ON THE INTEGRALITY OF THE TAYLOR COEFFICIENTS OF MIRROR MAPS

C. KRATTENTHALER† AND T. RIVOAL

Abstract. We show that the Taylor coefficients of the series $\mathbf{q}(z) = z\exp(\mathbf{G}(z)/\mathbf{F}(z))$ are integers, where $\mathbf{F}(z)$ and $\mathbf{G}(z) + \log(z)\mathbf{F}(z)$ are specific solutions of certain hypergeometric differential equations with maximal unipotent monodromy at $z = 0$. We also address the question of finding the largest integer u such that the Taylor coefficients of $(z^{-1}\mathbf{q}(z))^{1/u}$ are still integers. As consequences, we are able to prove numerous integrality results for the Taylor coefficients of mirror maps of Calabi–Yau complete intersections in weighted projective spaces, which improve and refine previous results by Lian and Yau, and by Zudilin. In particular, we prove the general "integrality" conjecture of Zudilin about these mirror maps.

1. Introduction and statement of results

1.1. Mirror maps. *Mirror maps* have appeared quite recently in mathematics and physics. Indeed, the term "mirror map" was coined in the late 1980s by physicists whose research in string theory led them to discover deep facts in algebraic geometry (e.g., given a Calabi–Yau threefold M, they constructed another Calabi–Yau threefold, the "mirror" of M, whose properties can be used to enumerate the rational curves on M).

The purpose of the present article is to prove rather sharp integrality assertions for the Taylor coefficients of mirror maps coming from certain hypergeometric differential equations, which are Picard–Fuchs equations of suitable one parameter families of Calabi–Yau complete intersections in weighted projective spaces. The corresponding results (see Theorems 1 and 2) encompass integrality results on these mirror maps which exist in the literature, improving and refining them in numerous instances.

In a sense, mirror maps can be viewed as higher order generalisations of certain classical modular forms (defined over various congruence sub-groups of $SL_2(\mathbb{Z})$), the latter appearing naturally at low order in Schwarz's theory of hypergeometric functions (see [30]). For

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integers $k \geq 1$ and $N \geq 1$, let us define the power series

$$F(z) := \sum_{m=0}^{\infty} \frac{(Nm)!^k}{m!^{kN}} z^m,$$

which converges for $|z| < 1/N^{kN}$. The function $F(z)$ is solution of a hypergeometric differential equation of degree kN, which is a special case of (1.5) below. The equation has maximal unipotent monodromy (MUM), i.e., $F(N^{-kN}z)$ is a hypergeometric function with only 1's as lower parameters (in other words, the roots of the indicial equation at $z = 0$ are all 0). A basis of solutions with at most logarithmic singularities around $z = 0$ can then be obtained by Frobenius' method; see [30]. In particular, there exists another solution of the form $G(z) + \log(z)F(z)$, where $G(z)$ is holomorphic around 0,

$$G(z) := \sum_{m=1}^{\infty} \frac{(Nm)!^k}{m!^{kN}} kN(H_{Nm} - H_m)\, z^m,$$

with $H_n := \sum_{i=1}^{n} \frac{1}{i}$ denoting the n-th harmonic number. In the context of mirror symmetry, the function $q(z) := z\exp(G(z)/F(z))$ is usually called *canonical coordinate*, and its compositional inverse $z(q)$ is the prototype of a *mirror map*.

In the case $k = N = 2$, one can express $z(q)$ explicitly in terms of the Legendre function $\lambda(q) := 16q\prod_{n=1}^{\infty}\big((1+q^{2n})/(1+q^{2n-1})\big)$, namely as $z(q) = \lambda(q)/16$, which is modular over $\Gamma(2)$. Moreover, if $k = 3$ and $N = 2$, we have $z(q) = \lambda(q)(1-\lambda(q))/16$. In particular, in both cases the power series $q(z)$ and $z(q)$ have *integral Taylor coefficients*. It is this fact that we will generalise in this paper. For other examples of mirror maps of modular origin, we refer to the discussion in [3, pp. 111–113] (and also for the importance of such facts in Diophantine approximation) and [20, Sec. 3].

The most famous (apparently non-modular) example of a mirror map arises in the case when $N = 5$ and $k = 1$, which was used in the epoch-making paper by the physicists Candelas et al. [7] in their study of the family $\mathbf{M}$ of quintic hypersurfaces in $\mathbb{P}^4(\mathbb{C})$ defined by $\sum_{k=1}^{5} x_k^5 - 5z\prod_{k=1}^{5} x_k = 0$, z being a complex parameter (see [23, 24, 28]).

The following conjecture belongs probably to the folklore of mirror symmetry theory.

Conjecture 1. *For any integers $k \geq 1$ and $N \geq 1$, we have $q(z) \in z\mathbb{Z}[[z]]$ and $z(q) \in q\mathbb{Z}[[q]]$.* ([1])

Lian and Yau [20, Sec. 5, Theorem 5.5] proved this conjecture for $k = 1$ and any N which is a prime number. Zudilin [31, Theorem 3] extended their result by proving the conjecture for any $k \geq 1$ and any N which is a prime power.

[1] In the number-theoretic study undertaken in the present paper, we are interested in the integrality of the coefficients of (roots of) mirror maps $z(q)$. In that context, the mirror map $z(q)$ and the corresponding canonical coordinate $q(z)$ play strictly the same role, because $(z^{-1}q(z))^{1/\tau} \in 1 + z\mathbb{Z}[[z]]$ for some integer τ implies that $(q^{-1}z(q))^{1/\tau} \in 1 + q\mathbb{Z}[[q]]$, and conversely. (See [19, Introduction].) We shall, in the sequel, formulate our integrality results exclusively for canonical coordinates, assuming tacitly that the reader keeps in mind that they automatically also hold for the corresponding mirror maps. It is also therefore that, by abuse of terminology, we shall often use the term "mirror map" for any canonical coordinate.

Our original goal was to settle Conjecture 1 in complete generality. In the present paper, we shall accomplish much more. We shall prove a more general conjecture by Zudilin [31] concerning the integrality of Taylor coefficients of a very large class of mirror maps (see Conjecture 2) and refinements of the corresponding integrality results in special cases. In the remainder of this introductory section, we describe these two sets of results (see Theorems 1 and 2). Their proofs will then be given in the subsequent sections.

1.2. **Zudilin's conjecture.** In order to state Zudilin's conjecture, we need to introduce some notation.

For a positive integer N, let $p_1, p_2, \dots, p_\ell$ denote its distinct prime factors. We define the vectors of integers

$$(\alpha_j)_{j=1,\dots,\mu} := \left(N, \frac{N}{p_{j_1}p_{j_2}}, \frac{N}{p_{j_1}p_{j_2}p_{j_3}p_{j_4}}, \dots\right)_{1\le j_1<j_2<\dots\le\ell} \tag{1.1}$$

and

$$(\beta_j)_{j=1,\dots,\eta} := \left(\frac{N}{p_{j_1}}, \frac{N}{p_{j_1}p_{j_2}p_{j_3}}, \dots, 1, 1, \dots, 1\right)_{1\le j_1<j_2<\dots\le\ell}, \tag{1.2}$$

where $\alpha_1 + \alpha_2 + \dots + \alpha_\mu = \beta_1 + \beta_2 + \dots + \beta_\eta$. We put $\mathbf{B}_1(m) := 1$ and $\mathbf{H}_1(m) := 0$, and, if $N \ge 2$,

$$\mathbf{B}_N(m) := \frac{\prod_{j=1}^{\mu}(\alpha_j m)!}{\prod_{j=1}^{\eta}(\beta_j m)!} \tag{1.3}$$

and [2]

$$\mathbf{H}_N(m) = \sum_{j=1}^{\mu} \alpha_j H_{\alpha_j m} - \sum_{j=1}^{\eta} \beta_j H_{\beta_j m}. \tag{1.4}$$

For example, we have

$$\mathbf{B}_4(m) = \frac{(4m)!}{(2m)!\, m!^2},\ \mathbf{B}_6(m) = \frac{(6m)!}{(3m)!\,(2m)!\,m!},\ \mathbf{B}_{30}(m) = \frac{(30m)!\,(5m)!\,(3m)!\,(2m)!}{(15m)!\,(10m)!\,(6m)!\,m!^9},$$

and, correspondingly,

$$\mathbf{H}_4(m) = 4H_{4m} - 2H_{2m} - 2H_m,\ \mathbf{H}_6(m) = 6H_{6m} - 3H_{3m} - 2H_{2m} - H_m,$$
$$\mathbf{H}_{30}(m) = 30H_{30m} + 5H_{5m} + 3H_{3m} + 2H_{2m} - 15H_{15m} - 10H_{10m} - 6H_{6m} - 9H_m.$$

Given a vector $\mathbf{N} = (N_1, N_2, \dots, N_k)$ of positive integers, we construct the power series

$$\mathbf{F_N}(z) := \sum_{m=0}^{\infty} \left(\prod_{j=1}^{k} \mathbf{B}_{N_j}(m)\right) z^m$$

[2] Zudilin's definition [31, Eq. (5)] of the quantity $\mathbf{H}_N(m)$ (he writes $D_N(m)$) is different from (1.4). We do not need it in our paper, but, for the sake of completeness, we prove the equivalence of the two definitions in Section 11.

and

$$\mathbf{G}_{\mathbf{N}}(z) := \sum_{m=1}^{\infty} \left(\sum_{j=1}^{k} \mathbf{H}_{N_j}(m) \right) \left(\prod_{j=1}^{k} \mathbf{B}_{N_j}(m) \right) z^m.$$

It can be shown (see [31], taking into account Lemma 15 in Section 11) that the series $\mathbf{F}_{\mathbf{N}}(z)$ and $\mathbf{G}_{\mathbf{N}}(z) + \log z \mathbf{F}_{\mathbf{N}}(z)$ are two solutions to the equation $\mathbf{L}y = 0$, where the hypergeometric differential operator $\mathbf{L}$ is defined by

$$\mathbf{L} := \left(z \frac{\mathrm{d}}{\mathrm{d}z} \right)^{\varphi(N_1)+\cdots+\varphi(N_k)} - C_{\mathbf{N}} z \prod_{j=1}^{k} \prod_{i=1}^{\varphi(N_j)} \left(z \frac{\mathrm{d}}{\mathrm{d}z} + \frac{r_{i,j}}{N_j} \right), \tag{1.5}$$

where $\varphi(\,.\,)$ is Euler's totient function, $C_{\mathbf{N}} := \prod_{j=1}^{k} C_{N_j}$ with $C_{N_j} := N_j^{\varphi(N_j)} \prod_{p \mid N_j} p^{\varphi(N_j)/(p-1)}$, and the $r_{i,j} \in \{1, 2, \ldots, N_j\}$ form the residue classes modulo N_j which are coprime to N_j. The differential equation $\mathbf{L}y = 0$ has MUM ([3]) at the origin because the roots of the indicial equation at $z = 0$ are visibly all 0.

We can now state Zudilin's conjecture from [31, p. 605]. (Zudilin's formulation is different. That our formulation is equivalent with Zudilin's follows from [31, Lemma 4], respectively (11.1), and from Lemma 15 in Section 11.)

Conjecture 2 (Zudilin). *For any positive integers $N_1, N_2, \ldots, N_k$, we have $\mathbf{q}_{\mathbf{N}}(z) := z \exp(\mathbf{G}_{\mathbf{N}}(z)/\mathbf{F}_{\mathbf{N}}(z)) \in z\mathbb{Z}[[z]]$.*

It can be seen (see [31, paragraph above Theorem 2], respectively (1.6)–(1.8) below) that Conjecture 1 is the special case of the above conjecture where k is replaced by $k \cdot d(N)$ (with $d(N)$ denoting the number of positive divisors of N), and where $\{N_1, \ldots, N_{k \cdot d(N)}\}$ is the multiset ([4]) in which each divisor of N appears exactly k times.

Zudilin proved that his conjecture holds under the condition that if a prime number divides $N_1 N_2 \cdots N_k$ then it also divides each N_j.

We claim that Conjecture 2 follows from the theorem below, which is the first main result of the paper. For the statement of the theorem, for an integer $L \geq 1$, we need to define

$$\mathbf{G}_{L,\mathbf{N}}(z) := \sum_{m=1}^{\infty} H_{Lm} \left(\prod_{j=1}^{k} \mathbf{B}_{N_j}(m) \right) z^m.$$

Theorem 1. *For any integers $N_1, N_2, \ldots, N_k \geq 1$ and $L \in \{1, 2, \ldots, \max(N_1, \ldots, N_k)\}$, we have $\mathbf{q}_{L,\mathbf{N}}(z) := \exp(\mathbf{G}_{L,\mathbf{N}}(z)/\mathbf{F}_{\mathbf{N}}(z)) \in \mathbb{Z}[[z]]$.*

An outline of the proof of this theorem is given in Section 2, with details being filled in in Sections 4–8.

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[3] This equation is the Picard–Fuchs equation of the mirror Calabi–Yau family of a one parameter family of Calabi–Yau complete intersections in a weighted projective space. See [8] and [13, Sec. 3].

[4] A multiset is a "set" where one allows repetitions of elements.

To see that Theorem 1 implies Conjecture 2, we observe that, by (1.4), $\sum_{j=1}^{k} \mathbf{H}_{N_j}(m)$ is a finite sum of terms of the form λH_{Lm}, where λ and L are integers with

$$L \in \{1, 2, \dots, \max(N_1, \dots, N_k)\}.$$

Thus, $z^{-1}\mathbf{q}_{\mathbf{N}}(z)$ is a product of series $\mathbf{q}_{L,\mathbf{N}}(z)$, each one raised to an integer power. It follows that Conjecture 2 implies Theorem 1, as claimed.

1.3. **Stronger integrality properties in special cases.** Let $N_1, N_2, \dots, N_k$ be given positive integers (not necessarily distinct). In the setting of Section 1.2 with k replaced by $k \cdot (d(N_1) + d(N_2) + \cdots + d(N_k))$, $d(N)$ again denoting the number of positive divisors of N, we consider the special case where the vector $\mathbf{N}$ can be partitioned into k blocks, the j-th block consisting of all the positive divisors of N_j, $j = 1, 2, \dots, k$. It can be seen (cf. [31, paragraph above Theorem 2]) that the functions $\mathbf{F}_{\mathbf{N}}(z)$, $\mathbf{G}_{\mathbf{N}}(z)$ and $\mathbf{G}_{L,\mathbf{N}}(z)$ then simplify to

$$\sum_{m=0}^{\infty} \left(\prod_{j=1}^{k} \frac{(N_j m)!}{m!^{N_j}} \right) z^m, \tag{1.6}$$

$$\sum_{m=1}^{\infty} \left(\sum_{j=1}^{k} N_j (H_{N_j m} - H_m) \right) \left(\prod_{j=1}^{k} \frac{(N_j m)!}{m!^{N_j}} \right) z^m, \tag{1.7}$$

$$\sum_{m=1}^{\infty} H_{Lm} \left(\prod_{j=1}^{k} \frac{(N_j m)!}{m!^{N_j}} \right) z^m, \tag{1.8}$$

respectively. In order to simplify notation, for the remainder of this subsection we "re-define" $\mathbf{N}$ by letting $\mathbf{N} = (N_1, N_2, \dots, N_k)$, and we denote the series in (1.6), (1.7), and (1.8) respectively by $F_{\mathbf{N}}(z)$, $G_{\mathbf{N}}(z)$, and $G_{L,\mathbf{N}}(z)$. Accordingly, we define $q_{\mathbf{N}}(z) := z \exp\big(G_{\mathbf{N}}(z)/F_{\mathbf{N}}(z)\big)$ and $q_{L,\mathbf{N}}(z) := \exp\big(G_{L,\mathbf{N}}(z)/F_{\mathbf{N}}(z)\big)$.

For the mirror map $q_{(N)}(z)$ arising for $k = 1$, physicists made the observation that, apparently,

$$\big(z^{-1} q_{(N)}(z)\big)^{1/N} \in \mathbb{Z}[[z]]. \tag{1.9}$$

This was proved by Lian and Yau [22] for any prime number N, thus strengthening their result from [20] mentioned after Conjecture 1. The observation (1.9) leads naturally to the more general question of determining the largest integer V such that $\big(z^{-1}q(z)\big)^{1/V} \in \mathbb{Z}[[z]]$ for the mirror map $q(z) = q_{(N)}(z)$ in (1.9) or other mirror maps. ([5]) While we are not able to give a precise answer, we shall prove the following result for a large class of mirror maps which, as we explain in [16], comes very close to being optimal for this class.

