-1\right\}$, we have $m_j=\lfloor k_j/p^s\rfloor$, $j=1,\dots,d$, and furthermore we have the partition

$$\left\{0,\dots,p^r-1\right\}^d = \bigcup_{\mathbf{0}\le\mathbf{m}\le(p^{r-s}-1)\mathbf{1}}\prod_{j=1}^d\left\{m_jp^s,\dots,(m_j+1)p^s-1\right\}. \tag{6.2}$$

Hence, it follows that

$$X_s = \sum_{\mathbf{0}\le\mathbf{k}\le(p^r-1)\mathbf{1}} Z\left(\left\lfloor\frac{k_1}{p^s}\right\rfloor p^s,\dots,\left\lfloor\frac{k_d}{p^s}\right\rfloor p^s\right) W(k_1,\dots,k_d).$$

Similarly, we have

$$Y_s = \sum_{\mathbf{0}\le\mathbf{k}\le(p^r-1)\mathbf{1}} Z\left(\left\lfloor\frac{k_1}{p^{s+1}}\right\rfloor p^{s+1},\dots,\left\lfloor\frac{k_d}{p^{s+1}}\right\rfloor p^{s+1}\right) W(k_1,\dots,k_d),$$

where we used that $\left\lfloor\frac{1}{p}\left\lfloor\frac{k}{p^s}\right\rfloor\right\rfloor = \left\lfloor\frac{k}{p^{s+1}}\right\rfloor$. We therefore have

$$\begin{aligned}\sum_{s=0}^{r-1}(X_s-Y_s) &= \sum_{\mathbf{0}\le\mathbf{k}\le(p^r-1)\mathbf{1}} W(k_1,\dots,k_d)\\ &\quad\times\sum_{s=0}^{r-1}\left(Z\left(\left\lfloor\frac{k_1}{p^s}\right\rfloor p^s,\dots,\left\lfloor\frac{k_d}{p^s}\right\rfloor p^s\right) - Z\left(\left\lfloor\frac{k_1}{p^{s+1}}\right\rfloor p^{s+1},\dots,\left\lfloor\frac{k_d}{p^{s+1}}\right\rfloor p^{s+1}\right)\right)\\ &= \sum_{\mathbf{0}\le\mathbf{k}\le(p^r-1)\mathbf{1}} W(\mathbf{k})\big(Z(\mathbf{k})-Z(\mathbf{0})\big),\end{aligned}$$

because the sum over s is a telescoping sum. Since

$$\sum_{\mathbf{0}\le\mathbf{k}\le(p^r-1)\mathbf{1}} W(\mathbf{k}) = \overline{W}_r(\mathbf{0}),$$

this completes the proof of the lemma. □

Second proof of Lemma 5. We observe that

$$\sum_{\mathbf{0}\le\mathbf{k}\le(p^r-1)\mathbf{1}} Z(\mathbf{k})W(\mathbf{k}) = \sum_{\mathbf{0}\le\mathbf{k}\le(p^r-1)\mathbf{1}} \left(Z(\mathbf{k}) - Z\left(\left\lfloor\tfrac{k_1}{p}\right\rfloor p,\dots,\left\lfloor\tfrac{k_d}{p}\right\rfloor p\right)\right) W(\mathbf{k}) + \sum_{\mathbf{0}\le\mathbf{k}'\le(p^{r-1}-1)\mathbf{1}} Z(p\mathbf{k}')\overline{W}_1(\mathbf{k}'),$$

where $\mathbf{k}' = \left(\left\lfloor\frac{k_1}{p}\right\rfloor,\left\lfloor\frac{k_2}{p}\right\rfloor,\dots,\left\lfloor\frac{k_d}{p}\right\rfloor\right)$. This construction is now iterated. □

7. Proof of Theorem 1

We adapt Dwork's proof [11, Theorem 1.1] of the special case $d=1$, that is, the case in which there is just one variable.

For integer vectors $\mathbf{k},\mathbf{K},\mathbf{v}\in\mathbb{Z}$ with $\mathbf{k}\ge\mathbf{0}$ and $0\le v_i<p$ for $i=1,2,\dots,d$, set

$$U(\mathbf{k},\mathbf{K}) = A(\mathbf{v}+p(\mathbf{K}-\mathbf{k}))A(\mathbf{k}) - A(\mathbf{v}+p\mathbf{k})A(\mathbf{K}-\mathbf{k}),$$

being 0 if $k_i > K_i$ for some i, which is the case in particular if $K_i < 0$ for some i. Furthermore, for a vector $\mathbf{m} \in \mathbb{Z}^d$ with $\mathbf{m} \geq \mathbf{0}$, set

$$H(\mathbf{m},\mathbf{K};s) = \sum_{p^s\mathbf{m} \leq \mathbf{k} \leq p^s(\mathbf{m}+\mathbf{1})-\mathbf{1}} U(\mathbf{k},\mathbf{K}), \tag{7.1}$$

being 0 if $K_i < 0$ for some i. (The reader should recall that, by definition, $\mathbf{1}$ is the all 1 vector.) We omit to indicate the dependence on p and $\mathbf{v}$ in order to not overload notation.

Lemma 6. *Let $\mathbf{k},\mathbf{K},\mathbf{v} \in \mathbb{Z}$ with $\mathbf{k} \geq \mathbf{0}$ and $0 \leq v_i < p$ for $i = 1, 2, \dots, d$. Then there hold the following three facts:*

(*i*) *We have $U(\mathbf{K}-\mathbf{k},\mathbf{K}) = -U(\mathbf{k},\mathbf{K})$.*

(*ii*) *For all integer vectors $\mathbf{M}$ with $p^{s+1}(\mathbf{M}+\mathbf{1}) > \mathbf{K}$, we have*

$$\sum_{\mathbf{0} \leq \mathbf{m} \leq \mathbf{M}} H(\mathbf{m},\mathbf{K};s) = 0.$$

(*iii*) *We have*

$$H(\mathbf{k},\mathbf{K};s+1) = \sum_{\mathbf{0} \leq \mathbf{i} \leq (p-1)\mathbf{1}} H(\mathbf{i}+p\mathbf{k},\mathbf{K};s).$$

Proof. The assertion (*i*) is obvious.

