On Dwork's p-adic formal congruences Theorem and hypergeometric mirror maps

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Abstract

Using Dwork's theory, we prove a broad generalization of his famous p-adic formal congruences theorem. This enables us to prove certain p-adic congruences for the generalized hypergeometric series with rational parameters; in particular, they hold for *any* prime number p and not only for almost all primes. Furthermore, using Christol's functions, we provide an explicit formula for the "Eisenstein constant" of any hypergeometric series with rational parameters.

As an application of these results, we obtain an arithmetic statement "on average" of a new type concerning the integrality of Taylor coefficients of the associated mirror maps. It contains all the similar univariate integrality results in the literature, with the exception of certain refinements that hold only in very particular cases.

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1. Introduction

Mirror maps are power series which occur in Mirror Symmetry as the inverse for composition of power series of the form $q(z) = \exp(\omega_2(z)/\omega_1(z))$, called canonical coordinates, where $\omega_1(z)$ and $\omega_2(z)$ are particular solutions of the Picard-Fuchs equation associated with certain one-parameter families of Calabi-Yau varieties. They can be viewed as higher dimensional generalizations of the classical modular forms, and in several cases, it has been observed that such mirror maps and canonical coordinates have integral Taylor coefficients at the origin.

The arithmetical study of mirror maps began with the famous example of a family of mirror manifolds for quintic threefolds in $\mathbb{P}^4$ given by Candelas et al. [6] and associated with the Picard-Fuchs equation

$$\theta^4\omega - 5z(5\theta+1)(5\theta+2)(5\theta+3)(5\theta+4)\omega = 0, \quad \theta = z\frac{d}{dz}.$$

This equation is (a rescaling of) a generalized hypergeometric differential equation with two linearly independent local solutions at $z = 0$ given by

$$\omega_1(z) = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5} z^n \quad \text{and} \quad \omega_2(z) = G(z) + \log(z)\omega_1(z),$$

where

$$G(z) = \sum_{n=1}^{\infty} \frac{(5n)!}{(n!)^5} \big(5H_{5n} - 5H_n\big) z^n \quad \text{and} \quad H_n := \sum_{k=1}^{n} \frac{1}{k}.$$

The corresponding canonical coordinate $\exp\big(\omega_2(z)/\omega_1(z)\big)$ occurs in enumerative geometry and in the Mirror Conjecture associated with quintic threefolds in $\mathbb{P}^4$ (see [25]). The integrality of its Taylor coefficients at the origin has been proved by Lian and Yau in [26].

In a more general context, Batyrev and van Straten conjectured the integrality of the Taylor coefficients at the origin of a large class of canonical coordinates [2, Conjecture 6.3.4] built on A-hypergeometric series (see [33] for an introduction to these series, which generalize the classical hypergeometric series to the multivariate case). Furthermore, they provided a lot of examples of univariate canonical coordinates whose Taylor coefficients were subsequently proved to be integers in many cases by Zudilin [34] and Krattenthaler and Rivoal [18].

In the sequel of this article, we say that a power series $f(z) \in \mathbb{C}[[z]]$ is N-integral if there exists $c \in \mathbb{Q}$ such that $f(cz) \in \mathbb{Z}[[z]]$. The constant c might be called the Eisenstein constant of f, in reference to Eisenstein's theorem that such a constant c exists when f is a holomorphic algebraic function over $\mathbb{Q}(z)$.

Motivated by the search for differential operators $\mathcal{L}$ associated with particular families of Calabi-Yau varieties, Almkvist et al. [1] and Bogner and Reiter [5] introduced the notion of "Calabi-Yau operators". Even if both notions slightly differ, both require that an irreducible differential operator $\mathcal{L} \in \mathbb{Q}(z)[d/dz]$ of Calabi-Yau type satisfies

(P_1) $\mathcal{L}$ has a solution $\omega_1(z) \in 1 + z\mathbb{C}[[z]]$ at $z = 0$ which is N-integral.

(P_2) $\mathcal{L}$ has a linearly independent solution $\omega_2(z) = G(z) + \log(z)\omega_1(z)$ at $z = 0$ with $G(z) \in z\mathbb{C}[[z]]$ and $\exp\big(\omega_2(z)/\omega_1(z)\big)$ is N-integral.