[5] Let $q(z)$ be a given power series in $\mathbb{Z}[[z]]$, and let V be the largest integer with the property that $q(z)^{1/V} \in \mathbb{Z}[[z]]$. Then V carries complete information about *all* integers with that property: namely, the set of integers U with $q(z)^{1/U} \in \mathbb{Z}[[z]]$ consists of all divisors of V. Indeed, it is clear that all divisors of V belong to this set. Moreover, if U_1 and U_2 belong to this set, then also $\mathrm{lcm}(U_1, U_2)$ does (cf. [12, Lemma 5] for a simple proof based on Bézout's lemma).

Theorem 2. *For any integers $N_1, \dots, N_k \ge 1$, let $M_{\mathbf{N}} = \prod_{i=1}^k N_i!$. Furthermore, let $\Theta_L := L!/\gcd(L!, L!\,H_L)$ be the denominator of H_L when written as a reduced fraction. Then, for all $L \in \{1, 2, \dots, \max(N_1, \dots, N_k)\}$, we have $q_{L,\mathbf{N}}(z)^{\frac{\Theta_L}{M_{\mathbf{N}}}} \in \mathbb{Z}[[z]]$.*

Remarks. (a) For any integer $s \ge 1$, we have $q_{L,\mathbf{N}}(z)^{1/s} = 1 + s^{-1} M_{\mathbf{N}} H_L z + \mathcal{O}(z^2)$, and hence Theorem 2 is optimal when $L = 1$. This is not necessarily the case for other values of L and, in particular, Theorem 2 can be improved when $L = N_1 = \dots = N_k$. See [16] for such results.

(b) It is natural to expect refinements of Theorem 1 in the spirit of Theorem 2. We did not make a systematic research in this direction, but it could be interesting to do so. For example, in the case $k = 1$ and $\mathbf{N} = (6)$, it seems that the following relations are best possible: $\mathbf{q}_{1,(6)}(z)^{1/60}$, $\mathbf{q}_{2,(6)}(z)^{1/6}$, $\mathbf{q}_{3,(6)}(z)^{1/2}$, $\mathbf{q}_{4,(6)}(z)$, $\mathbf{q}_{5,(6)}(z)$ and $\mathbf{q}_{6,(6)}(z)$ are in $\mathbb{Z}[[z]]$. As a first step towards such refinements, we prove in Lemma 14 in Section 5 that $\mathbf{B}_{\mathbf{N}}(1)$ always divides $\mathbf{B}_{\mathbf{N}}(m)$ for any $m \ge 1$ and any $\mathbf{N}$, where $\mathbf{B}_{\mathbf{N}}(m) := \prod_{j=1}^k \mathbf{B}_{N_j}(m)$. Our techniques enable us to deduce that $\mathbf{q}_{1,\mathbf{N}}(z)^{1/\mathbf{B}_{\mathbf{N}}(1)} \in \mathbb{Z}[[z]]$ (cf. Remark 1 in Section 2), which proves the above assertion that $\mathbf{q}_{1,(6)}(z)^{1/60} \in \mathbb{Z}[[z]]$. This is optimal because $\mathbf{q}_{1,\mathbf{N}}(z) = 1 + \mathbf{B}_{\mathbf{N}}(1) z + \mathcal{O}(z^2)$. It turns out that $\mathbf{B}_{\mathbf{N}}(1)$ is a natural generalisation of the quantity $M_{\mathbf{N}}$ which appears in Theorem 2. However, for larger values of the parameter L, we do not know what the analogue of the quantity $M_{\mathbf{N}}/\Theta_L$ appearing in Theorem 2 would be.

An outline of the proof of Theorem 2 is given in Section 3, with details being filled in in Sections 9–10.

Due to the equation

$$q_{(N,\dots,N)}(z) = z q_{N,(N,\dots,N)}(z)^{kN} q_{1,(N,\dots,N)}(z)^{-kN}, \tag{1.10}$$

(with k occurrences of N in $(N, \dots, N)$), Theorem 2 has the following consequence for the mirror map $q_{(N,\dots,N)}(z)$, thus improving significantly upon (1.9).

Corollary 1. *For all integers $k \ge 1$ and $N \ge 1$, we have*

$$\left(z^{-1} q_{(N,\dots,N)}(z)\right)^{\frac{\Theta_N}{N!^k kN}} \in \mathbb{Z}[[z]].$$

Remarks. (a) In particular, in the emblematic case of the mirror map $q_{(5)}(z)$ of the quintic (case $N = 5$, $k = 1$), we obtain that $\left(z^{-1} q_{(5)}(z)\right)^{1/10} \in \mathbb{Z}[[z]]$, which improves on (1.9) by a factor of 2.

(b) Also Corollary 1 can be improved. The corresponding result, which is optimal subject to a widely believed conjecture on harmonic numbers, is very technical. We refer the reader again to our article [16].

1.4. **Method of proof.** The basic idea is to transfer the original integrality assertions into a p-adic framework (by means of Lemma 8), to a point where we can employ Dwork's theory of formal congruences. This theory seems to provide the most powerful tools available for attacking integrality assertions of the type discussed in this paper. We recall the corner stones of Dwork's theory in Section 4.

We draw the reader's attention to the fact that, while the general line of our approach follows that of previous authors (particularly [22]), there does arise a crucial difference (other than just technical complications): the reduction and rearrangement of the sums $\mathbf{C}(a + Kp)$ in Section 2, respectively $C(a + Kp)$ in Section 3, via the congruence (2.3) requires a new reduction step, namely Lemma 1, respectively Lemma 5. In fact, the proofs of these two lemmas form the most difficult parts of our paper. (In previous work, the use of (2.3) sufficed because the corresponding authors restricted themselves to N being prime or a prime power.)

1.5. **Related work and perspectives.** One of the goals of the present paper is to highlight number-theoretic phenomena in the context of mirror symmetry theory. Our analysis is clearly on the number theory side. Nevertheless, we hope that our results contribute to the clarification of such phenomena. On the other hand, it should not be hidden that integrality phenomena should also have intrinsic significance, within mirror symmetry. One such example is the (apparent) coincidence of coefficients in Yukawa couplings with numbers of certain rational curves (cf. [11, 21]; see also the discussion in [28, p. 49]).

In the context of our results, it is possible to prove similar, but strictly weaker, statements by means of methods of arithmetic geometry. This is the case for the preprint [29], which is an elaborate version of [14]. The mirror maps that are considered in that paper comprise ours. When both approaches apply simultaneously, our results in Theorems 1 and 2 are stronger than Theorem 2 in [29, Sec. 1.3]. Indeed, we prove that certain mirror maps have integral Taylor coefficients, while in [29] the weaker statement is proved that mirror maps have Taylor coefficients in $\mathbb{Z}[1/n]$, where n is an integer parameter of geometric origin which is at least 2 (by assumption iii) just before Theorem 2 in [29]).

We also want to point out that the range of application of Dwork's ideas is not restricted to p-adic functions in one variable. In [15], we extend Dwork's theory as outlined in Section 4 to several variables. As applications, we obtain integrality properties of mirror maps in several variables arising in the context of the very general multivariable mirror maps coming from the Gelfand–Kapranov–Zelevinsky hypergeometric series (see [5, Sec. 7.1], [13] and [27, Sec. 8] for numerous examples related to Calabi–Yau manifolds which are complete intersections in products of weighted projective spaces). As a by-product, by appropriate specialisations, we are even able to prove (predicted) integrality of the Taylor coefficients of some mirror maps in one variable that do not fall under the scope of the results of the present paper, in particular many of those in the table presented in [1].

We therefore believe that Dwork's methods provide valuable insight in integrality properties of mirror maps. Since the power and range of applicability of these methods have apparently not yet been exhausted, their development should be further pursued. In particular, we hope to be able to test them against the difficult number-theoretical problems raised by Yukawa couplings.

Finally, we point out that the quantity $\mathbf{B}_N(m)$ defined in (1.3), which is the crucial building block for the series $\mathbf{F_N}(z)$, $\mathbf{G_N}(z)$, and $\mathbf{G}_{L,\mathbf{N}}(z)$, is in fact an integer for all N and m. This follows from the criterion [17] of Landau, which, applied to our case, says that, for a fixed N, $\mathbf{B}_N(m)$ is integral for all m if and only if

$$\sum_{j=1}^{\mu} \lfloor \alpha_j x \rfloor - \sum_{j=1}^{\eta} \lfloor \beta_j x \rfloor \geq 0$$

for all non-negative real numbers x. Indeed, the above quantity is exactly the quantity $\Delta(x)$ defined in Lemma 12 in Section 5, which plays a crucial role in the proof of Lemma 1 in Section 6, and, thus, in the proof of Theorem 1. More precisely, amongst its properties, we use many times the fact that it is weakly increasing on $[0, 1)$. We point out that such "Landau functions" were introduced in the theory of mirror symmetry of Calabi–Yau threefolds by Rodriguez-Villegas [25], who used them to prove that there are exactly 14 hypergeometric functions whose coefficients can be written as integral quotients of factorials (after rescaling the variable z to Cz for some C) and with the MUM property at the origin. In [26], he also used Landau functions to characterise algebraic hypergeometric series whose coefficients can be written as integral quotients of factorials after rescaling: using the theory of Beukers and Heckman [6], he proved that such series are algebraic if and only if $\Delta(x) \in \{0, 1\}$. Of course, it is not always possible to write the Taylor coefficients of a hypergeometric series as quotients of factorials.

It can be proved that amongst hypergeometric series whose Taylor coefficients are integral quotients of factorials, the weak increase of the associated Landau function Δ is equivalent to the MUM property. This is a consequence of more general results of Eric Delaygue, in a thesis currently written under the supervision of the second author. Thus, from the point of view of mirror symmetry where the MUM property is essential, our Theorem 1 is best possible in that class of hypergeometric series. The question then naturally arises which properties the Landau function Δ must have such that the mirror map formally associated to it (by forming functions analogous to $\mathbf{F_N}$ and $\mathbf{G_N}$) has integral Taylor coefficients (even when a mirror symmetry interpretation is not available; for example, when the MUM property does not hold). We believe that the following statement is true: the formal mirror map has integral Taylor coefficients if and only if $\Delta(x)$, defined on $[0, 1)$, remains ≥ 1 after its first jump from 0. New ideas are necessary to solve this problem because the approach used in this paper does not work in this more general setting; indeed there are counterexamples to almost all our lemmas when Δ is not weakly increasing. This is currently under investigation by Eric Delaygue.

1.6. **Structure of the paper.** We now briefly review the organisation of the rest of the paper.

The proofs of our theorems being highly complex, we start with brief outlines of the proofs of Theorems 1 and 2 in Sections 2 and 3, respectively. These outlines reduce the proofs to a certain number of lemmas. The reduction is heavily based on Dwork's theory of formal congruences, the relevant pieces of which being recalled in Section 4. The lemmas which are necessary for the proof of Theorem 1 are subsequently established in Sections 5–8, while those necessary for the proof of Theorem 2 are established in Sections 9–10.

2. Outline of the proof of Theorem 1

In this section, we provide a brief outline of the proof of Theorem 1. As we already said in the introduction, the proof follows the p-adic approach pioneered by Dwork [9, 10], of which we review its corner stones in Section 4. We show that this approach allows us to reduce the proof of Theorem 1 to Lemmas 1–3. These lemmas are subsequently proved in Sections 6–8, with four auxiliary lemmas being the subject of Section 5.

By Dwork's Lemma given in Section 4 (or rather its consequence given in Lemma 10), we want to prove that

$$\mathbf{F_N}(z)\mathbf{G}_{L,\mathbf{N}}(z^p) - p\mathbf{F_N}(z^p)\mathbf{G}_{L,\mathbf{N}}(z) \in pz\mathbb{Z}_p[[z]].$$

The $(a+Kp)$-th Taylor coefficient of $\mathbf{F_N}(z)\mathbf{G}_{L,\mathbf{N}}(z^p) - p\mathbf{F_N}(z^p)\mathbf{G}_{L,\mathbf{N}}(z)$ is

$$\mathbf{C}(a+Kp) := \sum_{j=0}^{K} \mathbf{B_N}(a+jp)\mathbf{B_N}(K-j)(H_{L(K-j)} - pH_{La+Ljp}), \tag{2.1}$$

where $\mathbf{B_N}(m) := \prod_{j=1}^{k} \mathbf{B}_{N_j}(m)$, the quantities $\mathbf{B}_{N_j}(m)$ being defined in (1.3). In view of Lemma 10, proving Theorem 1 is equivalent to proving that

$$\mathbf{C}(a+Kp) \in p\mathbb{Z}_p \tag{2.2}$$

for all primes p and non-negative integers a and K with $0 \le a < p$. Since

$$H_J = \sum_{j=1}^{\lfloor J/p \rfloor} \frac{1}{pj} + \sum_{\substack{j=1\\ p\nmid j}}^{J} \frac{1}{j},$$

we have

$$pH_J \equiv H_{\lfloor J/p \rfloor} \mod p\mathbb{Z}_p. \tag{2.3}$$

Applying this with $J = La + Ljp$, we get

$$pH_{La+Ljp} \equiv H_{\lfloor La/p \rfloor + Lj} \mod p\mathbb{Z}_p.$$

This implies that

$$\mathbf{C}(a+Kp) \equiv \sum_{j=0}^{K} \mathbf{B_N}(a+jp)\mathbf{B_N}(K-j)(H_{L(K-j)} - H_{\lfloor La/p \rfloor + Lj}) \mod p\mathbb{Z}_p. \tag{2.4}$$

We now want to transform the sum on the right-hand side of (2.4) to a more manageable expression. In particular, we want to get rid of the floor function $\lfloor La/p \rfloor$. In order to achieve this, we will prove the following lemma in Section 6.

Lemma 1. *For any prime p, non-negative integers a and j with $0 \le a < p$, positive integers $N_1, N_2, \dots, N_k$, and $L \in \{1, 2, \dots, \max(N_1, \dots, N_k)\}$, we have*

$$\mathbf{B_N}(a+pj)\big(H_{Lj+\lfloor La/p \rfloor} - H_{Lj}\big) \in p\mathbb{Z}_p. \tag{2.5}$$

It follows from Eq. (2.4) and Lemma 1 that

$$\mathbf{C}(a+Kp) \equiv \sum_{j=0}^{K} \mathbf{B_N}(a+jp)\mathbf{B_N}(K-j)\Big(H_{L(K-j)} - H_{Lj}\Big) \mod p\mathbb{Z}_p ,$$

which can be rewritten as

$$\mathbf{C}(a+Kp) \equiv -\sum_{j=0}^{K} H_{Lj}\Big(\mathbf{B_N}(a+jp)\mathbf{B_N}(K-j) - \mathbf{B_N}(j)\mathbf{B_N}(a+(K-j)p)\Big) \mod p\mathbb{Z}_p . \tag{2.6}$$

We now use a combinatorial lemma due to Dwork (see [10, Lemma 4.2]) which provides an alternative way to write the sum on the right-hand side of (2.6): namely, we have

$$\sum_{j=0}^{K} H_{Lj}\Big(\mathbf{B_N}(a+jp)\mathbf{B_N}(K-j) - \mathbf{B_N}(j)\mathbf{B_N}(a+(K-j)p)\Big) = \sum_{s=0}^{r}\sum_{m=0}^{p^{r+1-s}-1} \mathbf{Y}_{m,s}, \tag{2.7}$$

where r is such that $K < p^r$, and

$$\mathbf{Y}_{m,s} := \big(H_{Lmp^s} - H_{L\lfloor m/p\rfloor p^{s+1}}\big)\mathbf{S}(a,K,s,p,m),$$

the expression $\mathbf{S}(a,K,s,p,m)$ being defined by

$$\mathbf{S}(a,K,s,p,m) := \sum_{j=mp^s}^{(m+1)p^s-1} \big(\mathbf{B_N}(a+jp)\mathbf{B_N}(K-j) - \mathbf{B_N}(j)\mathbf{B_N}(a+(K-j)p)\big).$$

In this expression for $\mathbf{S}(a,K,s,p,m)$, it is assumed that $\mathbf{B_N}(n) = 0$ for negative integers n.