(*ii*) We have

$$\begin{aligned}
\sum_{\mathbf{0} \leq \mathbf{m} \leq \mathbf{M}} H(\mathbf{m},\mathbf{K};s) &= \sum_{\mathbf{0} \leq \mathbf{m} \leq \mathbf{M}} \left(\sum_{p^s\mathbf{m} \leq \mathbf{k} \leq p^s(\mathbf{m}+\mathbf{1})-\mathbf{1}} U(\mathbf{k},\mathbf{K}) \right) \\
&= \sum_{\mathbf{0} \leq \mathbf{k} \leq p^s(\mathbf{M}+\mathbf{1})-\mathbf{1}} U(\mathbf{k},\mathbf{K}) \\
&= \sum_{\mathbf{0} \leq \mathbf{k} \leq \mathbf{K}} U(\mathbf{k},\mathbf{K}) \\
&= 0.
\end{aligned}$$

Here, in order to pass from the second to the third line, we used the fact that $U(\mathbf{k},\mathbf{K}) = 0$ if $k_i > K_i$ for some i between 1 and d. To obtain the last line, we used the functional equation given in (*i*).

(*iii*) We have

$$\sum_{\mathbf{0} \leq \mathbf{i} \leq (p-1)\mathbf{1}} H(\mathbf{i}+p\mathbf{k},\mathbf{K};s) = \sum_{\mathbf{0} \leq \mathbf{i} \leq (p-1)\mathbf{1}} \left(\sum_{p^s(\mathbf{i}+p\mathbf{m}) \leq \mathbf{k} \leq p^s(\mathbf{i}+p\mathbf{m}+\mathbf{1})-\mathbf{1}} U(\mathbf{k},\mathbf{K}) \right),$$

and it is rather straightforward to see that this sum simply equals $H(\mathbf{m},\mathbf{K};s+1)$. □

Proof of Theorem 1. We define two assertions, denoted by α_s and $\beta_{t,s}$, in the following way: for all $s \geq 0$, α_s is the assertion that the congruence

$$H(\mathbf{m},\mathbf{K};s) \equiv 0 \mod p^{s+1}g(\mathbf{m})\mathbb{Z}_p$$

holds for all vectors $\mathbf{m}, \mathbf{K} \in \mathbb{Z}^d$ with $\mathbf{m} \geq \mathbf{0}$.

For all integers s and t with $0 \leq t \leq s$, $\beta_{t,s}$ is the assertion that the congruence

$$H(\mathbf{m}, \mathbf{K} + p^s\mathbf{m}; s) \equiv \sum_{\mathbf{0} \leq \mathbf{k} \leq (p^{s-t}-1)\mathbf{1}} \frac{A(\mathbf{k} + p^{s-t}\mathbf{m})}{A(\mathbf{k})} H(\mathbf{k}, \mathbf{K}; t) \mod p^{s+1} g(\mathbf{m})\mathbb{Z}_p \tag{7.2}$$

holds for all vectors $\mathbf{m}, \mathbf{K} \in \mathbb{Z}^d$ with $\mathbf{m} \geq 0$.

Moreover, we define three further assertions $A1, A2, A3$:

$A1$: for all vectors $\mathbf{k}, \mathbf{K} \in \mathbb{Z}^d$ with $\mathbf{k} \geq 0$, we have $U(\mathbf{k}, \mathbf{K}) \in pg(\mathbf{k})\mathbb{Z}_p$.

$A2$: for all vectors $\mathbf{m}, \mathbf{k}, \mathbf{K} \in \mathbb{Z}^d$ and integers $s \geq 0$ with $\mathbf{m} \geq \mathbf{0}$ and $0 \leq k_i < p^s$ for $i = 1, 2, \ldots, d$, we have

$$U(\mathbf{k} + p^s\mathbf{m}, \mathbf{K} + p^s\mathbf{m}) \equiv \frac{A(\mathbf{k} + p^s\mathbf{m})}{A(\mathbf{k})} U(\mathbf{k}, \mathbf{K}) \mod p^{s+1} g(\mathbf{m})\mathbb{Z}_p.$$

$A3$: for all integers s and t with $0 \leq t < s$, we have

"α_{s-1} and $\beta_{t,s}$ together imply $\beta_{t+1,s}$."

In the following, we shall first show that Assertions $A1$, $A2$, $A3$ altogether imply Theorem 1, see the "first step" below. Subsequently, in the "second step," we show that Assertions $A1$, $A2$, $A3$ hold indeed.

First step. We claim that Theorem 1 follows from $A1$, $A2$ and $A3$. So, from now on we shall assume that $A1$, $A2$ and $A3$ are true. Our goal is to show that α_s holds for all $s \geq 0$. We shall accomplish this by induction on $s \geq 0$.

We begin by establishing α_0. To do so, we observe that

$$H(\mathbf{m}, \mathbf{K}; 0) = U(\mathbf{m}, \mathbf{K}), \tag{7.3}$$

that is, that Assertion α_0 is equivalent to $A1$. Hence, Assertion α_0 is true.