The present paper is mainly concerned with arithmetic properties of mirror maps associated with generalized hypergeometric equations, that we shall now define.

We let $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_r)$ and $\boldsymbol{\beta} := (\beta_1, \dots, \beta_s)$ be tuples of parameters in $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$.

We introduce the generalized hypergeometric series

$$F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n} z^n, \tag{1.1}$$

where $(x)_n$ denotes the Pochhammer symbol $(x)_n = x(x+1)\cdots(x+n-1)$ if $n \geq 1$ and $(x)_0 = 1$ otherwise. If $\beta_s = 1$, then our definition (1.1) agrees with the classical notation

$$F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z) = {}_rF_{s-1}\left[\begin{array}{c}\alpha_1, \dots, \alpha_r \\ \beta_1, \dots, \beta_{s-1}\end{array}; z\right] := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_{s-1})_n} \frac{z^n}{n!}.$$

We also consider the series

$$G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z) := \sum_{n=1}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n} \left(\sum_{i=1}^{r} H_{\alpha_i}(n) - \sum_{j=1}^{s} H_{\beta_j}(n) \right),$$

where, for all $n \in \mathbb{N}$ and all $x \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$, $H_x(n) := \sum_{k=0}^{n-1} \frac{1}{x+k}$.

We define the canonical coordinate associated with $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ by

$$q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z) := z \exp\left(\frac{G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)}{F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)} \right) \in z\mathbb{Q}[[z]]. \tag{1.2}$$

The mirror map $z_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q) \in q\mathbb{Q}[[q]]$ associated with $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is, by definition, the compositional inverse of $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$.

This definition of $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$ is motivated by the fact that, if $\beta_{s-1} = \beta_s = 1$, then $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$ is the canonical coordinate associated to the generalized hypergeometric operator given by

$$\mathcal{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}} := \prod_{i=1}^{s} (\theta + \beta_i - 1) - z \prod_{i=1}^{r} (\theta + \alpha_i), \quad \theta = z\frac{d}{dz}.$$

Indeed, in this case, $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$ and $G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z) + \log(z)F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$ are formal solutions of $\mathcal{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and we have

$$q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z) = \exp\left(\frac{G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z) + \log(z)F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)}{F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)} \right).$$

We shall now give a brief overview of the content of the present paper, refering to Section 2 for the detailed statements of our main results.

We start with a study of the N-integrality properties of $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$. Whether or not $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$ is N-integral can be decided by using a criterion due to Christol. The first task undertaken in this paper is to study the minimal constant $C_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ in $\mathbb{Q}^+ \setminus \{0\}$ such that

$$F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C_{\boldsymbol{\alpha},\boldsymbol{\beta}}z) \in \mathbb{Z}[[z]].$$

In particular, we give an explicit formula for $C_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ when $r=s$, $\boldsymbol{\alpha}\in(0,1]^r$ and $\boldsymbol{\beta}\in(0,1]^s$. We refer to Section 2.1 for details.

We shall now introduce some notations. For all $x\in\mathbb{Q}$, we denote by $\langle x\rangle$ the unique element in $(0,1]$ such that $x-\langle x\rangle\in\mathbb{Z}$, and, for all $\boldsymbol{x}=(x_1,...,x_m)\in\mathbb{Q}^m$, we set $\langle\boldsymbol{x}\rangle=(\langle x_1\rangle,...,\langle x_m\rangle)$. Hence $\langle x\rangle$ is the fractional part of x if $x\notin\mathbb{Z}$, and $\langle x\rangle=1$ otherwise. We denote by $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ the least common multiple of the exact denominators ([1]) of the elements of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

In the rest of this section, we assume that, for all $(i,j)\in\{1,...,r\}\times\{1,...,s\}$, we have $\boldsymbol{\alpha}_i-\boldsymbol{\beta}_j\notin\mathbb{Z}$ ([2]), and that $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$ is N-integral.

We now come to the N-integrality of $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$. The N-integrality of $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$ is the exception, not the rule. This is illustrated in [**8**] where the N-integral mirror maps are classified when the parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are R-partitioned ([3]) and in [**31, 32**] where such a classification is obtained when $\beta_1=\cdots=\beta_s=1$. One of our main contribution is to exhibit an explicit condition, denoted by $H_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, on $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ such that the following properties are equivalent:

(1) $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$ is N-integral;
(2) assertion $H_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ holds, we have $r=s$ and, for all $a\in\{1,\dots,d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ coprime to $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, we have $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)=q_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(z)$.