It would suffice to prove that

$$\mathbf{Y}_{m,s} \in p\mathbb{Z}_p \tag{2.8}$$

for all m and s, because (2.6) and (2.7) would then imply that $\mathbf{C}(a+Kp) \in p\mathbb{Z}_p$, as desired.

We will prove (2.8) in the following manner. The expression for $\mathbf{S}(a,K,s,p,m)$ is of the form considered in Proposition 1 in Section 4. The proposition will enable us to prove the following fact in Section 7.

Lemma 2. *For all primes p and non-negative integers a, m, s, K with $0 \le a < p$, we have*

$$\mathbf{S}(a,K,s,p,m) \in p^{s+1}\mathbf{B_N}(m)\mathbb{Z}_p.$$

Furthermore, in Section 8 we shall prove the following lemma.

Lemma 3. *For all primes p, non-negative integers m, positive integers $N_1, N_2, \dots, N_k$, and $L \in \{1, 2, \dots, \max(N_1, \dots, N_k)\}$, we have*

$$\mathbf{B_N}(m)\left(H_{Lmp^s} - H_{L\lfloor m/p\rfloor p^{s+1}}\right) \in \frac{1}{p^s}\,\mathbb{Z}_p. \tag{2.9}$$

It is clear that Lemmas 2 and 3 imply (2.8). This completes the outline of the proof of Theorem 1.

Remark 1. To prove the refinement announced at the end of the Introduction that $\mathbf{q}_{1,\mathbf{N}}(z)^{1/\mathbf{B}_\mathbf{N}(1)} \in \mathbb{Z}[[z]]$, by Lemma 10 we should show that $\mathbf{C}(a+Kp) \in p\mathbf{B}_\mathbf{N}(1)\mathbb{Z}_p$ instead of the weaker (2.2), which means to show that

$$\mathbf{C}(a+Kp) \equiv \sum_{j=0}^{K} \mathbf{B}_\mathbf{N}(a+jp)\mathbf{B}_\mathbf{N}(K-j)(H_{L(K-j)} - H_{\lfloor La/p\rfloor+Lj}) \mod p\mathbf{B}_\mathbf{N}(1)\mathbb{Z}_p \quad (2.10)$$

instead of the weaker (2.4), that $\mathbf{B}_\mathbf{N}(a+pj)\big(H_{j+\lfloor a/p\rfloor} - H_j\big) \in p\mathbf{B}_\mathbf{N}(1)\mathbb{Z}_p$ instead of (2.5) (but this is trivial because $H_{j+\lfloor a/p\rfloor} - H_j = 0$), and that

$$\mathbf{B}_\mathbf{N}(m)\left(H_{Lmp^s} - H_{L\lfloor m/p\rfloor p^{s+1}}\right) \in \frac{\mathbf{B}_\mathbf{N}(1)}{p^s}\mathbb{Z}_p \quad (2.11)$$

instead of the weaker (2.9). To establish (2.10) one would apply the same type of argument as the one establishing (3.3), however with Lemma 4 replaced by Lemma 14, the latter lemma being proved in Section 5. To prove (2.11), Lemma 14 must be used in (8.2).

3. Outline of the proof of Theorem 2

This section is devoted to an outline of the proof of Theorem 2, reducing it to Lemmas 5–7. These lemmas are subsequently proved in Sections 9–10.

We follow the strategy that we used in Section 2 to prove Theorem 1; that is, by the consequence of Dwork's Lemma given in Lemma 10, we want to prove that

$$F_\mathbf{N}(z)G_{L,\mathbf{N}}(z^p) - pF_\mathbf{N}(z^p)G_{L,\mathbf{N}}(z) \in p\frac{M_\mathbf{N}}{\Theta_L}z\mathbb{Z}_p[[z]].$$

The $(a+Kp)$-th Taylor coefficient of $F_\mathbf{N}(z)G_{L,\mathbf{N}}(z^p) - pF_\mathbf{N}(z^p)G_{L,\mathbf{N}}(z)$ is

$$C(a+Kp) := \sum_{j=0}^{K} B_\mathbf{N}(a+jp)B_\mathbf{N}(K-j)(H_{L(K-j)} - pH_{La+Ljp}), \quad (3.1)$$

where $B_\mathbf{N}(m) = \prod_{j=1}^k B_{N_j}(m)$ with $B_N(m) := \frac{(Nm)!}{m!^N}$ (not to be confused with $\mathbf{B}_\mathbf{N}(m)$ and $\mathbf{B}_N(m)$). In view of Lemma 10, proving Theorem 2 is equivalent to proving that

$$C(a+Kp) \in p\frac{M_\mathbf{N}}{\Theta_L}\mathbb{Z}_p \quad (3.2)$$

for all primes p and non-negative integers a and K with $0 \le a < p$.

The following simple lemma will be frequently used in the sequel.

Lemma 4. *For all integers $m \ge 1$ and $N \ge 1$, we have*

$$B_N(m) \in N!\,\mathbb{Z}.$$

Proof. Set $U_m(N) = \frac{(Nm)!}{m!^N N!}$. For any $m, N \ge 1$, we have the trivial relation

$$U_m(N+1) = \binom{Nm+m-1}{m-1}U_m(N).$$

Therefore, since $U_m(1) = 1$, the result follows by induction on N. ∎

We deduce in particular that $B_{\mathbf{N}}(m) \in M_{\mathbf{N}}\mathbb{Z}$ for any $m \geq 1$.

Using this together with (2.3) specialised to $J = La + Ljp$, we infer

$$C(a+Kp) \equiv \sum_{j=0}^{K} B_{\mathbf{N}}(a+jp)B_{\mathbf{N}}(K-j)(H_{L(K-j)} - H_{\lfloor La/p \rfloor + Lj}) \mod pM_{\mathbf{N}}\mathbb{Z}_p. \tag{3.3}$$

Indeed, if $K \geq 1$ or $a \geq 1$, this is because $a + jp$ and $K - j$ cannot be simultaneously zero and therefore at least one of $B_{\mathbf{N}}(a + jp)$ or $B_{\mathbf{N}}(K - j)$ is divisible by $M_{\mathbf{N}}$ by Lemma 4. In the remaining case $K = a = j = 0$, we note that the difference of harmonic numbers in (3.3) is equal to 0, and therefore the congruence (3.3) holds trivially because $C(0) = 0$.

The analogue of Lemma 1 in the present context, which allows us to get rid of the floor function $\lfloor La/p \rfloor$ and rearrange the sum over j, is the following lemma. The proof can be found in Section 9.

Lemma 5. *For any prime p, non-negative integers a and j with $0 \leq a < p$, positive integers $N_1, N_2, \ldots, N_k$, and $L \in \{1, 2, \ldots, \max(N_1, \ldots, N_k)\}$, we have*

$$B_{\mathbf{N}}(a+pj)\left(H_{Lj+\lfloor La/p \rfloor} - H_{Lj}\right) \in p\frac{M_{\mathbf{N}}}{\Theta_L}\mathbb{Z}_p. \tag{3.4}$$

We now do the same rearrangements as those after Lemma 1 to conclude that

$$C(a+Kp) \equiv -\sum_{s=0}^{r} \sum_{m=0}^{p^{r+1-s}-1} Y_{m,s} \mod p\frac{M_{\mathbf{N}}}{\Theta_L}\mathbb{Z}_p,$$

where r is such that $K < p^r$, and

$$Y_{m,s} := \left(H_{Lmp^s} - H_{L\lfloor m/p \rfloor p^{s+1}}\right) S(a, K, s, p, m),$$

the expression $S(a, K, s, p, m)$ being defined by

$$S(a, K, s, p, m) := \sum_{j=mp^s}^{(m+1)p^s-1} \left(B_{\mathbf{N}}(a+jp)B_{\mathbf{N}}(K-j) - B_{\mathbf{N}}(j)B_{\mathbf{N}}(a+(K-j)p)\right).$$

In this expression for $S(a, K, s, p, m)$, it is assumed that $B_{\mathbf{N}}(n) = 0$ for negative integers n.

If we prove that

$$Y_{m,s} \in p\frac{M_{\mathbf{N}}}{\Theta_L}\mathbb{Z}_p, \tag{3.5}$$

then $C(a + Kp) \in p\frac{M_{\mathbf{N}}}{\Theta_L}\mathbb{Z}_p$, as desired.

Now, the last assertion follows from the following two lemmas. Lemma 6 is the special case of Lemma 2 where the vector $\mathbf{N}$ is specialised in the way described at the beginning of Section 1.3 (in which case the quantity $\mathbf{B}_{\mathbf{N}}(m)$ of Lemma 2 reduces to $B_{\mathbf{N}}(m)$, and hence $\mathbf{S}(a, K, s, p, m)$ to $S(a, K, s, p, m)$). On the other hand, Lemma 7 is the analogue of Lemma 3. Its proof can be found in Section 10.

Lemma 6. *For all primes p and non-negative integers a, m, s, K with $0 \le a < p$, we have*

$$S(a,K,s,p,m) \in p^{s+1} B_{\mathbf{N}}(m)\mathbb{Z}_p. \tag{3.6}$$

Lemma 7. *For all primes p, non-negative integers m, positive integers $N_1, N_2, \dots, N_k$, and $L \in \{1, 2, \dots, \max(N_1, \dots, N_k)\}$, we have*

$$B_{\mathbf{N}}(m)\big(H_{Lmp^s} - H_{L\lfloor m/p\rfloor p^{s+1}}\big) \in \frac{M_{\mathbf{N}}}{p^s\Theta_L}\mathbb{Z}_p\ . \tag{3.7}$$

It is clear that (3.6) and (3.7) imply (3.5). This completes the outline of the proof of Theorem 2.

4. Dwork's theory of formal congruences

In this section, we review those aspects of Dwork's theory on which the arguments of the proofs of Theorems 1 and 2 (see Sections 2 and 3) are based.

First, consider a formal power series $S(z) \in \mathbb{Q}[[z]]$ and suppose that we want to prove that $S(z) \in \mathbb{Z}[[z]]$.

Lemma 8. *Let $S(z)$ be a power series in $\mathbb{Q}[[z]]$. If $S(z) \in \mathbb{Z}_p[[z]]$ for any prime number p, then $S(z) \in \mathbb{Z}[[z]]$.*

This is a consequence of the fact that, given $x \in \mathbb{Q}$, we have $x \in \mathbb{Z}$ if and only if $x \in \mathbb{Z}_p$ for all prime numbers p. Hence we can work in $\mathbb{Q}_p$ for any fixed prime p.

Lemma 9 ("Dwork's Lemma")**.** *Let $S(z) \in 1 + z\mathbb{Q}_p[[z]]$. Then, we have $S(z) \in 1 + z\mathbb{Z}_p[[z]]$ if and only if*

$$\frac{S(z^p)}{S(z)^p} \in 1 + pz\mathbb{Z}_p[[z]].$$

Proof. The proof is neither difficult nor long and can for example be found in the book of Lang [18, Ch. 14, p. 76]. Lang attributes this lemma to Dieudonné and Dwork. ☐

We now suppose that $S(z) = \exp(T(z)/\tau)$ for some $T(z) \in z\mathbb{Q}[[z]]$ and some integer $\tau \ge 1$. Dwork's Lemma implies the following result: τ being any fixed positive integer, we have $\exp\big(T(z)/\tau\big) \in 1 + z\mathbb{Z}_p[[z]]$ if and only if $T(z^p) - pT(z) \in p\tau z\mathbb{Z}_p[[z]]$. (See [20, Corollary 6.7] for a proof.) Since we will be interested in the case when $T(z) = g(z)/f(z)$ with $f(z) \in 1 + z\mathbb{Z}[[z]]$ and $g(z) \in z\mathbb{Q}[[z]]$, we state this result as follows.

Lemma 10. *Given two formal power series $f(z) \in 1 + z\mathbb{Z}[[z]]$ and $g(z) \in z\mathbb{Q}[[z]]$ and an integer $\tau \ge 1$, we have* $\exp\big(g(z)/(\tau f(z))\big) \in 1 + z\mathbb{Z}_p[[z]]$ *if and only if*

$$f(z)g(z^p) - p\,f(z^p)g(z) \in p\tau z\mathbb{Z}_p[[z]]. \tag{4.1}$$

Because of the special form of the functions which will play the role of $f(z)$ and $g(z)$, we will be able to deduce (4.1) from the following crucial result, also due to Dwork (see [10, Theorem 1.1]).

Proposition 1 ("Dwork's Formal Congruences Theorem"). *Let* $A : \mathbb{Z}_{\geq 0} \to \mathbb{Q}_p^\times$, $g : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_p \setminus \{0\}$ *be mappings such that*

(*i*) $|A(0)|_p = 1$;

(*ii*) $A(m) \in g(m)\mathbb{Z}_p$;

(*iii*) *for all integers* $u, v, n, s \geq 0$ *such that* $0 \leq u < p^s$ *and* $0 \leq v < p$, *we have*

$$\frac{A(v + up + np^{s+1})}{A(v + up)} - \frac{A(u + np^s)}{A(u)} \in p^{s+1} \frac{g(n)}{g(v + up)} \mathbb{Z}_p. \tag{4.2}$$

Furthermore, let $F(z) = \sum_{m=0}^{\infty} A(m) z^m$, *and*

$$F_{m,s}(z) = \sum_{j=mp^s}^{(m+1)p^s - 1} A(j) z^j.$$

Then, for any integers $m, s \geq 0$, *we have*

$$F(z^p) F_{m,s+1}(z) - F(z) F_{m,s}(z^p) \in p^{s+1} g(m) \mathbb{Z}_p[[z]], \tag{4.3}$$

or, equivalently,

$$\sum_{j=mp^s}^{(m+1)p^s - 1} \big(A(a + jp) A(K - j) - A(j) A(a + (K - j)p) \big) \in p^{s+1} g(m) \mathbb{Z}_p \tag{4.4}$$

for all a *and* K *with* $0 \leq a < p$ *and* $K \geq 0$.

Remarks. (a) Dwork's original theorem is in fact more general in that it contains families of functions A_r and g_r, $r = 0, 1, 2, \ldots$, (which are all equal to A, respectively to g, in the above specialisation). Moreover, Dwork proved his theorem with $A_r : \mathbb{Z}_{\geq 0} \to \mathbb{C}_p^\times$, $g_r : \mathbb{Z}_{\geq 0} \to \mathcal{O}_p \setminus \{0\}$ and $A_r(m) \in g_r(m)\mathcal{O}_p$ (where $\mathcal{O}_p$ is the ring of integers in $\mathbb{C}_p$). He obtained a result similar to (4.3) and (4.4), with $\mathcal{O}_p$ instead of $\mathbb{Z}_p$. In our more restrictive setting, (4.3) and (4.4) hold because $\big(p^{s+1} g(m) \mathcal{O}_p\big) \cap \mathbb{Q}_p = p^{s+1} g(m) \mathbb{Z}_p$.

(b) For any integers a and K with $0 \leq a < p$ and $K \geq 0$, the sum

$$\sum_{j=mp^s}^{(m+1)p^s - 1} \big(A(a + jp) A(K - j) - A(j) A(a + (K - j)p) \big) \tag{4.5}$$

is exactly the $(a + pK)$-th Taylor coefficient of $F(z^p) F_{m,s+1}(z) - F(z) F_{m,s}(z^p)$, which explains the equivalence between the formal congruence (4.3) and the congruence (4.4). Note that in (4.5) the value of A at negative integers must be taken as 0.

(c) Most authors chose $g(m) = 1$ or a constant in m. We will use instead $g(m) = A(m)$: this choice has already been made by Dwork in [9, Sec. 2, p. 37].

(d) Dwork also applied his methods to the problems considered in the present paper. Indeed, he proved a result, namely [10, p. 311, Theorem 4.1], which implies that for any prime p that does not divide $N_1 N_2 \cdots N_k$, the mirror maps $\mathbf{q}_{\mathbf{N}}(z)$ have Taylor coefficients in $\mathbb{Z}_p$ (see [31, Proposition 2] for details). This fact was used by the authors of [19, 20, 31] who focussed essentially on the remaining case when p divides $N_1 N_2 \cdots N_k$. Our approach is different, for we make no distinction of this kind between prime numbers.