We now suppose that α_{s-1} is true. We shall show by induction on $t \geq 0$ that $\beta_{t,s}$ is true for all $t \leq s$. Because of $A3$, it suffices to prove that $\beta_{0,s}$ is true. To do so, we see that

$$\begin{aligned}
\sum_{\mathbf{0} \leq \mathbf{k} \leq (p^s-1)\mathbf{1}} & \frac{A(\mathbf{k} + p^s\mathbf{m})}{A(\mathbf{k})} H(\mathbf{k}, \mathbf{K}; 0) \\
&= \sum_{\mathbf{0} \leq \mathbf{k} \leq (p^s-1)\mathbf{1}} \frac{A(\mathbf{k} + p^s\mathbf{m})}{A(\mathbf{k})} U(\mathbf{k}, \mathbf{K}) \\
&\equiv \sum_{\mathbf{0} \leq \mathbf{k} \leq (p^s-1)\mathbf{1}} U(\mathbf{k} + p^s\mathbf{m}, \mathbf{K} + p^s\mathbf{m}) \mod p^{s+1} g(\mathbf{m})\mathbb{Z}_p \\
&\equiv H(\mathbf{m}, \mathbf{K} + p^s\mathbf{m}; s) \mod p^{s+1} g(\mathbf{m})\mathbb{Z}_p.
\end{aligned} \tag{7.4}$$

Here, the first equality results from (7.3), the subsequent congruence results from $A2$, and the last line is obtained by remembering the definition (7.1) of H (there holds in fact equality between the last two lines). The congruence (7.4) is nothing else but Assertion $\beta_{0,s}$, which is therefore proved under our assumptions.

The above argument shows in particular that Assertion $\beta_{s,s}$ is true, which means that we have the congruence

$$H(\mathbf{m},\mathbf{K}+p^s\mathbf{m};s)\equiv\frac{A(\mathbf{m})}{A(\mathbf{0})}\,H(\mathbf{0},\mathbf{K};s)\mod p^{s+1}g(\mathbf{m})\mathbb{Z}_p. \tag{7.5}$$

Let us now consider the property $\gamma_{\mathbf{K}}$ defined by

$$\gamma_{\mathbf{K}}:\quad H(\mathbf{0},\mathbf{K};s)\equiv 0\mod p^{s+1}\mathbb{Z}_p.$$

This property holds certainly if $K_i<0$ for some i because in that case each term of the multiple sum that defines H vanishes. We want to show that the assertion also holds when $\mathbf{K}\geq\mathbf{0}$. Let $\mathbf{K}'$ be one of the vectors of non-negative integers (if there is at all) such that $|\mathbf{K}'|=K_1'+K_2'+\cdots+K_d'$ is minimal and $\gamma_{\mathbf{K}'}$ does *not* hold. Let $\mathbf{m}\in\mathbb{Z}^d$ be a vector with $\mathbf{m}\geq\mathbf{0}$ and $|\mathbf{m}|>0$, and set $\mathbf{K}=\mathbf{K}'-p^s\mathbf{m}$. Since $|\mathbf{K}|<|\mathbf{K}|'$, we have

$$H(\mathbf{0},\mathbf{K};s)\equiv 0\mod p^{s+1}\mathbb{Z}_p$$

because $\gamma_{\mathbf{K}}$ holds by minimality of $\mathbf{K}'$. Since $A(\mathbf{m})/A(\mathbf{0})\in\mathbb{Z}_p$ by Properties (i) and (ii) in the statement of Theorem 1, it follows from (7.5) that

$$H(\mathbf{m},\mathbf{K}';s)\equiv 0\mod p^{s+1}\mathbb{Z}_p \tag{7.6}$$

provided $\mathbf{m}\geq\mathbf{0}$ et $|\mathbf{m}|>0$.

However, by Lemma 6, (ii), we know that

$$\sum_{\mathbf{0}\leq\mathbf{m}\leq\mathbf{M}}H(\mathbf{m},\mathbf{K}';s)=0$$

if one chooses $\mathbf{M}$ sufficiently large. Isolating the term $H(\mathbf{0},\mathbf{K}';s)$, this equation can be rewritten as

$$H(\mathbf{0},\mathbf{K}';s)=-\sum_{\substack{\mathbf{0}\leq\mathbf{m}\leq\mathbf{M}\\|\mathbf{m}|>0}}H(\mathbf{m},\mathbf{K}';s).$$

The sum on the right-hand side is congruent to 0 mod p^{s+1} by (7.6), whence

$$H(\mathbf{0},\mathbf{K}';s)\equiv 0\mod p^{s+1}.$$

This means that $\gamma_{\mathbf{K}'}$ is true, which is absurd. Assertion $\gamma_{\mathbf{K}}$ is therefore true for all $\mathbf{K}\in\mathbb{Z}^d$.

Let us now return to Assertion $\beta_{s,s}$, which is displayed explicitly in (7.5). We have just shown that $H(\mathbf{0},\mathbf{K};s)\equiv 0\mod p^{s+1}$, while $A(\mathbf{m})/A(\mathbf{0})\in g(\mathbf{m})\mathbb{Z}_p$ by Properties (i) and (ii) in the statement of Theorem 1. Hence, we have also

$$H(\mathbf{m},\mathbf{K}+p^s\mathbf{m};s)\equiv 0\mod p^{s+1}g(\mathbf{m})\mathbb{Z}_p.$$

By replacing $\mathbf{K}$ by $\mathbf{K}-p^s\mathbf{m}$ (which is possible because $\mathbf{K}$ can be chosen freely from $\mathbb{Z}^d$), we see that this is nothing else but Assertion α_s. Thus, Theorem 1 follows indeed from the truth of $A1, A2$ and $A3$.

Second step. It remains to prove Assertions $A1$, $A2$ and $A3$ themselves, which we shall do in this order.