During the proofs of Lemma 2 in Section 7, we will also use certain properties of the p-adic gamma function Γ_p. This function is defined on integers $n \geq 1$ by

$$\Gamma_p(n) = (-1)^n \prod_{\substack{k=1 \\ (k,p)=1}}^{n-1} k.$$

We will not consider its extension to $\mathbb{Q}_p$. In the following lemma, we collect the results on Γ_p that we shall need later on.

Lemma 11. (i) *For all integers* $n \geq 1$*, we have*

$$\frac{(np)!}{n!} = (-1)^{np+1} p^n \Gamma_p(1 + np).$$

(ii) *For all integers* $k \geq 1, n \geq 1, s \geq 0$*, we have*

$$\Gamma_p(k + np^s) \equiv \Gamma_p(k) \mod p^s.$$

Proof. See [31, Lemma 7] for (i) and [18, p. 71, Lemma 1.1] for (ii). □

5. Auxiliary lemmas

In this section, we establish three auxiliary results. The first one, Lemma 12, is required for the proof of Lemma 1 in Section 6, while the second one, Lemma 13, is required for the proof of Lemma 2 in Section 7. The third result, Lemma 14, justifies an assertion made in the Introduction (see item (b) in the remarks after Theorem 2). Moreover, the proofs of Lemmas 13 and 14 make themselves use of Lemma 12.

Lemma 12. *For any integer* $N \geq 1$ *with associated parameters* $\alpha_i, \beta_i, \mu, \eta$*, the function*

$$\Delta(x) := \sum_{i=1}^{\mu} \lfloor \alpha_i x \rfloor - \sum_{i=1}^{\eta} \lfloor \beta_i x \rfloor$$

has the following properties:

- (i) Δ *is* 1*-periodic. In particular,* $\Delta(n) = 0$ *for all integers* n.
- (ii) *For all integers* n*,* Δ *is weakly increasing on intervals of the form* $[n, n+1)$.
- (iii) *For all real numbers* x*, we have* $\Delta(x) \geq 0$.
- (iv) *For all rational numbers* $r \neq 0$ *whose denominator is an element of* $\{2, 3, \ldots, N\}$*, we have* $\Delta(r) \geq 1$.

Remark. Clearly, the function Δ is a step function. The proof below shows that, in fact, all the jumps of Δ at non-integral places have the value $+1$ and occur exactly at rational numbers of the form r/N, with r coprime to N.

Proof. Property (i) follows from the equality $\sum_{i=1}^{\mu} \alpha_i = \sum_{i=1}^{\eta} \beta_i$ and the trivial fact that $\Delta(0) = 0$.

We turn our attention to property (ii). For convenience of notation, let

$$N = p_1^{e_1} p_2^{e_2} \cdots p_\ell^{e_\ell}$$

be the prime factorisation of N, where, as before, $p_1, p_2, \dots, p_\ell$ are the distinct prime factors of N, and where $e_1, e_2, \dots, e_\ell$ are positive integers.

As we already observed in the remark above, the function Δ is a step function. Moreover, jumps of Δ can only occur at values of x where some of the $\alpha_i x$, $1 \le i \le \mu$, or some of the $\beta_j x$, $1 \le j \le \eta$, (or both) are integers. At these values of x, the value of a (possible) jump is the difference between the number of i's for which $\alpha_i x$ is integral and the number of j's for which $\beta_j x$ is integral. In symbols, the value of the jump is

$$\#\{i : 1 \le i \le \mu \text{ and } \alpha_i x \in \mathbb{Z}\} - \#\{j : 1 \le j \le \eta \text{ and } \beta_j x \in \mathbb{Z}\}. \tag{5.1}$$

Let X be the place of a jump, X not being an integer. Then we can write X as

$$X = \frac{Z}{p_1^{f_1} p_2^{f_2} \cdots p_\ell^{f_\ell}},$$

where $f_1, f_2, \dots, f_\ell$ are non-negative integers, not all zero, and where Z is a non-zero integer relatively prime to $p_1^{f_1} p_2^{f_2} \cdots p_\ell^{f_\ell}$. Given

$$\alpha_i = p_1^{a_1} p_2^{a_2} \cdots p_\ell^{a_\ell}$$

with $e_1 + e_2 + \dots + e_\ell - (a_1 + a_2 + \dots + a_\ell)$ even and $0 \le e_k - a_k \le 1$ for each $k = 1, 2, \dots, \ell$, the number $\alpha_i X$ will be integral if and only if $a_k \ge f_k$ for all $k \in \{1, 2, \dots, \ell\}$. Similarly, given

$$\beta_j = p_1^{b_1} p_2^{b_2} \cdots p_\ell^{b_\ell}$$

with $e_1 + e_2 + \dots + e_\ell - (b_1 + b_2 + \dots + b_\ell)$ odd and $0 \le e_k - b_k \le 1$ for each $k = 1, 2, \dots, \ell$, the number $\beta_j X$ will be integral if and only if $b_k \ge f_k$ for all $k \in \{1, 2, \dots, \ell\}$. We do not have to take into account the β_j's which are 1, because $1 \cdot X = X$ is not an integer by assumption. For the generating function of vectors $(c_1, c_2, \dots, c_\ell)$ with $e_k \ge c_k \ge f_k$ and $e_k - c_k \le 1$, we have

$$\sum_{c_1 = \max\{e_1 - 1, f_1\}}^{e_1} \cdots \sum_{c_\ell = \max\{e_\ell - 1, f_\ell\}}^{e_\ell} z^{e_1 + \dots + e_\ell - (c_1 + \dots + c_\ell)} = \prod_{k=1}^{\ell} \left(1 + z \cdot \min\{1, e_k - f_k\}\right).$$

We obtain the difference in (5.1) (with X in place of x) by putting $z = -1$ on the left-hand side of this relation. The product on the right-hand side tells us that this difference is 0 in case that $e_k \ne f_k$ for some k, while it is 1 otherwise. Thus, all the jumps of the function Δ at non-integral places have the value $+1$.

Property (*iii*) follows now easily from (*i*) and (*ii*).

In order to prove (*iv*), we observe that the first jump of Δ in the interval $[0, 1)$ occurs at $x = 1/N$. Thus, $\Delta(x) \ge 1$ for all x in $[1/N, 1)$. This implies in particular that $\Delta(r) \ge 1$ for all the above rational numbers r in the interval $[1/N, 1)$. That the same assertion holds in fact for *all* the above rational numbers r (not necessarily restricted to $[1/N, 1)$) follows now from the 1-periodicity of the function Δ. ☐

Lemma 13. *For any integers $m, r, w \ge 0$ such that $0 \le w < p^r$, we have*

$$\frac{\mathbf{B_N}(w + mp^r)}{\mathbf{B_N}(m)} \in \mathbb{Z}_p, \tag{5.2}$$

where $\mathbf{B_N}(m)$ *is the quantity defined after* (2.1)*.*

Proof. We first show that we can assume that m is coprime to p. Indeed, let us write $m = hp^t$ with $\gcd(h,p) = 1$. We have to prove that

$$\frac{\mathbf{B_N}(w + hp^{r+t})}{\mathbf{B_N}(hp^t)} \in \mathbb{Z}_p.$$

Since $v_p\big(\mathbf{B_N}(hp^t)/\mathbf{B_N}(h)\big) = 0$ (as can be easily seen from (1.3) and Legendre's formula $v_p(n!) = \sum_{k=1}^{\infty} \lfloor \frac{n}{p^k} \rfloor$), this amounts to prove that

$$\frac{\mathbf{B_N}(w + hp^{r+t})}{\mathbf{B_N}(h)} \in \mathbb{Z}_p,$$

which is the content of the lemma with $r + t$ instead of r and h instead of m, with $w < p^r < p^{r+t}$.

Therefore, from now on, we assume that $\gcd(m,p) = 1$ (however, this assumption will only be used after (5.5)). Since $v_p\big(\mathbf{B_N}(mp^r)/\mathbf{B_N}(m)\big) = 0$, we have to prove that

$$v_p\left(\frac{\mathbf{B_N}(w + mp^r)}{\mathbf{B_N}(mp^r)}\right) \geq 0$$

or, in an equivalent form, that

$$\sum_{j=1}^{k}\sum_{\ell=1}^{\infty}\left(\left(\sum_{i=1}^{\mu_j}\left\lfloor\frac{\alpha_{i,j}(w+mp^r)}{p^\ell}\right\rfloor - \sum_{i=1}^{\eta_j}\left\lfloor\frac{\beta_{i,j}(w+mp^r)}{p^\ell}\right\rfloor\right) - \left(\sum_{i=1}^{\mu_j}\left\lfloor\frac{\alpha_{i,j}mp^r}{p^\ell}\right\rfloor - \sum_{i=1}^{\eta_j}\left\lfloor\frac{\beta_{i,j}mp^r}{p^\ell}\right\rfloor\right)\right) \geq 0, \quad (5.3)$$

where $\alpha_{i,j}, \beta_{i,j}, \mu_j, \eta_j$ are the parameters associated to N_j.

If $\ell \leq r$, then for any $j \in \{1, 2, \dots, k\}$,

$$\sum_{i=1}^{\mu_j}\left\lfloor\frac{\alpha_{i,j}mp^r}{p^\ell}\right\rfloor - \sum_{i=1}^{\eta_j}\left\lfloor\frac{\beta_{i,j}mp^r}{p^\ell}\right\rfloor = mp^{r-\ell}\left(\sum_{i=1}^{\mu_j}\alpha_{i,j} - \sum_{i=1}^{\eta_j}\beta_{i,j}\right) = 0.$$

Moreover,

$$\sum_{i=1}^{\mu_j}\left\lfloor\frac{\alpha_{i,j}(w+mp^r)}{p^\ell}\right\rfloor - \sum_{i=1}^{\eta_j}\left\lfloor\frac{\beta_{i,j}(w+mp^r)}{p^\ell}\right\rfloor \geq 0$$

because of Lemma 12(*iii*) with $N = N_j$. It therefore suffices to show

$$\sum_{j=1}^{k}\sum_{\ell=r+1}^{\infty}\left(\left(\sum_{i=1}^{\mu_j}\left\lfloor\frac{\alpha_{i,j}(w+mp^r)}{p^\ell}\right\rfloor - \sum_{i=1}^{\eta_j}\left\lfloor\frac{\beta_{i,j}(w+mp^r)}{p^\ell}\right\rfloor\right) - \left(\sum_{i=1}^{\mu_j}\left\lfloor\frac{\alpha_{i,j}mp^r}{p^\ell}\right\rfloor - \sum_{i=1}^{\eta_j}\left\lfloor\frac{\beta_{i,j}mp^r}{p^\ell}\right\rfloor\right)\right) \geq 0, \quad (5.4)$$

(The reader should note the difference to (5.3) occurring in the summation bounds for ℓ.) For $\ell > r$, set $x_\ell = \{mp^r/p^\ell\}$ and $y_\ell = \{(w+mp^r)/p^\ell\}$. Using again $\sum_{i=1}^{\mu_j} \alpha_{i,j} - \sum_{i=1}^{\eta_j} \beta_{i,j} = 0$, we see that the left-hand side of (5.4) is equal to

$$\sum_{j=1}^{k} \sum_{\ell=r+1}^{\infty} \left(\left(\sum_{i=1}^{\mu_j} \lfloor \alpha_{i,j} y_\ell \rfloor - \sum_{i=1}^{\eta_j} \lfloor \beta_{i,j} y_\ell \rfloor \right) - \left(\sum_{i=1}^{\mu_j} \lfloor \alpha_{i,j} x_\ell \rfloor - \sum_{i=1}^{\eta_j} \lfloor \beta_{i,j} x_\ell \rfloor \right) \right). \tag{5.5}$$

We now claim that $x_\ell \le y_\ell$ for $\ell > r$. To see this, we begin by the observation that, since m and p are coprime and $\ell > r$, the rational number $m/p^{\ell-r}$ is not an integer. It follows that

$$x_\ell + \frac{1}{p^{\ell-r}} = \left\{ \frac{m}{p^{\ell-r}} \right\} + \frac{1}{p^{\ell-r}} \le 1.$$

Hence, since $w < p^r$, we infer that

$$x_\ell + \frac{w}{p^\ell} < 1.$$

On the other hand, we have

$$y_\ell = \left\{ \frac{w}{p^\ell} + \left\lfloor \frac{m}{p^{\ell-r}} \right\rfloor + x_\ell \right\} = \left\{ \frac{w}{p^\ell} + x_\ell \right\} = \frac{w}{p^\ell} + x_\ell.$$

Since $w \ge 0$, we obtain indeed $y_\ell \ge x_\ell$, as we claimed.

Using $x_\ell \le y_\ell$ together with Lemma 12(ii), we see that, for $\ell > r$ and $j = 1, 2, \dots k$, we have

$$\sum_{i=1}^{\mu_j} \lfloor \alpha_{i,j} y_\ell \rfloor - \sum_{i=1}^{\eta_j} \lfloor \beta_{i,j} y_\ell \rfloor \ge \sum_{i=1}^{\mu_j} \lfloor \alpha_{i,j} x_\ell \rfloor - \sum_{i=1}^{\eta_j} \lfloor \beta_{i,j} x_\ell \rfloor,$$

which shows that the expression in (5.5) is non-negative, thus establishing (5.4) and also (5.3). This finishes the proof of the lemma. □

We conclude this section with a result which was announced in item (b) of the remarks after Theorem 2. It is used nowhere else, but we give it here for the sake of completeness. It is a generalisation of Lemma 4. By the same techniques used to prove Theorem 2, it enables one to prove that $\mathbf{q}_{1,\mathbf{N}}(z)^{1/\mathbf{B}_\mathbf{N}(1)} \in \mathbb{Z}[[z]]$ (see Remark 1 in Section 2).

Lemma 14. *For any vector* $\mathbf{N}$ *and any integer* $m \ge 1$*, we have that* $\mathbf{B}_\mathbf{N}(1)$ *divides* $\mathbf{B}_\mathbf{N}(m)$*, where* $\mathbf{B}_\mathbf{N}(m)$ *is the quantity defined after* (2.1)*.*

Proof. Obviously, it is sufficient to prove the assertion for $k = 1$ and $\mathbf{N} = (N)$. Let Δ be the function associated to N as defined in Lemma 12. We want to prove that, for any prime p, we have $v_p(\mathbf{B}_N(m)) \ge v_p(\mathbf{B}_N(1))$. We can assume that m and p are coprime because $v_p(\mathbf{B}_N(mp^t)) = v_p(\mathbf{B}_N(m))$ for any integers $m, t \ge 0$.

Now, when $\gcd(m, p) = 1$, we have that

$$\begin{aligned} v_p(\mathbf{B}_N(m)) &= \sum_{\ell=1}^{\infty} \Delta(m/p^\ell) = \sum_{\ell=1}^{\infty} \Delta(\{m/p^\ell\}) \\ &\geq \sum_{\ell=1}^{\infty} \Delta(1/p^\ell) = v_p(\mathbf{B}_N(1)). \end{aligned}$$

Here, we used the 1-periodicity of Δ for the second equality. For the inequality, we used that $\{m/p^\ell\} \geq 1/p^\ell$ (because $\gcd(m, p) = 1$ implies that m/p^ℓ is not an integer) and the (partial) monotonicity of Δ described in Lemma 12(ii). □

6. Proof of Lemma 1

The assertion is trivially true if $\lfloor La/p \rfloor = 0$, that is, if $0 \leq a < p/L$. We may hence assume that $p/L \leq a < p$ from now on.

We write $\alpha_{i,m}$, $\beta_{i,m}$, μ_m, and η_m for the parameters associated to N_m, $m = 1, 2, \ldots, k$. We may assume that, without loss of generality, $\max(N_1, \ldots, N_k) = N_k$. Then, using again Lemma 12(iii),

$$\begin{aligned} v_p\big(\mathbf{B}_{\mathbf{N}}(a + pj)\big) &= \sum_{m=1}^{k} \sum_{\ell=1}^{\infty} \left(\sum_{i=1}^{\mu_m} \left\lfloor \frac{\alpha_{i,m}(a + pj)}{p^\ell} \right\rfloor - \sum_{i=1}^{\eta_m} \left\lfloor \frac{\beta_{i,m}(a + pj)}{p^\ell} \right\rfloor \right) \\ &\geq \sum_{\ell=1}^{\infty} \left(\sum_{i=1}^{\mu_k} \left\lfloor \frac{\alpha_{i,k}(a + pj)}{p^\ell} \right\rfloor - \sum_{i=1}^{\eta_k} \left\lfloor \frac{\beta_{i,k}(a + pj)}{p^\ell} \right\rfloor \right) = \sum_{\ell=1}^{\infty} \Delta_k \left(\frac{a + jp}{p^\ell} \right) \end{aligned}$$

with

$$\Delta_k(x) := \sum_{i=1}^{\mu_k} \lfloor \alpha_{i,k} x \rfloor - \sum_{i=1}^{\eta_k} \lfloor \beta_{i,k} x \rfloor .$$

On the other hand, by definition of the harmonic numbers, we have

$$H_{Lj + \lfloor La/p \rfloor} - H_{Lj} = \frac{1}{Lj + 1} + \frac{1}{Lj + 2} + \cdots + \frac{1}{Lj + \lfloor La/p \rfloor}.$$

It therefore suffices to show that

$$v_p\big(\mathbf{B}_{\mathbf{N}}(a + pj)\big) \geq 1 + v_p(Lj + \varepsilon) \tag{6.1}$$

for any integer ε such that $1 \leq \varepsilon \leq \lfloor La/p \rfloor$. We have

$$\frac{a + jp}{p^\ell} = \frac{a - p\varepsilon/L}{p^\ell} + \frac{pj + p\varepsilon/L}{p^\ell}.$$

6.1. **First step.** We claim that

$$\Delta_k\left(\frac{a+jp}{p^\ell}\right) \geq \Delta_k\left(\frac{pj+p\varepsilon/L}{p^\ell}\right). \tag{6.2}$$

To see this, we first observe that

$$\Delta_k\left(\frac{a+jp}{p^\ell}\right) = \Delta_k\left(\frac{a-p\varepsilon/L}{p^\ell}+\frac{pj+p\varepsilon/L}{p^\ell}\right) = \Delta_k\left(\frac{a-p\varepsilon/L}{p^\ell}+\left\{\frac{pj+p\varepsilon/L}{p^\ell}\right\}\right)$$

because Δ_k is 1-periodic.

We now claim that

$$0 \leq \frac{a-p\varepsilon/L}{p^\ell}+\left\{\frac{pj+p\varepsilon/L}{p^\ell}\right\} < 1. \tag{6.3}$$

Indeed, positivity is clear and we now concentrate on the upper bound. We write $j = up^{\ell-1}+v$ with $0 \leq v < p^{\ell-1}$. Hence,

$$\left\{\frac{pj+p\varepsilon/L}{p^\ell}\right\} = \left\{u+\frac{pv+p\varepsilon/L}{p^\ell}\right\} = \left\{\frac{v}{p^{\ell-1}}+\frac{p\varepsilon/L}{p^\ell}\right\}.$$

Since $0 \leq \varepsilon \leq \lfloor La/p \rfloor < L$, we have $0 \leq \frac{p\varepsilon/L}{p^\ell} < 1/p^{\ell-1}$ and therefore

$$0 \leq \frac{v}{p^{\ell-1}}+\frac{p\varepsilon/L}{p^\ell} < \frac{v}{p^{\ell-1}}+\frac{1}{p^{\ell-1}} \leq 1$$

(where the last inequality holds by definition of v), whence

$$\left\{\frac{pj+p\varepsilon/L}{p^\ell}\right\} = \frac{pv+p\varepsilon/L}{p^\ell}.$$

Therefore, we have

$$\frac{a-p\varepsilon/L}{p^\ell}+\left\{\frac{pj+p\varepsilon/L}{p^\ell}\right\} = \frac{a-p\varepsilon/L}{p^\ell}+\frac{pv+p\varepsilon/L}{p^\ell} = \frac{a}{p^\ell}+\frac{v}{p^{\ell-1}}.$$

Since $\frac{v}{p^{\ell-1}} < 1$ and $a < p$, we necessarily have

$$\frac{a}{p^\ell}+\frac{v}{p^{\ell-1}} < 1,$$

as desired.

Since $\frac{a-p\varepsilon/L}{p^\ell} \geq 0$, it follows from Lemma 12(i),(ii) (with $\Delta = \Delta_k$) and (6.3) that

$$\Delta_k\left(\frac{a-p\varepsilon/L}{p^\ell}+\left\{\frac{pj+p\varepsilon/L}{p^\ell}\right\}\right) \geq \Delta_k\left(\left\{\frac{pj+p\varepsilon/L}{p^\ell}\right\}\right) = \Delta_k\left(\frac{pj+p\varepsilon/L}{p^\ell}\right).$$

Thus, we have proved the claim (6.2) made at the beginning of this step.

6.2. **Second step.** Let us write $Lj+\varepsilon=\beta p^d$, where $d=v_p(Lj+\varepsilon)$, so that

$$\frac{pj+p\varepsilon/L}{p^\ell}=\frac{\beta p^{d+1-\ell}}{L}.$$

We have proved in the first step that

$$v_p\big(\mathbf{B}_\mathbf{N}(a+pj)\big)\geq\sum_{\ell=1}^\infty\Delta_k\left(\frac{\beta p^{d+1-\ell}}{L}\right). \tag{6.4}$$

Now we claim that $\beta p^{d+1-\ell}/L$ cannot be an integer. Indeed, if it were, then $L\gamma p^{\ell-1}=\beta p^d=Lj+\varepsilon$ for a suitable integer γ. It would follow that L divides ε, contradicting $1\leq\varepsilon\leq La/p<L$. Furthermore, for $\ell\leq d+1$, the denominator of $\frac{\beta p^{d+1-\ell}}{L}$ is obviously at most L. Since $L\leq N_k$, it follows then from Lemma 12(iv), again with $\Delta=\Delta_k$, that $\Delta_k(\beta p^{d+1-\ell}/L)\geq 1$ for any ℓ in $\{1,2,\dots,d+1\}$. Use of this estimation in (6.4) gives

$$v_p\big(\mathbf{B}_\mathbf{N}(a+pj)\big)\geq d+1=1+v_p(Lj+\varepsilon).$$

This completes the proof of (6.1) and, hence, of Lemma 1.

7. Proof of Lemma 2

We want to use Proposition 1 with $A(m)=g(m)=\mathbf{B}_\mathbf{N}(m)$. Clearly, the proposition would imply that $\mathbf{S}(a,K,s,p,m)\in p^{s+1}\mathbf{B}_\mathbf{N}(m)\mathbb{Z}_p$, and, thus, the claim. So, we need to verify the conditions (i)–(iii) in the statement of the proposition.

Condition (i) is true since $\mathbf{B}_\mathbf{N}(0)=1$. Condition (ii) holds by the definitions of $A(m)$ and $g(m)$. To check that Condition (iii) holds is more involved. The proof will be decomposed in three steps. The reader should recall that

$$\mathbf{B}_\mathbf{N}(m):=\prod_{j=1}^k\mathbf{B}_{N_j}(m),$$

where $\mathbf{B}_{N_j}(m)$ is given by (1.3), or, alternativaly (cf. [31, Lemma 4], respectively (11.1) below) as

$$\mathbf{B}_{N_j}(m)=C_{N_j}^m\prod_{\ell=1}^{\varphi(N_j)}\frac{(r_{\ell,j}/N_j)_m}{m!}, \tag{7.1}$$

where C_{N_j} and the $r_{\ell,j}$'s are defined as in Subsection 1.2. Expression (7.1) will be useful in the first step below, while the direct use of Expression (1.3) would lead to much more involved computations.

7.1. **First step.** Let us fix $j\in\{1,2,\dots,k\}$. We set $D_{N_j}:=N_j^{-\varphi(N_j)}C_{N_j}$, which is an integer.

We claim that

$$\frac{\mathbf{B}_{N_j}(v+up+np^{s+1})}{\mathbf{B}_{N_j}(up+np^{s+1})}=\frac{\mathbf{B}_{N_j}(v+up)}{\mathbf{B}_{N_j}(up)}+\mathcal{O}(p^{s+1}), \tag{7.2}$$

where $\mathcal{O}(R)$ denotes an element of $R\mathbb{Z}_p$. To prove (7.2), we observe that ([6])

$$\frac{\mathbf{B}_{N_j}(v+up+np^{s+1})}{\mathbf{B}_{N_j}(up+np^{s+1})} = \frac{D_{N_j}^v \prod_{\ell=1}^{\varphi(N_j)} \prod_{i=1}^{v} \big(r_{\ell,j}+(i-1)N_j+uN_jp+nN_jp^{s+1}\big)}{\big((v+up+np^{s+1})(v-1+up+np^{s+1})\cdots(1+up+np^{s+1})\big)^{\varphi(N_j)}}, \tag{7.3}$$

$$= \frac{\Big(D_{N_j}^v \prod_{\ell=1}^{\varphi(N_j)} \prod_{i=1}^{v} \big(r_{\ell,j}+(i-1)N_j+uN_jp\big)\Big)+\mathcal{O}(p^{s+1})}{\big((v+up)(v-1+up)\cdots(1+up)\big)^{\varphi(N_j)}+\mathcal{O}(p^{s+1})}.$$

If $v=0$, then (7.2) holds trivially. If $v>0$, then, together with the hypothesis $v<p$, we infer that $(v+up)(v-1+up)\cdots(1+up)$ is not divisible by p, and thus we have

$$\frac{1}{\big((v+up)(v-1+up)\cdots(1+up)\big)^{\varphi(N_j)}+\mathcal{O}(p^{s+1})} = \frac{1}{\big((v+up)(v-1+up)\cdots(1+up)\big)^{\varphi(N_j)}}\big(1+\mathcal{O}(p^{s+1})\big).$$

Hence,

$$\frac{\Big(D_{N_j}^v \prod_{\ell=1}^{\varphi(N_j)} \prod_{i=1}^{v} \big(r_{\ell,j}+(i-1)N_j+uN_jp\big)\Big)+\mathcal{O}(p^{s+1})}{\big((v+up)(v-1+up)\cdots(1+up)\big)^{\varphi(N_j)}+\mathcal{O}(p^{s+1})} = \frac{D_{N_j}^v \prod_{\ell=1}^{\varphi(N_j)} \prod_{i=1}^{v} \big(r_{\ell,j}+(i-1)N_j+uN_jp\big)}{\big((v+up)(v-1+up)\cdots(1+up)\big)^{\varphi(N_j)}} + \frac{\mathcal{O}(p^{s+1})}{\big((v+up)(v-1+up)\cdots(1+up)\big)^{\varphi(N_j)}},$$

which proves (7.2) because

$$\frac{1}{(v+up)(v-1+up)\cdots(1+up)} \in \mathbb{Z}_p \tag{7.4}$$

and

$$\frac{D_{N_j}^v \prod_{\ell=1}^{\varphi(N_j)} \prod_{i=1}^{v} \big(r_{\ell,j}+(i-1)N_j+uN_jp\big)}{\big((v+up)(v-1+up)\cdots(1+up)\big)^{\varphi(N_j)}} = \frac{\mathbf{B}_{N_j}(v+up)}{\mathbf{B}_{N_j}(up)}. \tag{7.5}$$

[6]Identities (7.3) and (7.5) are immediate consequences of the alternative form (7.1) of $\mathbf{B}_{N_j}$. Zudilin used them in his proof of the following stronger version of (7.2):

$$\frac{\mathbf{B}_{N_j}(v+up+np^{s+1})}{\mathbf{B}_{N_j}(up+np^{s+1})} = \frac{\mathbf{B}_{N_j}(v+up)}{\mathbf{B}_{N_j}(up)}\big(1+\mathcal{O}(p^{s+1})\big).$$

However, for this, he assumes that p divides N_j (see [31, Eq. (35)]). Here, we do not assume that p divides N_j, and therefore we obtain the weaker equality (7.2), which is fortunately enough for our purposes.

A side result of (7.5) (which was actually used to prove (7.2)) is that

$$\frac{\mathbf{B}_{N_j}(v+up)}{\mathbf{B}_{N_j}(up)} \in \mathbb{Z}_p.$$

We deduce from this fact and from (7.2) that

$$\prod_{j=1}^{k} \frac{\mathbf{B}_{N_j}(v+up+np^{s+1})}{\mathbf{B}_{N_j}(up+np^{s+1})} = \prod_{j=1}^{k} \left(\frac{\mathbf{B}_{N_j}(v+up)}{\mathbf{B}_{N_j}(up)} + \mathcal{O}(p^{s+1}) \right) = \prod_{j=1}^{k} \frac{\mathbf{B}_{N_j}(v+up)}{\mathbf{B}_{N_j}(up)} + \mathcal{O}(p^{s+1}),$$

or, in other words,

$$\frac{\mathbf{B}_{\mathbf{N}}(v+up+np^{s+1})}{\mathbf{B}_{\mathbf{N}}(up+np^{s+1})} = \frac{\mathbf{B}_{\mathbf{N}}(v+up)}{\mathbf{B}_{\mathbf{N}}(up)} + \mathcal{O}(p^{s+1}). \tag{7.6}$$

7.2. **Second step.** Let us fix $j \in \{1, 2, \dots, k\}$. The properties of Γ_p imply that

$$\frac{\mathbf{B}_{N_j}(up+np^{s+1})}{\mathbf{B}_{N_j}(u+np^{s})} = (-1)^{\mu_j-\eta_j} \frac{\prod_{i=1}^{\mu_j} \Gamma_p\big(1+\alpha_{i,j}(up+np^{s+1})\big)}{\prod_{i=1}^{\eta_j} \Gamma_p\big(1+\beta_{i,j}(up+np^{s+1})\big)} \tag{7.7}$$

$$= (-1)^{\mu_j-\eta_j} \frac{\prod_{i=1}^{\mu_j} \Gamma_p(1+\alpha_{i,j}up) + \mathcal{O}(p^{s+1})}{\prod_{i=1}^{\eta_j} \Gamma_p(1+\beta_{i,j}up) + \mathcal{O}(p^{s+1})} \tag{7.8}$$

$$= (-1)^{\mu_j-\eta_j} \frac{\prod_{i=1}^{\mu_j} \Gamma_p(1+\alpha_{i,j}up)}{\prod_{i=1}^{\eta_j} \Gamma_p(1+\beta_{i,j}up)} \big(1+\mathcal{O}(p^{s+1})\big) \tag{7.9}$$

$$= \frac{\mathbf{B}_{N_j}(up)}{\mathbf{B}_{N_j}(u)} \big(1+\mathcal{O}(p^{s+1})\big), \tag{7.10}$$

where (i) of Lemma 11 is used to see (7.7) and (7.10), and (ii) is used for (7.8). Equation (7.9) holds because $\Gamma_p(m)$ is never divisible by p for any integer m.