Proof of $A1$. The assertion holds if $k_i > K_i$ or if $K_i < 0$ for some i. If $\mathbf{K} \geq \mathbf{k} \geq \mathbf{0}$, we have

$$U(\mathbf{k},\mathbf{K}) = A(\mathbf{K}-\mathbf{k})A(\mathbf{v})\left(\frac{A(\mathbf{v}+p\mathbf{k})}{A(\mathbf{v})} - \frac{A(\mathbf{k})}{A(\mathbf{0})}\right) + A(\mathbf{k})A(\mathbf{v})\left(\frac{A(\mathbf{K}-\mathbf{k})}{A(\mathbf{0})} - \frac{A(\mathbf{v}+p(\mathbf{K}-\mathbf{k}))}{A(\mathbf{v})}\right).$$

Property (iii) in the statement of Theorem 1 with $\mathbf{u}=\mathbf{0}$, $\mathbf{n}=\mathbf{k}$, $s=0$ says that

$$\frac{A(\mathbf{v}+p\mathbf{k})}{A(\mathbf{v})} - \frac{A(\mathbf{k})}{A(\mathbf{0})} \in p\,\frac{g(\mathbf{k})}{g(\mathbf{v})}\,\mathbb{Z}_p$$

while its special case in which $\mathbf{u}=\mathbf{0}$, $\mathbf{n}=\mathbf{K}-\mathbf{k}$, $s=0$ reads

$$\frac{A(\mathbf{K}-\mathbf{k})}{A(\mathbf{0})} - \frac{A(\mathbf{v}+p(\mathbf{K}-\mathbf{k}))}{A(\mathbf{v})} \in p\,\frac{g(\mathbf{K}-\mathbf{k})}{g(\mathbf{v})}\,\mathbb{Z}_p.$$

Hence,

$$A(\mathbf{K}-\mathbf{k})A(\mathbf{v})\left(\frac{A(\mathbf{v}+p\mathbf{k})}{A(\mathbf{v})} - \frac{A(\mathbf{k})}{A(\mathbf{0})}\right) \in p\,g(\mathbf{k})A(\mathbf{K}-\mathbf{k})\frac{A(\mathbf{v})}{g(\mathbf{v})}\,\mathbb{Z}_p \subseteq pg(\mathbf{k})\,\mathbb{Z}_p$$

and

$$A(\mathbf{k})A(\mathbf{v})\left(\frac{A(\mathbf{K}-\mathbf{k})}{A(\mathbf{0})} - \frac{A(\mathbf{v}+p(\mathbf{K}-\mathbf{k}))}{A(\mathbf{v})}\right) \in pg(\mathbf{k})g(\mathbf{K}-\mathbf{k})\frac{A(\mathbf{k})}{g(\mathbf{k})}\frac{A(\mathbf{v})}{g(\mathbf{v})}\,\mathbb{Z}_p \subseteq pg(\mathbf{k})\,\mathbb{Z}_p,$$

where the inclusion relations result from Property (ii) in the statement of Theorem 1. It therefore follows that

$$U(\mathbf{k},\mathbf{K}) \in pg(\mathbf{k})\mathbb{Z}_p,$$

which proves Assertion $A1$.

Proof of $A2$. By a straightforward calculation, we have

$$U(\mathbf{k}+p^s\mathbf{m},\mathbf{K}+p^s\mathbf{m}) - \frac{A(\mathbf{k}+p^s\mathbf{m})}{A(\mathbf{k})}U(\mathbf{k},\mathbf{K}) = -A(\mathbf{K}-\mathbf{k})A(\mathbf{v}+p\mathbf{k})\left(\frac{A(\mathbf{v}+p\mathbf{k}+p^{s+1}\mathbf{m})}{A(\mathbf{v}+p\mathbf{k})} - \frac{A(\mathbf{k}+p^s\mathbf{m})}{A(\mathbf{k})}\right).$$

If $K_i < 0$ for some i, the right-hand side is zero since $A(\mathbf{K}-\mathbf{k}) = 0$, whence Assertion $A2$ is trivially true. If $\mathbf{K} \geq \mathbf{0}$, by Properties (iii) and (ii) in the statement of Theorem 1, the right-hand side is an element of

$$A(\mathbf{K}-\mathbf{k})A(\mathbf{v}+p\mathbf{k})\frac{g(\mathbf{m})}{g(\mathbf{v}+p\mathbf{k})}\,p^{s+1}\mathbb{Z}_p \subseteq g(\mathbf{m})p^{s+1}\mathbb{Z}_p,$$

which proves Assertion $A2$ in this case as well.

Proof of $A3$. Let $0 \leq t < s$, and assume that α_{s-1} and $\beta_{t,s}$ are true. Under these assumptions, we must deduce the truth of Assertion $\beta_{t+1,s}$.

In the assertion $\beta_{t,s}$, we replace the summation index $\mathbf{k}$ in the sum on the right-hand side of (7.2) by $\mathbf{i}+p\mathbf{u}$, where $0 \le i_\ell < p-1$ and $0 \le u_\ell < p^{s-t-1}$ for $\ell = 1,2,\dots,d$. Thus, we obtain that

$$H(\mathbf{m},\mathbf{K}+p^s\mathbf{m};s) \equiv \sum_{\mathbf{0}\le\mathbf{i}\le(p-1)\mathbf{1}} \left(\sum_{\mathbf{0}\le\mathbf{u}\le(p^{s-t-1}-1)\mathbf{1}} \frac{A(\mathbf{i}+p\mathbf{u}+p^{s-t}\mathbf{m})}{A(\mathbf{i}+p\mathbf{u})} H(\mathbf{i}+p\mathbf{u},\mathbf{K};t) \right) \mod p^{s+1}g(\mathbf{m})\mathbb{Z}_p. \quad (7.7)$$