Hence, taking the product over $j = 1, 2, \dots, k$, we obtain

$$\frac{\mathbf{B}_{\mathbf{N}}(up+np^{s+1})}{\mathbf{B}_{\mathbf{N}}(u+np^{s})} = \frac{\mathbf{B}_{\mathbf{N}}(up)}{\mathbf{B}_{\mathbf{N}}(u)} \big(1+\mathcal{O}(p^{s+1})\big). \tag{7.11}$$

7.3. **Third step.** We now multiply the right-hand and left-hand sides of (7.6) and (7.11). After simplification, we get

$$\frac{\mathbf{B}_{\mathbf{N}}(v+up+np^{s+1})}{\mathbf{B}_{\mathbf{N}}(u+np^{s})} = \frac{\mathbf{B}_{\mathbf{N}}(v+up)}{\mathbf{B}_{\mathbf{N}}(u)} \big(1+\mathcal{O}(p^{s+1})\big) + \frac{\mathbf{B}_{\mathbf{N}}(up)}{\mathbf{B}_{\mathbf{N}}(u)} \mathcal{O}(p^{s+1}).$$

We can rewrite this as

$$\begin{aligned} \frac{\mathbf{B}_{\mathbf{N}}(v+up+np^{s+1})}{\mathbf{B}_{\mathbf{N}}(v+up)} &= \frac{\mathbf{B}_{\mathbf{N}}(u+np^{s})}{\mathbf{B}_{\mathbf{N}}(u)} \big(1+\mathcal{O}(p^{s+1})\big) + \frac{\mathbf{B}_{\mathbf{N}}(up)}{\mathbf{B}_{\mathbf{N}}(u)} \cdot \frac{\mathbf{B}_{\mathbf{N}}(u+np^{s})}{\mathbf{B}_{\mathbf{N}}(v+up)} \mathcal{O}(p^{s+1}) \\ &= \frac{\mathbf{B}_{\mathbf{N}}(u+np^{s})}{\mathbf{B}_{\mathbf{N}}(u)} + \frac{\mathbf{B}_{\mathbf{N}}(u+np^{s})}{\mathbf{B}_{\mathbf{N}}(u)} \mathcal{O}(p^{s+1}) + \frac{\mathbf{B}_{\mathbf{N}}(u+np^{s})}{\mathbf{B}_{\mathbf{N}}(v+up)} \mathcal{O}(p^{s+1}), \end{aligned} \tag{7.12}$$

where the last line holds because $v_p\big(\mathbf{B}_{\mathbf{N}}(up)/\mathbf{B}_{\mathbf{N}}(u)\big) = 0$.

If we compare (4.2) (with $A(m) = g(m) = \mathbf{B_N}(m)$) and (7.12), we see that it only remains to prove that we have

$$\frac{\mathbf{B_N}(u+np^s)}{\mathbf{B_N}(u)} \in \frac{\mathbf{B_N}(n)}{\mathbf{B_N}(v+up)}\mathbb{Z}_p \quad \text{and} \quad \frac{\mathbf{B_N}(u+np^s)}{\mathbf{B_N}(v+up)} \in \frac{\mathbf{B_N}(n)}{\mathbf{B_N}(v+up)}\mathbb{Z}_p. \tag{7.13}$$

The first assertion in (7.13) can be rewritten as

$$\frac{\mathbf{B_N}(u+np^s)}{\mathbf{B_N}(n)} \cdot \frac{\mathbf{B_N}(v+up)}{\mathbf{B_N}(u)} \in \mathbb{Z}_p, \tag{7.14}$$

while the second assertion can be rewritten as

$$\frac{\mathbf{B_N}(u+np^s)}{\mathbf{B_N}(n)} \in \mathbb{Z}_p. \tag{7.15}$$

Now, the assertion (7.15) is the special case $w = u$, $m = n$ and $r = s$ of Lemma 13, while (7.14) follows from (7.15) combined with the special case $w = v$, $m = u$ and $r = 1$ of Lemma 13.

This completes the proof of the lemma.

8. Proof of Lemma 3

The claim is trivially true if p divides m. We may therefore assume that p does not divide m for the rest of the proof. Let us write $m = a + pj$, with $0 < a < p$. Then comparison with (2.5) shows that we are in a very similar situation here. Indeed, we may derive (2.9) from Lemma 1. In order to see this, we observe that

$$\begin{aligned} H_{Lmp^s} - H_{L\lfloor m/p\rfloor p^{s+1}} &= \sum_{\varepsilon=1}^{Lap^s} \frac{1}{Ljp^{s+1}+\varepsilon} \\ &= \sum_{\varepsilon=1}^{\lfloor La/p\rfloor} \frac{1}{Ljp^{s+1}+\varepsilon p^{s+1}} + \sum_{\substack{\varepsilon=1\\ p^{s+1}\nmid\varepsilon}}^{Lap^s} \frac{1}{Ljp^{s+1}+\varepsilon} \\ &= \frac{1}{p^{s+1}}(H_{Lj+\lfloor La/p\rfloor} - H_{Lj}) + \sum_{\substack{\varepsilon=1\\ p^{s+1}\nmid\varepsilon}}^{Lap^s} \frac{1}{Ljp^{s+1}+\varepsilon}. \end{aligned}$$

Because of $v_p(x+y) \ge \min\{v_p(x), v_p(y)\}$, this implies

$$v_p(H_{Lmp^s} - H_{L\lfloor m/p\rfloor p^{s+1}}) \ge \min\{-1-s+v_p(H_{Lj+\lfloor La/p\rfloor} - H_{Lj}), -s\}. \tag{8.1}$$

It follows that

$$\begin{aligned} v_p&\Big(\mathbf{B_N}(m)\big(H_{Lmp^s} - H_{L\lfloor m/p\rfloor p^{s+1}}\big)\Big) \\ &\ge -1-s+\min\Big\{v_p\Big(\mathbf{B_N}(a+pj)(H_{Lj+\lfloor La/p\rfloor} - H_{Lj})\Big), 1+v_p\big(\mathbf{B_N}(a+pj)\big)\Big\}. \end{aligned} \tag{8.2}$$

Use of Lemma 1 then completes the proof.

9. Proof of Lemma 5

We follow the same approach as the one of the proof of Lemma 1 in Section 6. In particular, the first part below is completely parallel to the proof of Lemma 1. We nevertheless include it here for the sake of readability and for later reference. On the other hand, since Lemma 5 makes a stronger divisibility assertion then Lemma 1, much more work is needed to arrive there: the corresponding arguments form the contents of the second and third part of this proof.

We start again by observing that the assertion (3.4) is trivially true if $\lfloor La/p\rfloor = 0$, that is, if $0 \le a < p/L$. We may hence assume that $p/L \le a < p$ from now on. A further assumption upon which we agree without loss of generality for the rest of the proof is that $N_k = \max(N_1, \dots, N_k)$.

9.1. **First part: a weak version of Lemma 5.** In a first step, we prove that

$$B_{\mathbf{N}}(a+pj)\left(H_{Lj+\lfloor La/p\rfloor} - H_{Lj}\right) \in p\mathbb{Z}_p\ . \tag{9.1}$$

(The reader should note the absence of the term $M_{\mathbf{N}}/\Theta_L$ in comparison with (3.4).)

For the proof of (9.1), we note that the p-adic valuation of $B_{\mathbf{N}}(a+pj)$ is equal to

$$v_p\big(B_{\mathbf{N}}(a+pj)\big) = \sum_{i=1}^{k}\sum_{\ell=1}^{\infty}\left(\left\lfloor \frac{N_i(a+pj)}{p^\ell}\right\rfloor - N_i\left\lfloor\frac{a+pj}{p^\ell}\right\rfloor\right).$$

Obviously, all the summands in this sum are non-negative, whence, in particular,

$$v_p\big(B_{\mathbf{N}}(a+pj)\big) \ge \sum_{\ell=1}^{\infty}\left(\left\lfloor \frac{N_k(a+pj)}{p^\ell}\right\rfloor - N_k\left\lfloor\frac{a+pj}{p^\ell}\right\rfloor\right). \tag{9.2}$$

On the other hand, by definition of the harmonic numbers, we have

$$H_{Lj+\lfloor La/p\rfloor} - H_{Lj} = \frac{1}{Lj+1} + \frac{1}{Lj+2} + \cdots + \frac{1}{Lj+\lfloor La/p\rfloor}.$$

It therefore suffices to show that

$$v_p\big(B_{\mathbf{N}}(a+pj)\big) \ge 1 + \max_{1\le\varepsilon\le\lfloor La/p\rfloor} v_p(Lj+\varepsilon). \tag{9.3}$$

The lower bound on the right-hand side of (9.2) can, in fact, be simplified since $0 \le a < p$; namely, we have

$$\left\lfloor\frac{a+pj}{p^\ell}\right\rfloor = \left\lfloor\frac{j}{p^{\ell-1}}\right\rfloor. \tag{9.4}$$

For a given integer ε with $1 \le \varepsilon \le \lfloor La/p\rfloor$, let $Lj+\varepsilon = p^d\beta$, where $d = v_p(Lj+\varepsilon)$. If we use this notation in (9.2), together with (9.4), we obtain

$$v_p\big(B_{\mathbf{N}}(a+pj)\big) \ge \sum_{\ell=1}^{\infty}\left(\left\lfloor \frac{N_k a}{p^\ell} - \frac{N_k\varepsilon}{Lp^{\ell-1}} + \frac{N_k\beta}{L}p^{d+1-\ell}\right\rfloor - N_k\left\lfloor -\frac{\varepsilon}{Lp^{\ell-1}} + \frac{\beta}{L}p^{d+1-\ell}\right\rfloor\right). \tag{9.5}$$

Since $\varepsilon \le \lfloor La/p \rfloor$, we have $\frac{N_k a}{p^\ell} - \frac{N_k \varepsilon}{L p^{\ell-1}} \ge 0$, whence

$$\left\lfloor \frac{N_k a}{p^\ell} - \frac{N_k \varepsilon}{L p^{\ell-1}} + \frac{N_k \beta}{L} p^{d+1-\ell} \right\rfloor \ge \left\lfloor \frac{N_k \beta}{L} p^{d+1-\ell} \right\rfloor . \tag{9.6}$$

Clearly, we also have

$$\left\lfloor - \frac{\varepsilon}{L p^{\ell-1}} + \frac{\beta}{L} p^{d+1-\ell} \right\rfloor \le \left\lfloor \frac{\beta}{L} p^{d+1-\ell} \right\rfloor . \tag{9.7}$$

If we use (9.6) and (9.7) in (9.5), then we obtain

$$v_p\big(B_{\mathbf{N}}(a+pj)\big) \ge \sum_{\ell=1}^{\infty} \left(\left\lfloor \frac{N_k \beta}{L} p^{d+1-\ell} \right\rfloor - N_k \left\lfloor \frac{\beta}{L} p^{d+1-\ell} \right\rfloor \right) . \tag{9.8}$$

By the same argument as the one that we used in the second step of the proof of Lemma 1 in Section 6, the rational number $\frac{\beta p^{d+1-\ell}}{L}$ is not an integer. However, the fact that $\beta p^{d+1-\ell}/L$ is not an integer entails that

$$\frac{\beta}{L} p^{d+1-\ell} - \left\lfloor \frac{\beta}{L} p^{d+1-\ell} \right\rfloor \ge \frac{1}{L},$$

as long as $\ell \le d+1$. Multiplication of both sides of this inequality by N_k leads to the chain of inequalities

$$\frac{N_k \beta}{L} p^{d+1-\ell} - N_k \left\lfloor \frac{\beta}{L} p^{d+1-\ell} \right\rfloor \ge \frac{N_k}{L} \ge 1$$

(it is here where we use the assumption $L \le N_k = \max(N_1, \dots, N_k)$), whence

$$\left\lfloor \frac{N_k \beta}{L} p^{d+1-\ell} \right\rfloor - N_k \left\lfloor \frac{\beta}{L} p^{d+1-\ell} \right\rfloor \ge 1,$$

provided $\ell \le d+1$. Use of this estimation in (9.8) gives

$$v_p\big(B_{\mathbf{N}}(a+pj)\big) \ge d+1 = 1 + v_p(Lj+\varepsilon).$$

This completes the proof of (9.3), and, hence, of (9.1).

For later use, we record that we have in particular shown that for any

$$D \le 1 + \max_{1 \le \varepsilon \le \lfloor La/p \rfloor} v_p(Lj+\varepsilon)$$

we have

$$\sum_{\ell=2}^{D} \left(\left\lfloor \frac{N_k(a+pj)}{p^\ell} \right\rfloor - N_k \left\lfloor \frac{a+pj}{p^\ell} \right\rfloor \right) \ge D-1. \tag{9.9}$$

We now embark on the proof of (3.4) itself.

9.2. **Second part: the case $j=0$.** In this case, we want to prove that

$$B_{\mathbf{N}}(a)H_{\lfloor La/p\rfloor}\in p\frac{M_{\mathbf{N}}}{\Theta_L}\mathbb{Z}_p\ , \tag{9.10}$$

or, using (2.3) (in the other direction), equivalently

$$B_{\mathbf{N}}(a)H_{La}\in \frac{M_{\mathbf{N}}}{\Theta_L}\mathbb{Z}_p\ , \tag{9.11}$$

The reader should keep in mind that we still assume that $p/L\le a<p$, so that, in particular, $a>0$.

If $p>N_k=\max(N_1,\dots,N_k)$, then our claim, in the form (9.10), reduces to $B_{\mathbf{N}}(a)H_{\lfloor La/p\rfloor}\in p\mathbb{Z}_p$, which is indeed true because of (9.1) with $j=0$.

Now let $p\le N_k$. By Lemma 4 and the definition of Θ_L, our claim, this time in the form (9.11), holds for $a=1$. So, let $a\ge 2$ from now on.

In a similar way as we did for the expression in (9.1), we bound the p-adic valuation of the expression in (9.11) from below:

$$\begin{aligned}
v_p\big(B_{\mathbf{N}}(a)H_{La}\big) &= \sum_{i=1}^{k}\sum_{\ell=1}^{\infty}\left(\left\lfloor\frac{N_ia}{p^\ell}\right\rfloor - N_i\left\lfloor\frac{a}{p^\ell}\right\rfloor\right)+v_p(H_{La})\\
&\ge \sum_{i=1}^{k}\sum_{\ell=1}^{\infty}\left\lfloor\frac{N_ia}{p^\ell}\right\rfloor - \left\lfloor\log_p La\right\rfloor\\
&\ge \sum_{i=1}^{k}\sum_{\ell=1}^{\infty}\left\lfloor\frac{2N_i}{p^\ell}\right\rfloor - \left\lfloor\log_p Lp\right\rfloor\\
&\ge \left\lfloor\frac{2N_k}{p}\right\rfloor+\sum_{\ell=2}^{\infty}\left\lfloor\frac{2N_k}{p^\ell}\right\rfloor+\sum_{i=1}^{k-1}\sum_{\ell=1}^{\infty}\left\lfloor\frac{2N_i}{p^\ell}\right\rfloor - \left\lfloor\log_p L\right\rfloor-1\\
&\ge \left\lfloor\frac{N_k}{p}\right\rfloor+\sum_{i=1}^{k}\sum_{\ell=1}^{\infty}\left\lfloor\frac{N_i}{p^\ell}\right\rfloor - \left\lfloor\log_p L\right\rfloor-1 \qquad (9.12)\\
&\ge \max\{1,\lfloor L/p\rfloor\}+\sum_{i=1}^{k}v_p(N_i!) - \left\lfloor\log_p L\right\rfloor-1. \qquad (9.13)
\end{aligned}$$

If $p=2$, then we can continue the estimation (9.13) as

$$v_2\big(B_{\mathbf{N}}(a)H_{La}\big)\ge\sum_{i=1}^{k}v_2(N_i!)-\lfloor\log_2 L\rfloor = v_2\big(M_{\mathbf{N}}/\Theta_L\big), \tag{9.14}$$

where we used the simple fact $v_2(H_L)=-\lfloor\log_2 L\rfloor$ to obtain the equality. (In fact, at this point it was not necessary to consider the case $p=2$ because $a<p$ and because we assumed $a\ge 2$. However, we shall re-use the present estimations later in the third part of the current proof, in a context where $a=1$ is allowed.)

From now on let $p \ge 3$. We use the fact that

$$x \ge \lfloor \log_p x \rfloor + 2 \tag{9.15}$$

for all integers $x \ge 2$ and primes $p \ge 3$. Thus, in the case that $L \ge 2p$, the estimation (9.13) can be continued as

$$v_p\big(B_{\mathbf{N}}(a)H_{La}\big) \ge 1 + \big\lfloor \log_p \lfloor L/p \rfloor \big\rfloor + \sum_{i=1}^{k} v_p(N_i!) - \big\lfloor \log_p L \big\rfloor \ge \sum_{i=1}^{k} v_p(N_i!) = v_p(M_{\mathbf{N}}),$$

implying (9.11) in this case. If $p \le L < 2p$, then the estimation (9.13) can be continued as

$$v_p\big(B_{\mathbf{N}}(a)H_{La}\big) \ge 1 + \sum_{i=1}^{k} v_p(N_i!) - 2 = v_p(M_{\mathbf{N}}/\Theta_L),$$

implying (9.11) in this case also. Finally, if $L < p$, it follows from (9.13) that

$$v_p\big(B_{\mathbf{N}}(a)H_{La}\big) \ge 1 + \sum_{i=1}^{k} v_p(N_i!) - 1 = v_p(M_{\mathbf{N}}),$$

implying (9.11) also in this final case. Thus, (9.10) is established.