Define

$$X := H(\mathbf{m},\mathbf{K}+p^s\mathbf{m};s) - \sum_{\mathbf{0}\le\mathbf{u}\le(p^{s-t-1}-1)\mathbf{1}} \frac{A(\mathbf{u}+p^{s-t-1}\mathbf{m})}{A(\mathbf{u})} \sum_{\mathbf{0}\le\mathbf{i}\le(p-1)\mathbf{1}} H(\mathbf{i}+p\mathbf{u},\mathbf{K};t).$$

Since $\beta_{t,s}$ (in the form (7.7)) is true, we have

$$X \equiv \sum_{\mathbf{0}\le\mathbf{i}\le(p-1)\mathbf{1}} \left(\sum_{\mathbf{0}\le\mathbf{u}\le(p^{s-t-1}-1)\mathbf{1}} H(\mathbf{i}+p\mathbf{u},\mathbf{K};t) \times \left(\frac{A(\mathbf{i}+p\mathbf{u}+p^{s-t}\mathbf{m})}{A(\mathbf{i}+p\mathbf{u})} - \frac{A(\mathbf{u}+p^{s-t-1}\mathbf{m})}{A(\mathbf{u})} \right) \right) \mod p^{s+1}g(\mathbf{m})\,\mathbb{Z}_p.$$

Since $u_i < p^{s-t-1}$ for all i, Property (iii) in the statement of Theorem 1 implies that

$$\frac{A(\mathbf{i}+p\mathbf{u}+p^{s-t}\mathbf{m})}{A(\mathbf{i}+p\mathbf{u})} - \frac{A(\mathbf{u}+p^{s-t-1}\mathbf{m})}{A(\mathbf{u})} \in p^{s+1}\frac{g(\mathbf{m})}{g(\mathbf{i}+p\mathbf{u})}\mathbb{Z}_p. \quad (7.8)$$

Moreover, since $t<s$, Assertion α_{s-1} implies that

$$H(\mathbf{i}+p\mathbf{u},\mathbf{K};t) \in p^{t+1}g(\mathbf{i}+p\mathbf{u})\mathbb{Z}_p. \quad (7.9)$$

It now follows from (7.8) and (7.9) that $X \equiv 0 \mod p^{s+1}g(\mathbf{m})\mathbb{Z}_p$.

However, by Lemma 6, (iii), we know that

$$\sum_{\mathbf{0}\le\mathbf{i}\le(p-1)\mathbf{1}} H(\mathbf{i}+p\mathbf{u},\mathbf{K};t) = H(\mathbf{u},\mathbf{K};t+1),$$

which can be used to simplify X to

$$X = H(\mathbf{m},\mathbf{K}+p^s\mathbf{m};s) - \sum_{\mathbf{0}\le\mathbf{u}\le(p^{s-t-1}-1)\mathbf{1}} \frac{A(\mathbf{u}+p^{s-t-1}\mathbf{m})}{A(\mathbf{u})} H(\mathbf{u},\mathbf{K};t+1).$$

Since $X \equiv 0 \mod p^{s+1}g(\mathbf{m})\mathbb{Z}_p$, the preceding identity shows that

$$H(\mathbf{m},\mathbf{K}+p^s\mathbf{m};s) \equiv \sum_{\mathbf{0}\le\mathbf{u}\le(p^{s-t-1}-1)\mathbf{1}} \frac{A(\mathbf{u}+p^{s-t-1}\mathbf{m})}{A(\mathbf{u})} H(\mathbf{u},\mathbf{K};t+1) \mod p^{s+1}g(\mathbf{m})\mathbb{Z}_p.$$

This is nothing else but Assertion $\beta_{t+1,s}$. Hence, Assertion $A3$ is established.

This completes the proof of Theorem 1. □

8. Theorem 1 implies Theorem 2

We want to prove that Theorem 1 can be applied for $A = g = B_{\mathbf{N}}$. In order to see this, we first establish some intermediary lemmas, extending corresponding auxiliary results in Section 7 of [14] to higher dimensions.

Lemma 7. *Under the assumptions of Theorem 1, we have*

$$\frac{B_{\mathbf{N}}(\mathbf{v}+p\mathbf{u}+p^{s+1}\mathbf{n})}{B_{\mathbf{N}}(p\mathbf{u}+p^{s+1}\mathbf{n})} = \frac{B_{\mathbf{N}}(\mathbf{v}+p\mathbf{u})}{B_{\mathbf{N}}(p\mathbf{u})} + \mathcal{O}\big(p^{s+1}\big),$$

where $\mathcal{O}(R)$ denotes an element of $R\mathbb{Z}_p$.

Proof. Recalling the definition of $B(\mathbf{N}^{(j)},\mathbf{m})$ in (2.1), we have

$$\begin{aligned}
&\frac{B(\mathbf{N}^{(j)},\mathbf{v}+p\mathbf{u}+p^{s+1}\mathbf{n})}{B(\mathbf{N}^{(j)},p\mathbf{u}+p^{s+1}\mathbf{n})}\\
&\quad= \frac{\Big(\sum_{i=1}^d N_i^{(j)}(pu_i+p^{s+1}n_i)+\sum_{i=1}^d N_i^{(j)}v_i\Big)\cdots\Big(\sum_{i=1}^d N_i^{(j)}(pu_i+p^{s+1}n_i)+1\Big)}{\prod_{i=1}^d\Big((v_i+pu_i+p^{s+1}n_i)\cdots(1+pu_i+p^{s+1}n_i)\Big)^{N_i^{(j)}}}\\
&\quad= \frac{\Big(\sum_{i=1}^d N_i^{(j)}(pu_i)+\sum_{i=1}^d N_i^{(j)}v_i\Big)\cdots\Big(\sum_{i=1}^d N_i^{(j)}(pu_i)+1\Big)+\mathcal{O}(p^{s+1})}{\prod_{i=1}^d\Big((v_i+pu_i)\cdots(1+pu_i)\Big)^{N_i^{(j)}}+\mathcal{O}(p^{s+1})}.
\end{aligned}$$