9.3. **Third part: the case $j > 0$.** Now let $j > 0$. If $p > N_k = \max(N_1, \dots, N_k)$, then (3.4) reduces to

$$B_{\mathbf{N}}(a + pj)\,\big(H_{Lj+\lfloor La/p \rfloor} - H_{Lj}\big) \in p\mathbb{Z}_p\ ,$$

which is again true because of (9.1).

Now let $p \le N_k$. The reader should keep in mind that we still assume that $p/L \le a < p$, so that, in particular, $a > 0$. In a similar way as we did for the expression in (9.1), we bound the p-adic valuation of the expression in (3.4) from below. For the sake of convenience, we write T_1 for $\max_{1 \le \varepsilon \le \lfloor La/p \rfloor} v_p(Lj + \varepsilon)$ and T_2 for $\big\lfloor \log_p(a + pj) \big\rfloor$. Since it is somewhat hidden where our assumption $j > 0$ enters the subsequent considerations, we point out to the reader that $j > 0$ implies that $T_2 \ge 1$; without this property the split of the sum over ℓ into subsums in the chain of inequalities below would be impossible. So, using the above notation, we have (the detailed explanations for the various steps are given immediately after the following chain of estimations)

$$\begin{aligned}
&v_p\Big(B_{\mathbf{N}}(a + pj)\,\big(H_{Lj+\lfloor La/p \rfloor} - H_{Lj}\big)\Big)\\
&\quad= \sum_{i=1}^{k} \sum_{\ell=1}^{\infty} \left(\left\lfloor \frac{N_i(a + pj)}{p^\ell} \right\rfloor - N_i \left\lfloor \frac{a + pj}{p^\ell} \right\rfloor \right) + v_p\big(H_{Lj+\lfloor La/p \rfloor} - H_{Lj}\big)\\
&\quad= \left\lfloor \frac{N_k(a + pj)}{p} \right\rfloor - N_k \left\lfloor \frac{a + pj}{p} \right\rfloor + \sum_{\ell=2}^{\min\{1+T_1, T_2\}} \left(\left\lfloor \frac{N_k(a + pj)}{p^\ell} \right\rfloor - N_k \left\lfloor \frac{a + pj}{p^\ell} \right\rfloor \right)\\
&\qquad\quad+ \sum_{\ell=\min\{1+T_1, T_2\}+1}^{\infty} \left(\left\lfloor \frac{N_k(a + pj)}{p^\ell} \right\rfloor - N_k \left\lfloor \frac{a + pj}{p^\ell} \right\rfloor \right)
\end{aligned}$$

$$+\sum_{i=1}^{k-1}\sum_{\ell=1}^{\infty}\left(\left\lfloor\frac{N_i(a+pj)}{p^\ell}\right\rfloor-N_i\left\lfloor\frac{a+pj}{p^\ell}\right\rfloor\right)+v_p\big(H_{Lj+\lfloor La/p\rfloor}-H_{Lj}\big)$$

$$\geq\left\lfloor\frac{N_ka}{p}\right\rfloor+\min\{1+T_1,T_2\}-1+\sum_{i=1}^{k}\sum_{\ell=T_2+1}^{\infty}\left(\left\lfloor\frac{N_i(a+pj)}{p^\ell}\right\rfloor-N_i\left\lfloor\frac{a+pj}{p^\ell}\right\rfloor\right)$$

$$+v_p\big(H_{Lj+\lfloor La/p\rfloor}-H_{Lj}\big) \tag{9.16}$$

$$\geq\left\lfloor\frac{N_ka}{p}\right\rfloor+T_1+v_p\big(H_{Lj+\lfloor La/p\rfloor}-H_{Lj}\big)+\min\{0,T_2-T_1-1\}$$

$$+\sum_{i=1}^{k}\sum_{\ell=\lfloor\log_p(a+pj)\rfloor+1}^{\infty}\left(\left\lfloor\frac{N_i(a+pj)}{p^\ell}\right\rfloor-N_i\left\lfloor\frac{a+pj}{p^\ell}\right\rfloor\right) \tag{9.17}$$

$$\geq\max\{1,\lfloor L/p\rfloor\}+\min\{0,T_2-T_1-1\}+\sum_{i=1}^{k}\sum_{\ell=1}^{\infty}\left\lfloor\frac{N_i}{p^\ell}\cdot\frac{a+pj}{p^{\lfloor\log_p(a+pj)\rfloor}}\right\rfloor \tag{9.18}$$

$$\geq\max\{1,\lfloor L/p\rfloor\}+\big\lfloor\log_p(a+pj)\big\rfloor-\big\lfloor\log_p\big(Lj+\lfloor La/p\rfloor\big)\big\rfloor-1$$

$$+\sum_{i=1}^{k}\sum_{\ell=1}^{\infty}\left\lfloor\frac{N_i}{p^\ell}\cdot\frac{a+pj}{p^{\lfloor\log_p(a+pj)\rfloor}}\right\rfloor \tag{9.19}$$

$$\geq\max\{1,\lfloor L/p\rfloor\}+\big\lfloor\log_p j\big\rfloor-\big\lfloor\log_p\big(Lj+\lfloor La/p\rfloor\big)\big\rfloor+\sum_{i=1}^{k}\sum_{\ell=1}^{\infty}\left\lfloor\frac{N_i}{p^\ell}\right\rfloor \tag{9.20}$$

$$\geq\max\{1,\lfloor L/p\rfloor\}+\big\lfloor\log_p j\big\rfloor-\big\lfloor\log_p L\big\rfloor-\left\lfloor\log_p\left(j+\frac{1}{L}\lfloor La/p\rfloor\right)\right\rfloor-1$$

$$+\sum_{i=1}^{k}v_p(N_i!) \tag{9.21}$$

$$\geq\max\{1,\lfloor L/p\rfloor\}-\big\lfloor\log_p L\big\rfloor-1+v_p(M_{\mathbf{N}}). \tag{9.22}$$

Here, we used (9.9) in order to get (9.16). To get (9.18), we used the inequalities

$$\left\lfloor\frac{N_ka}{p}\right\rfloor\geq\left\lfloor\frac{N_k}{p}\right\rfloor\geq\max\{1,\lfloor L/p\rfloor\} \tag{9.23}$$

and

$$T_1+v_p\big(H_{Lj+\lfloor La/p\rfloor}-H_{Lj}\big)\geq 0. \tag{9.24}$$

To get (9.19), we used that

$$T_2-T_1-1\geq\big\lfloor\log_p(a+pj)\big\rfloor-\big\lfloor\log_p\big(Lj+\lfloor La/p\rfloor\big)\big\rfloor-1$$

and

$$\big\lfloor\log_p(a+pj)\big\rfloor-\big\lfloor\log_p\big(Lj+\lfloor La/p\rfloor\big)\big\rfloor-1=\big\lfloor\log_p j\big\rfloor-\big\lfloor\log_p\big(Lj+\lfloor La/p\rfloor\big)\big\rfloor\leq 0,$$

so that

$$\min\{0, T_2 - T_1 - 1\} \ge \lfloor \log_p(a+pj) \rfloor - \lfloor \log_p\left(Lj + \lfloor La/p \rfloor\right) \rfloor - 1. \tag{9.25}$$

Next, to get (9.20), we used

$$\left\lfloor \frac{N_i}{p^\ell} \cdot \frac{a+pj}{p^{\lfloor \log_p(a+pj) \rfloor}} \right\rfloor \ge \left\lfloor \frac{N_i}{p^\ell} \right\rfloor. \tag{9.26}$$

To get (9.21), we used

$$\lfloor \log_p\left(Lj + \lfloor La/p \rfloor\right) \rfloor \le \lfloor \log_p L \rfloor + \left\lfloor \log_p\left(j + \frac{1}{L}\lfloor La/p \rfloor\right) \right\rfloor + 1. \tag{9.27}$$

Finally, we used $\frac{1}{L}\lfloor La/p \rfloor < 1$ in order to get (9.22).

If we now repeat the arguments after (9.13), then we see that the estimation (9.22) implies

$$v_p\Big(B_{\mathbf{N}}(a+pj)\left(H_{Lj+\lfloor La/p \rfloor} - H_{Lj}\right)\Big) \ge v_p\big(M_{\mathbf{N}}/\Theta_L\big). \tag{9.28}$$

This almost proves (3.4), our lower bound on the p-adic valuation of the number in (3.4) is just by 1 too low.

In order to establish that (3.4) is indeed correct, we assume by contradiction that all the inequalities in the estimations leading to (9.22) and finally to (9.28) are in fact equalities. In particular, the estimations in (9.23) hold with equality only if $a = 1$ and, if L should be at least p, also $\lfloor N_k/p \rfloor = \lfloor L/p \rfloor$. We shall henceforth assume both of these two conditions.

If we examine the arguments after (9.13) that led us from (9.22) to (9.28), then we see that they prove in fact

$$v_p\Big(B_{\mathbf{N}}(a+pj)\left(H_{Lj+\lfloor La/p \rfloor} - H_{Lj}\right)\Big) \ge 1 + v_p\big(M_{\mathbf{N}}/\Theta_L\big) \tag{9.29}$$

except if:

Case 1: $p = 2$ and $\lfloor L/2 \rfloor = 1$;
Case 2: $p \ge 3$ and $p \le L < 2p$;
Case 3: $p = 3$ and $\lfloor L/3 \rfloor = 2$;
Case 4: $L < p$.

In all other cases, there holds either strict inequality in (9.15) with $x = \lfloor L/p \rfloor$, or one has $v_p(\Theta_L) \ge 1$ and is able to show

$$v_p\Big(B_{\mathbf{N}}(a+pj)\left(H_{Lj+\lfloor La/p \rfloor} - H_{Lj}\right)\Big) \ge v_p\big(M_{\mathbf{N}}\big),$$

so that (9.29) is satisfied, as desired. We now show that (9.29) holds as well in Cases 1–4, thus completing the proof of (3.4).

Case 1. Let first $p = 2$ and $L = 2$. We then have

$$\begin{aligned}\min\{0, T_2 - T_1 - 1\} &= \min\{0, \lfloor \log_2(2j+1) \rfloor - v_2(2j+1) - 1\} \\ &= \min\{0, \lfloor \log_2(2j+1) \rfloor - 1\} = 0 > -1,\end{aligned}$$

in contradiction to having equality in (9.25).

On the other hand, if $p = 2$ and $L = 3$ then, because of equality in the second estimation in (9.23), we must have $N_k = 3$. We have

$$H_{Lj+\lfloor La/p \rfloor} - H_{Lj} = H_{3j+1} - H_{3j} = \frac{1}{3j+1}.$$

If there holds equality in (9.25), then $Lj + \lfloor La/p \rfloor = 3j + 1$ must be a power of 2, say $3j + 1 = 2^e$ or, equivalently, $j = (2^e - 1)/3$. It follows that

$$\left\lfloor \frac{N_k}{p} \cdot \frac{a+pj}{p^{\lfloor \log_p(a+pj) \rfloor}} \right\rfloor = \left\lfloor \frac{3}{2} \cdot \frac{1+2j}{2^{\lfloor \log_2(1+2j) \rfloor}} \right\rfloor = \left\lfloor \frac{3}{2} \cdot \frac{2^{e+1}+1}{3 \cdot 2^{e-1}} \right\rfloor = 2 > 1 = \left\lfloor \frac{3}{2} \right\rfloor = \left\lfloor \frac{N_k}{p} \right\rfloor,$$

in contradiction to having equality in (9.26) with $\ell = 1$.

Case 2. Our assumptions $p \geq 3$ and $p \leq L < 2p$ imply

$$H_{Lj+\lfloor La/p \rfloor} - H_{Lj} = H_{Lj+1} - H_{Lj} = \frac{1}{Lj+1}.$$

Arguing as in the previous case, in order to have equality in (9.25), we must have $Lj+1 = f \cdot p^e$ for some positive integers e and f with $0 < f < p$. Thus, $j = (f \cdot p^e - 1)/L$ and $p < L$. (If $p = L$ then j would be non-integral.) It follows that

$$\left\lfloor \frac{N_k}{p} \cdot \frac{a+pj}{p^{\lfloor \log_p(a+pj) \rfloor}} \right\rfloor = \left\lfloor \frac{N_k}{p} \cdot \frac{f \cdot p^{e+1} + L - p}{L \cdot p^{\lfloor \log_p((f \cdot p^{e+1}+L-p)/L) \rfloor}} \right\rfloor. \tag{9.30}$$

If $f = 1$, then we obtain from (9.30) that

$$\left\lfloor \frac{N_k}{p} \cdot \frac{a+pj}{p^{\lfloor \log_p(a+pj) \rfloor}} \right\rfloor = \left\lfloor \frac{N_k}{p} \cdot \frac{p^{e+1} + L - p}{L \cdot p^{e-1}} \right\rfloor \geq \left\lfloor \frac{p^{e+1} + L - p}{p^e} \right\rfloor > 1 = \left\lfloor \frac{L}{p} \right\rfloor = \left\lfloor \frac{N_k}{p} \right\rfloor,$$

in contradiction with having equality in (9.26) with $\ell = 1$.

On the other hand, if $f \geq 2$, then we obtain from (9.30) that

$$\left\lfloor \frac{N_k}{p} \cdot \frac{a+pj}{p^{\lfloor \log_p(a+pj) \rfloor}} \right\rfloor \geq \left\lfloor \frac{f \cdot p^{e+1} + L - p}{p^{e+1}} \right\rfloor \geq f > 1 = \left\lfloor \frac{L}{p} \right\rfloor = \left\lfloor \frac{N_k}{p} \right\rfloor,$$

again in contradiction with having equality in (9.26) with $\ell = 1$.

Case 3. Our assumptions $p = 3$ and $\lfloor L/3 \rfloor = 2$ imply

$$H_{Lj+\lfloor La/p \rfloor} - H_{Lj} = H_{Lj+2} - H_{Lj} = \frac{1}{Lj+1} + \frac{1}{Lj+2}.$$

Similar to the previous cases, in order to have equality in (9.25), we must have $Lj+\varepsilon = f \cdot 3^e$ for some positive integers ε, e, f with $0 < \varepsilon, f < 3$. The arguments from Case 2 can now be repeated almost verbatim. We leave the details to the reader.

Case 4. If $L < p$, then $p/L > 1 = a$, a contradiction to the assumption that we made at the very beginning of this section.

This completes the proof of the lemma.

10. Proof of Lemma 7

We proceed in the same way as in the proof of Lemma 3 in Section 8. Again, the claim is trivially true if p divides m, so that we may assume that p does not divide m for the rest of the proof. Let us write $m = a + pj$, with $0 < a < p$. Then comparison with (3.4) shows that we are in a very similar situation here. Indeed, we may derive (3.7) from Lemma 5. In order to see this, we use (8.1) to deduce

$$v_p\Big(B_{\mathbf{N}}(m)\big(H_{Lmp^s} - H_{L\lfloor m/p\rfloor p^{s+1}}\big)\Big)\\ \geq -1 - s + \min\Big\{v_p\Big(B_{\mathbf{N}}(a+pj)(H_{Lj+\lfloor La/p\rfloor} - H_{Lj})\Big), 1 + v_p\big(B_{\mathbf{N}}(a+pj)\big)\Big\}.$$

Use of Lemmas 4 and 5 then completes the proof.

11. The equivalence of Zudilin's and our definition of $\mathbf{H}_N(m)$

Zudilin's definition of the quantity $\mathbf{H}_N(m)$ deviates from (1.4). In this final section, we show that our definition is equivalent to Zudilin's.

Lemma 15. *Let m be a non-negative integer, and let N be a positive integer with associated parameters $\alpha_i, \beta_i, \mu, \eta$ (that is, given by* (1.1) *and* (1.2)*, respectively). Then*

$$\mathbf{H}_N(m) = \sum_{j=1}^{\varphi(N)} H(r_j/N, m) - \varphi(N)H(1, m),$$

where $H(x, m) := \sum_{n=0}^{m-1} \frac{1}{x+n}$, and where $r_j \in \{1, 2, \ldots, N\}$ form the residue classes modulo N which are coprime to N. As before, $\varphi(\,.\,)$ denotes Euler's totient function.

Proof. For $N = 1$, we have $\mathbf{H}_1(m) = 0$, so that the assertion of the lemma holds trivially. Therefore, from now on, we assume $N \geq 2$.

We claim that, for any real number $m \geq 0$, we have

$$\frac{C_N^m}{\Gamma(m+1)^{\varphi(N)}} \prod_{j=1}^{\varphi(N)} \frac{\Gamma(m + r_j/N)}{\Gamma(r_j/N)} = \frac{\prod_{j=1}^{\mu} \Gamma(\alpha_j m + 1)}{\prod_{j=1}^{\eta} \Gamma(\beta_j m + 1)}, \tag{11.1}$$

where $\Gamma(x)$ denotes the gamma function. This generalises Zudilin's identity (1.3) to real values of m. We essentially extend his proof to real m, using the well-known formula [4, p. 23, Theorem 1.5.2]

$$\Gamma(a)\,\Gamma\left(a + \frac{1}{n}\right)\Gamma\left(a + \frac{2}{n}\right)\cdots\Gamma\left(a + \frac{n-1}{n}\right) = n^{\frac{1}{2}-an}(2\pi)^{(n-1)/2}\Gamma(an), \tag{11.2}$$

valid for real numbers a and positive integers n such that aN is not an integer ≤ 0. Indeed, as in the Introduction, let $p_1, p_2, \ldots, p_\ell$ denote the distinct prime factors of N. (It should be noted that there is at least one such prime factor due to our assumption $N \geq 2$.) Furthermore, for a subset J of $\{1, 2, \ldots, \ell\}$, let p_J denote the product $\prod_{j\in J} p_j$

of corresponding prime factors of N. (In the case that $J = \emptyset$, the empty product must be interpreted as 1.) Then, by the principle of inclusion-exclusion, we can rewrite the left-hand side of (11.1) in the form

$$\frac{C_N^m}{\Gamma(m+1)^{\varphi(N)}} \cdot \frac{\prod_{\substack{J\subseteq\{1,2,\dots,\ell\}\\ |J| \text{ even}}} \prod_{i=1}^{N/p_J} \Gamma\left(m+\frac{ip_J}{N}\right)}{\prod_{\substack{J\subseteq\{1,2,\dots,\ell\}\\ |J| \text{ odd}}} \prod_{i=1}^{N/p_J} \Gamma\left(m+\frac{ip_J}{N}\right)} \cdot \frac{\prod_{\substack{J\subseteq\{1,2,\dots,\ell\}\\ |J| \text{ odd}}} \prod_{i=1}^{N/p_J} \Gamma\left(\frac{ip_J}{N}\right)}{\prod_{\substack{J\subseteq\{1,2,\dots,\ell\}\\ |J| \text{ even}}} \prod_{i=1}^{N/p_J} \Gamma\left(\frac{ip_J}{N}\right)}.$$

To each of the products over i, formula (11.2) can be applied. As a result, we obtain the expression

$$\frac{C_N^m}{\Gamma(m+1)^{\varphi(N)}} \cdot \frac{\prod_{\substack{J\subseteq\{1,2,\dots,\ell\}\\ |J| \text{ even}}} \left(\frac{N}{p_J}\right)^{-\left(m+\frac{p_J}{N}\right)\frac{N}{p_J}} \Gamma\left(m\frac{N}{p_J}+1\right)}{\prod_{\substack{J\subseteq\{1,2,\dots,\ell\}\\ |J| \text{ odd}}} \left(\frac{N}{p_J}\right)^{-\left(m+\frac{p_J}{N}\right)\frac{N}{p_J}} \Gamma\left(m\frac{N}{p_J}+1\right)} \cdot \frac{\prod_{\substack{J\subseteq\{1,2,\dots,\ell\}\\ |J| \text{ odd}}} \Gamma(1)}{\prod_{\substack{J\subseteq\{1,2,\dots,\ell\}\\ |J| \text{ even}}} \Gamma(1)}$$

$$= \frac{C_N^m}{\Gamma(m+1)^{\varphi(N)}} \cdot \frac{\prod_{\substack{J\subseteq\{1,2,\dots,\ell\}\\ |J| \text{ even}}} \left(\frac{N}{p_J}\right)^{-mN/p_J} \Gamma\left(m\frac{N}{p_J}+1\right)}{\prod_{\substack{J\subseteq\{1,2,\dots,\ell\}\\ |J| \text{ odd}}} \left(\frac{N}{p_J}\right)^{-mN/p_J} \Gamma\left(m\frac{N}{p_J}+1\right)}, \quad (11.3)$$

where the simplification in the exponent of N/p_J is due to the fact that there are as many subsets of even cardinality of a given non-empty set as there are subsets of odd cardinality. Since, again by inclusion-exclusion,

$$\sum_{\substack{J\subseteq\{1,2,\dots,\ell\}\\ |J| \text{ even}}} \frac{N}{p_J} - \sum_{\substack{J\subseteq\{1,2,\dots,\ell\}\\ |J| \text{ odd}}} \frac{N}{p_J} = N \prod_{p \mid N} \left(1-\frac{1}{p}\right) = \varphi(N), \quad (11.4)$$

we have

$$\frac{1}{\Gamma(m+1)^{\varphi(N)}} \cdot \frac{\prod_{\substack{J\subseteq\{1,2,\dots,\ell\}\\ |J| \text{ even}}} \Gamma\left(m\frac{N}{p_J}+1\right)}{\prod_{\substack{J\subseteq\{1,2,\dots,\ell\}\\ |J| \text{ odd}}} \Gamma\left(m\frac{N}{p_J}+1\right)} = \frac{\prod_{j=1}^{\mu} \Gamma(\alpha_j m+1)}{\prod_{j=1}^{\eta} \Gamma(\beta_j m+1)}$$

and

$$\frac{\prod_{\substack{J\subseteq\{1,2,\dots,\ell\}\\ |J| \text{ even}}} N^{-mN/p_J}}{\prod_{\substack{J\subseteq\{1,2,\dots,\ell\}\\ |J| \text{ odd}}} N^{-mN/p_J}} = N^{-m\varphi(N)}.$$

Finally, consider a fixed prime number dividing N, p_j say. Then, using again (11.4), we see that the exponent of p_j in the expression (11.3) is

$$-\frac{m}{p_j}\sum_{\substack{J\subseteq\{1,2,\dots,\ell\}\\ |J|\text{ odd},\, j\notin J}}\frac{N}{p_J}+\frac{m}{p_j}\sum_{\substack{J\subseteq\{1,2,\dots,\ell\}\\ |J|\text{ even},\, j\notin J}}\frac{N}{p_J}=\frac{mN}{p_j}\prod_{\substack{p\mid N\\ p\neq p_j}}\left(1-\frac{1}{p}\right)=\frac{m}{p_j}\frac{\varphi(N)}{1-\frac{1}{p_j}}=\frac{m\varphi(N)}{p_j-1}.$$

If all these observations are used in (11.3), we arrive at the right-hand side of (11.1).

Now, let us call $b(m)$ the function defined by both sides of (11.1), and let $\psi(x)=\Gamma'(x)/\Gamma(x)$ be the digamma function. We will use the well-known property (see [4, p. 13, Theorem 1.2.7]) that $\psi(x+n)-\psi(x)=H(x,n)$ for real numbers $x>0$ and integers $n\geq 0$.

By taking the logarithmic derivative of the right-hand side of (11.1), we have

$$\begin{aligned}\frac{b'(m)}{b(m)}&=\sum_{j=1}^{\mu}\alpha_j\psi(\alpha_jm+1)-\sum_{j=1}^{\eta}\beta_j\psi(\beta_jm+1)\\&=\sum_{j=1}^{\mu}\alpha_j\big(\psi(1)+H_{\alpha_jm}\big)-\sum_{j=1}^{\eta}\beta_j\big(\psi(1)+H_{\beta_jm}\big)\\&=\sum_{j=1}^{\mu}\alpha_jH_{\alpha_jm}-\sum_{j=1}^{\eta}\beta_jH_{\beta_jm}\\&=\mathbf{H}_N(m),\end{aligned}\tag{11.5}$$

because $\sum_{j=1}^{\mu}\alpha_j=\sum_{j=1}^{\eta}\beta_j$. It also follows that $b'(0)/b(0)=0$.

On the other hand, by taking the logarithmic derivative of the left-hand side of (11.1), we also have

$$\frac{b'(m)}{b(m)}=\log(C_N)+\sum_{j=1}^{\varphi(N)}\psi(m+r_j/N)-\varphi(N)\psi(m+1).$$

Since $b'(0)/b(0)=0$, we have $\log(C_N)=-\sum_{j=1}^{\varphi(N)}\psi(r_j/N)+\varphi(N)\psi(1)$ and therefore,

$$\begin{aligned}\frac{b'(m)}{b(m)}&=\sum_{j=1}^{\varphi(N)}\big(\psi(m+r_j/N)-\psi(r_j/N)\big)-\varphi(N)\big(\psi(m+1)-\psi(1)\big)\\&=\sum_{j=1}^{\varphi(N)}H(r_j/N,m)-\varphi(N)H(1,m).\end{aligned}\tag{11.6}$$

The lemma follows by equating the expressions (11.5) and (11.6) obtained for $b'(m)/b(m)$. □

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Bibliography

[1] G. Almkvist, C. van Enckevort, D. van Straten and W. Zudilin, *Tables of Calabi–Yau equations*, unpublished manuscript; `http://arχiv.org/abs/0507430`.

[2] G. Almkvist and W. Zudilin, *Differential equations, mirror maps and zeta values*, in: Mirror Symmetry V, N. Yui, S.-T. Yau, and J.D. Lewis (eds.), AMS/IP Studies in Advanced Mathematics **38** (2007), International Press & Amer. Math. Soc., 481–515.

[3] Y. André, *G-fonctions et transcendance*, J. reine angew. Math. **476** (1996), 95–125.

[4] G. E. Andrews, R. A. Askey et R. Roy, *Special Functions*, The Encyclopedia of Mathematics and Its Applications, vol. **71**, (G.-C. Rota, ed.), Cambridge University Press, Cambridge (1999).

[5] V. V. Batyrev and D. van Straten, *Generalized hypergeometric functions and rational curves on Calabi–Yau complete intersections in toric varieties*, Comm. Math. Phys. **168** (1995), 493–533.

[6] F. Beukers and G. Heckman, *Monodromy for the hypergeometric function* ${}_nF_{n-1}$, Invent. Math. **95** (1989), 325–354.

[7] P. Candelas, X. de la Ossa, P. Green and L. Parkes, *A pair of Calabi–Yau manifolds as an exactly soluble superconformal theory*, Nucl. Phys. **B359** (1991), 21–74.

[8] A. Corti and V. Golyshev, *Hypergeometric equations and weighted projective spaces*, preprint 2006; available at `http://arχiv.org/abs/0607.5016`.

[9] B. Dwork, *p-Adic cycles*, Publ. I.H.E.S. **37** (1969), 27–115.

[10] B. Dwork, *On p-adic differential equations IV: generalized hypergeometric functions as p-adic analytic functions in one variable*, Ann. Sci. É.N.S. (4) **6**, no. 3 (1973), 295–316.

[11] A. B. Givental, *Equivariant Gromov–Witten invariants*, Internat. Math. Res. Notices **13** (1996), 613–663.

[12] N. Heninger, E. M. Rains and N. J. A. Sloane, *On the integrality of nth roots of generating functions*, J. Combin. Theory Ser. A **113** (2006), 1732–1745.

[13] S. Hosono, A. Klemm, S. Theisen, S.-T. Yau, *Mirror symmetry, mirror map and applications to complete intersection Calabi–Yau spaces*, Nuclear Phys. B **43**, no. 3 (1995), 501–552.

[14] M. Kontsevich, A. Schwarz and V. Vologodsky, *Integrality of instanton numbers and p-adic B-model*, Phys. Lett. B **637** (2006), 97–101.

[15] C. Krattenthaler and T. Rivoal, *On the integrality of Taylor coefficients of mirror maps in several variables*, preprint; `http://arχiv.org/abs/0804.3049`.

[16] C. Krattenthaler and T. Rivoal, *On the integrality of Taylor coefficients of mirror maps, II*, preprint 2008.

[17] E. Landau, *Sur les conditions de divisibilité d'un produit de factorielles par un autre*, Nouv. Ann. (3) **19** (1900), 344–362, 576.

[18] S. Lang, *Cyclotomic fields II*, Graduate Texts in Mathematics **69**, Springer–Verlag, 1980.

[19] B. H. Lian and S.-T. Yau, *Arithmetic properties of mirror map and quantum coupling*, Comm. Math. Phys. **176**, no. 1 (1996), 163–191.

[20] B. H. Lian and S.-T. Yau, *Mirror maps, modular relations and hypergeometric series I*, appeared as *Integrality of certain exponential series*, in: Lectures in Algebra and Geometry, Proceedings of the International Conference on Algebra and Geometry, Taipei, 1995, M.-C. Kang (ed.), Int. Press, Cambridge, MA, 1998, pp. 215–227.

[21] B. H. Lian, K. Liu and S.-T. Yau, *Mirror principle. I.*, Asian J. Math. **1**, no. 4 (1997), 729–763.

[22] B. H. Lian and S.-T. Yau, *The nth root of the mirror map*, in: Calabi–Yau varieties and mirror symmetry, Proceedings of the Workshop on Arithmetic, Geometry and Physics around Calabi–Yau Varieties and Mirror Symmetry, Toronto, ON, 2001, N. Yui and J. D. Lewis (eds.), Fields Inst. Commun., **38**, Amer. Math. Soc., Providence, RI, 2003, pp. 195–199.

[23] D. R. Morrison, *Mirror symmetry and rational curves on quintic threefolds: a guide for mathematicians*, J. Amer. Math. Soc. **6** (1993), 223–247.

[24] R. Pandharipande, *Rational curves on hypersurfaces (after A. Givental)*, Séminaire Bourbaki. Vol. 1997/98. Astérisque No. **252** (1998), Exp. No. 848, 5, 307–340.

[25] F. Rodriguez–Villegas, *Hypergeometric families of Calabi–Yau manifolds*, in: Proceedings of the Workshop on Arithmetic, Geometry and Physics around Calabi–Yau Varieties and Mirror Symmetry, Toronto, ON, 2001, N. Yui and J. D. Lewis (eds.), Fields Inst. Commun., **38**, Amer. Math. Soc., Providence, R.I., 2003, 223–231.

[26] F. Rodriguez–Villegas, *Integral ratios of factorials and algebraic hypergeometric functions*, in: Oberwolfach Reports, vol. 2, issue 3, European Math. Soc., Publ. House, Zürich, 2005; electronically available at `http://www.math.utexas.edu/~villegas/publications/oberwolfach-05.pdf`.

[27] J. Stienstra, *GKZ Hypergeometric Structures*, Proceedings of the Summer School "Algebraic Geometry and Hypergeometric Functions", Istanbul, June 2005, preprint 2005; available at `http://arχiv.org/abs/0511.5351`.

[28] C. Voisin, *Mirror symmetry*, SMF/AMS Texts and Monographs, vol. 1, American Mathematical Society, Providence, RI; Société Mathématique de France, Paris, 1999.

[29] V. Vologodsky, *Integrality of instanton numbers*, preprint 2007; `http://arχiv.org/abs/0707.4617`.

[30] M. Yoshida, *Fuchsian differential equations*, Aspects of Mathematics **11**, Vieweg, 1987.

[31] W. Zudilin, *Integrality of power expansions related to hypergeometric series*, Mathematical Notes **71**.5 (2002), 604–616.

C. Krattenthaler, Fakultät für Mathematik, Universität Wien, Nordbergstrasse 15, A-1090 Vienna, Austria. WWW: `http://www.mat.univie.ac.at/~kratt`.

T. Rivoal, Institut Fourier, CNRS UMR 5582, Université Grenoble 1, 100 rue des Maths, BP 74, 38402 Saint-Martin d'Hères cedex, France.
WWW: `http://www-fourier.ujf-grenoble.fr/~rivoal`.