We claim that this implies

$$\begin{aligned}
&\frac{B(\mathbf{N}^{(j)},\mathbf{v}+p\mathbf{u}+p^{s+1}\mathbf{n})}{B(\mathbf{N}^{(j)},p\mathbf{u}+p^{s+1}\mathbf{n})}\\
&\quad= \frac{\Big(\sum_{i=1}^d N_i^{(j)}(pu_i)+\sum_{i=1}^d N_i^{(j)}v_i\Big)\cdots\Big(\sum_{i=1}^d N_i^{(j)}(pu_i)+1\Big)}{\prod_{i=1}^d\Big((v_i+pu_i)\cdots(1+pu_i)\Big)^{N_i^{(j)}}}+\mathcal{O}(p^{s+1})\\
&\quad= \frac{B(\mathbf{N}^{(j)},\mathbf{v}+p\mathbf{u})}{B(\mathbf{N}^{(j)},p\mathbf{u})}+\mathcal{O}\big(p^{s+1}\big).
\end{aligned}$$

Indeed, if $\mathbf{v}=\mathbf{0}$, then this holds trivially. If $\mathbf{v}>\mathbf{0}$, then, together with the hypothesis $v_i<p$, we infer that $(v_i+pu_i)(v_i+pu_i-1)\cdots(1+pu_i)$ is not divisible by p, which implies in particular that $B(\mathbf{N}^{(j)},\mathbf{v}+p\mathbf{u})/B(\mathbf{N}^{(j)},p\mathbf{u})\in\mathbb{Z}_p$. This allows us to arrive at the above conclusion in the same style as in Section 7.1 in [14].

By taking products, we deduce

$$\prod_{j=1}^k \frac{B(\mathbf{N}^{(j)},\mathbf{v}+p\mathbf{u}+p^{s+1}\mathbf{n})}{B(\mathbf{N}^{(j)},p\mathbf{u}+p^{s+1}\mathbf{n})} = \prod_{j=1}^k\left(\frac{B(\mathbf{N}^{(j)},\mathbf{v}+p\mathbf{u})}{B(\mathbf{N}^{(j)},p\mathbf{u})}+\mathcal{O}\big(p^{s+1}\big).\right).$$

By expanding the product on the right-hand side and using that $\dfrac{B(\mathbf{N}^{(j)},\mathbf{v}+p\mathbf{u})}{B(\mathbf{N}^{(j)},p\mathbf{u})}\in\mathbb{Z}_p$, we obtain the assertion of the lemma. □

For the proof of Lemma 9 below, we will use the p-adic gamma function, which is defined on integers $n\geq 1$ by

$$\Gamma_p(n)=(-1)^n\prod_{\substack{k=1\\(k,p)=1}}^{n-1}k.$$

In the following lemma, we collect some facts about Γ_p.

Lemma 8. (i) *For all integers* $n\geq 1$, *we have*

$$\frac{(np)!}{n!}=(-1)^{np+1}p^n\Gamma_p(1+np).$$

(ii) *For all integers* $k\geq 1, n\geq 1, s\geq 0$, *we have*

$$\Gamma_p(k+np^s)\equiv\Gamma_p(k)\mod p^s.$$

The above two properties of the p-adic gamma function are now used in the proof of the following result.

Lemma 9. *We have*

$$\frac{B_{\mathbf{N}}(p\mathbf{u}+p^{s+1}\mathbf{n})}{B_{\mathbf{N}}(\mathbf{u}+p^s\mathbf{n})}=\frac{B_{\mathbf{N}}(p\mathbf{u})}{B_{\mathbf{N}}(\mathbf{u})}\big(1+\mathcal{O}(p^{s+1})\big).$$

Proof. We have

$$\frac{B(\mathbf{N}^{(j)},p\mathbf{u}+p^{s+1}\mathbf{n})}{B(\mathbf{N}^{(j)},\mathbf{u}+p^s\mathbf{n})}=(-1)^{1+|\mathbf{N}^{(j)}|}\frac{\Gamma_p\big(1+\mathbf{N}^{(j)}\cdot(p\mathbf{u}+p^{s+1}\mathbf{n})\big)}{\prod_{i=1}^d\Gamma_p\big(1+pu_i+p^{s+1}n_i\big)^{N_i^{(j)}}} \tag{8.1}$$

$$=(-1)^{1+|\mathbf{N}^{(j)}|}\frac{\Gamma_p\big(1+p\mathbf{N}^{(j)}\cdot\mathbf{u}\big)+\mathcal{O}\big(p^{s+1}\big)}{\prod_{i=1}^d\Gamma_p\big(1+pu_i\big)^{N_i^{(j)}}+\mathcal{O}\big(p^{s+1}\big)} \tag{8.2}$$

$$=(-1)^{1+|\mathbf{N}^{(j)}|}\frac{\Gamma_p\big(1+p\mathbf{N}^{(j)}\cdot\mathbf{u}\big)}{\prod_{i=1}^d\Gamma_p\big(1+pu_i\big)^{N_i^{(j)}}}\big(1+\mathcal{O}(p^{s+1})\big) \tag{8.3}$$

$$=\frac{B(\mathbf{N}^{(j)},p\mathbf{u})}{B(\mathbf{N}^{(j)},\mathbf{u})}\big(1+\mathcal{O}(p^{s+1})\big). \tag{8.4}$$

where (i) of Lemma 8 is used to see (8.1) and (8.4), and (ii) is used for (8.2). Equation (8.3) holds because $\Gamma_p(1+pu_i)$ and $\Gamma_p(1+p\mathbf{N}^{(j)}\cdot\mathbf{u})$ are both not divisible by p. Taking the product over $j=1,2,\dots,k$, we obtain the assertion of the lemma. □

Before proceeding, we remark that $v_p\big(B(\mathbf{N}^{(j)},p^s\mathbf{u})/B(\mathbf{N}^{(j)},\mathbf{u})\big)=0$ for any integer $s\geq 0$, which can be proved in the same way as Lemma 13 in [14]. This property will be used twice below.

We now multiply both sides of the congruences obtained in Lemmas 7 and 9. Thus, we obtain

$$\begin{aligned}\frac{B_{\mathbf{N}}(\mathbf{v}+p\mathbf{u}+\mathbf{n}p^{s+1})}{B_{\mathbf{N}}(\mathbf{u}+\mathbf{n}p^s)} &= \frac{B_{\mathbf{N}}(\mathbf{v}+p\mathbf{u})}{B_{\mathbf{N}}(\mathbf{u})}\big(1+\mathcal{O}(p^{s+1})\big) + \frac{B_{\mathbf{N}}(p\mathbf{u})}{B_{\mathbf{N}}(\mathbf{u})}\mathcal{O}\big(p^{s+1}\big)\\ &= \frac{B_{\mathbf{N}}(\mathbf{v}+p\mathbf{u})}{B_{\mathbf{N}}(\mathbf{u})}\big(1+\mathcal{O}(p^{s+1})\big) + \mathcal{O}\big(p^{s+1}\big)\end{aligned}$$

(since $v_p\big(B_{\mathbf{N}}(p\mathbf{u})/B_{\mathbf{N}}(\mathbf{u})\big)=0$ by the remark above), which, in its turn, can be rewritten as

$$\frac{B_{\mathbf{N}}(\mathbf{v}+p\mathbf{u}+\mathbf{n}p^{s+1})}{B_{\mathbf{N}}(\mathbf{v}+p\mathbf{u})} = \frac{B_{\mathbf{N}}(\mathbf{u}+\mathbf{n}p^s)}{B_{\mathbf{N}}(\mathbf{u})} + \frac{B_{\mathbf{N}}(\mathbf{u}+\mathbf{n}p^s)}{B_{\mathbf{N}}(\mathbf{u})}\mathcal{O}\big(p^{s+1}\big) + \frac{B_{\mathbf{N}}(\mathbf{u}+\mathbf{n}p^s)}{B_{\mathbf{N}}(\mathbf{v}+p\mathbf{u})}\mathcal{O}\big(p^{s+1}\big).$$

It remains to show that

$$\frac{B_{\mathbf{N}}(\mathbf{u}+\mathbf{n}p^s)}{B_{\mathbf{N}}(\mathbf{u})} \in \frac{B_{\mathbf{N}}(\mathbf{n})}{B_{\mathbf{N}}(\mathbf{v}+p\mathbf{u})}\,\mathbb{Z}_p \tag{8.5}$$

and

$$\frac{B_{\mathbf{N}}(\mathbf{u}+\mathbf{n}p^s)}{B_{\mathbf{N}}(\mathbf{v}+p\mathbf{u})} \in \frac{B_{\mathbf{N}}(\mathbf{n})}{B_{\mathbf{N}}(\mathbf{v}+p\mathbf{u})}\,\mathbb{Z}_p. \tag{8.6}$$

These two facts will follow from the next lemma.

Lemma 10. *For all non-negative integers s, all integer vectors $\mathbf{n}\in\mathbb{Z}^d$ with $\mathbf{n}\geq\mathbf{0}$, and all integer vectors $\mathbf{u}\in\mathbb{Z}^d$ with $0\le u_i<p^s$, $i=1,2,\dots,d$, we have*

$$\frac{B_{\mathbf{N}}(\mathbf{u}+\mathbf{n}p^s)}{B_{\mathbf{N}}(\mathbf{u})} \in B_{\mathbf{N}}(\mathbf{n})\mathbb{Z}_p.$$

Proof. We have

$$\frac{B(\mathbf{N}^{(j)},\mathbf{u}+\mathbf{n}p^s)}{B(\mathbf{N}^{(j)},\mathbf{u})} = \frac{\binom{\sum_{i=1}^d N_i^{(j)}(u_i+n_ip^s)}{\sum_{i=1}^d N_i^{(j)}u_i}}{\prod_{i=1}^d \binom{u_i+n_ip^s}{u_i}^{N_i^{(j)}}}\cdot\frac{B(\mathbf{N}^{(j)},\mathbf{n}p^s)}{B(\mathbf{N}^{(j)},\mathbf{n})}\cdot B(\mathbf{N}^{(j)},\mathbf{n}).$$

On the right-hand side, the term $B(\mathbf{N}^{(j)},\mathbf{n}p^s)/B(\mathbf{N}^{(j)},\mathbf{n})$ and the binomial coefficients $\binom{u_i+n_ip^s}{u_i}$ have vanishing p-adic valuation (this has already been observed in the paragraph after the end of the proof of Lemma 9). Thus we have

$$\frac{B(\mathbf{N}^{(j)},\mathbf{u}+\mathbf{n}p^s)}{B(\mathbf{N}^{(j)},\mathbf{u})} \in B(\mathbf{N}^{(j)},\mathbf{n})\mathbb{Z}_p. \tag{8.7}$$

The lemma follows by taking the product over $j\in\{1,\dots,k\}$ of both sides of (8.7). $\square$

The preceding lemma implies

$$\frac{B_{\mathbf{N}}(\mathbf{u}+\mathbf{n}p^s)}{B_{\mathbf{N}}(\mathbf{u})} \in B_{\mathbf{N}}(\mathbf{n})\mathbb{Z}_p \subseteq \frac{B_{\mathbf{N}}(\mathbf{n})}{B_{\mathbf{N}}(\mathbf{v}+p\mathbf{u})}\,\mathbb{Z}_p,$$

which proves (8.5). Moreover, still due to Lemma 10, we have

$$\frac{B_{\mathbf{N}}(\mathbf{u}+\mathbf{n}p^s)}{B_{\mathbf{N}}(\mathbf{v}+p\mathbf{u})} = \frac{B_{\mathbf{N}}(\mathbf{u}+\mathbf{n}p^s)}{B_{\mathbf{N}}(\mathbf{u})} \cdot B_{\mathbf{N}}(\mathbf{u}) \cdot \frac{1}{B_{\mathbf{N}}(\mathbf{v}+p\mathbf{u})}$$
$$\in B_{\mathbf{N}}(\mathbf{u}) \cdot \frac{B_{\mathbf{N}}(\mathbf{n})}{B_{\mathbf{N}}(\mathbf{v}+p\mathbf{u})}\,\mathbb{Z}_p \subseteq \frac{B_{\mathbf{N}}(\mathbf{n})}{B_{\mathbf{N}}(\mathbf{v}+p\mathbf{u})}\,\mathbb{Z}_p,$$

which proves (8.6). Therefore,

$$\frac{B_{\mathbf{N}}(\mathbf{v}+p\mathbf{u}+\mathbf{n}p^{s+1})}{B_{\mathbf{N}}(\mathbf{v}+p\mathbf{u})} - \frac{B_{\mathbf{N}}(\mathbf{u}+\mathbf{n}p^s)}{B_{\mathbf{N}}(\mathbf{u})} \in p^{s+1}\,\frac{B_{\mathbf{N}}(\mathbf{n})}{B_{\mathbf{N}}(\mathbf{v}+p\mathbf{u})}\,\mathbb{Z}_p,$$

which shows that Property (*iii*) of Theorem 1 is satisfied. Since Properties (*i*) and (*ii*) are trivially true, we can hence apply the latter theorem.

9. Proof of Lemma 4

The claim is trivially true if p divides m_i for all i. We may therefore assume that p does not divide m_i for some i between 1 and d for the rest of the proof. Let us write $\mathbf{m} = \mathbf{a} + p\mathbf{j}$, with $0 \le a_i < p$ for all i (but at least one a_i is positive). We are apparently in a similar situation as in Lemma 3. Indeed, we may derive Lemma 4 from Lemma 3. In order to see this, we observe that

$$\begin{aligned} H_{\sum_{i=1}^d L_i m_i p^s} - H_{\sum_{i=1}^d L_i \lfloor \frac{m_i}{p} \rfloor p^{s+1}} &= \sum_{\varepsilon=1}^{p^s \mathbf{L}\cdot\mathbf{a}} \frac{1}{p^{s+1}\mathbf{L}\cdot\mathbf{j}+\varepsilon} \\ &= \sum_{\varepsilon=1}^{\lfloor \mathbf{L}\cdot\mathbf{a}/p \rfloor} \frac{1}{p^{s+1}\mathbf{L}\cdot\mathbf{j}+p^{s+1}\varepsilon} + \sum_{\substack{\varepsilon=1\\ p^{s+1}\nmid\varepsilon}}^{p^s \mathbf{L}\cdot\mathbf{a}} \frac{1}{p^{s+1}\mathbf{L}\cdot\mathbf{j}+\varepsilon} \\ &= \frac{1}{p^{s+1}}\big(H_{\mathbf{L}\cdot\mathbf{j}+\lfloor \mathbf{L}\cdot\mathbf{a}/p \rfloor} - H_{\mathbf{L}\cdot\mathbf{j}}\big) + \sum_{\substack{\varepsilon=1\\ p^{s+1}\nmid\varepsilon}}^{p^s \mathbf{L}\cdot\mathbf{a}} \frac{1}{p^{s+1}\mathbf{L}\cdot\mathbf{j}+\varepsilon}. \end{aligned}$$

Because of $v_p(x+y) \ge \min\{v_p(x), v_p(y)\}$, this implies

$$v_p\big(H_{\sum_{i=1}^d L_i m_i p^s} - H_{\sum_{i=1}^d L_i \lfloor \frac{m_i}{p} \rfloor p^{s+1}}\big) \ge \min\{-1-s+v_p(H_{\mathbf{L}\cdot\mathbf{j}+\lfloor \mathbf{L}\cdot\mathbf{a}/p \rfloor} - H_{\mathbf{L}\cdot\mathbf{j}}), -s\}.$$

It follows that

$$\begin{aligned} &v_p\Big(B_{\mathbf{N}}(\mathbf{m})\big(H_{\sum_{i=1}^d L_i m_i p^s} - H_{\sum_{i=1}^d L_i \lfloor \frac{m_i}{p} \rfloor p^{s+1}}\big)\Big) \\ &\qquad \ge -1-s+\min\Big\{v_p\Big(B_{\mathbf{N}}(\mathbf{a}+p\mathbf{j})(H_{\mathbf{L}\cdot\mathbf{j}+\lfloor \mathbf{L}\cdot\mathbf{a}/p \rfloor} - H_{\mathbf{L}\cdot\mathbf{j}})\Big), 1+v_p\Big(B_{\mathbf{N}}(\mathbf{a}+p\mathbf{j})\Big)\Big\}. \end{aligned}$$

Use of Lemma 3 then completes the proof. □

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