Remark 1. *For all $C\in\mathbb{Q}$, we have $z_{\boldsymbol{\alpha},\boldsymbol{\beta}}(Cq)\in\mathbb{Z}[[q]]$ if and only if $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(Cz)\in\mathbb{Z}[[z]]$. In particular, $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is N-integral if and only if $z_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is N-integral.*

Moreover, if one of the above equivalent properties holds true, then we prove that

$$(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)^{-1}q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)\in\mathbb{Z}[[z]],$$

where $C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}=2C_{\langle\boldsymbol{\alpha}\rangle,\langle\boldsymbol{\beta}\rangle}$ or $C_{\langle\boldsymbol{\alpha}\rangle,\langle\boldsymbol{\beta}\rangle}$. We refer to Theorem 8 for details. Actually, we are even able to improve this integrality result by considering roots of $(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)^{-1}q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)$; see Theorem 10 and Corollary 14.

Instead of considering $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$ itself, which is not N-integral in general, we also study

$$\widetilde{q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)=z\prod_{\substack{a=1\\ \gcd(a,d_{\boldsymbol{\alpha},\boldsymbol{\beta}})=1}}^{d_{\boldsymbol{\alpha},\boldsymbol{\beta}}}z^{-1}q_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(z).$$

We prove that, if $H_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ holds true and $r=s$, then $\widetilde{q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$ is N-integral, and that

$$(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)^{-1}\widetilde{q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)\in\mathbb{Z}[[z]].$$

[1]Consider $x\in\mathbb{Q}$. There exists a unique $(a,b)\in\mathbb{Z}\times\mathbb{Z}_{\geq 1}$ such that $x=a/b$ and $\gcd(a,b)=1$. We will call b the exact denominator of x.

[2]This is equivalent to the irreducibility of $\mathcal{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ on $\mathbb{C}(z)$.

[3]Throughout this article, we say that $\boldsymbol{x}\in\mathbb{Q}^n$ is R-partitioned if, up to permutation of its coordinates, $\boldsymbol{\alpha}$ is the concatenation of tuples of the form $(b/m)_{b\in\{1,...,m\},\gcd(b,m)=1}$ for $m\in\mathbb{Z}_{\geq 1}$.

Actually, we improve this N-integrality result by considering roots of $(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)^{-1}\widetilde{q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)$: if hypothesis $H_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ holds true and $r=s$, then we exhibit some integer $\mathfrak{n}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ such that

$$\big((C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)^{-1}\widetilde{q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)\big)^{\frac{1}{\mathfrak{n}_{\boldsymbol{\alpha},\boldsymbol{\beta}}}} \in \mathbb{Z}[[z]]. \tag{1.3}$$

We refer to Theorem 12 for details.

We shall now say a few words about the proof of (1.3); this will lead us to the main technical ingredient of this paper (some generalizations of Dwork's congruences detailed in Section 2.2). The starting point is the following classical result (see [**34**, Lemma 5], [**17**, Chap. IV, Sec. 2, Lemma 3], [**30**, p. 409, Theorem]).

Proposition 2 (Dieudonné-Dwork's lemma). *Given a prime p and $f(z)\in z\mathbb{Q}[[z]]$, we have* $\exp\big(f(z)\big)\in 1+z\mathbb{Z}_p[[z]]$ *if and only if $f(z^p)-pf(z)\in pz\mathbb{Z}_p[[z]]$, where $\mathbb{Z}_p$ is the ring of p-adic integers.*

Since

$$\big((C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)^{-1}\widetilde{q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)\big)^{\frac{1}{\mathfrak{n}_{\boldsymbol{\alpha},\boldsymbol{\beta}}}} = \exp\left(\frac{1}{\mathfrak{n}_{\boldsymbol{\alpha},\boldsymbol{\beta}}}\sum_{\substack{a=1\\ \gcd(a,d_{\boldsymbol{\alpha},\boldsymbol{\beta}})=1}}^{d_{\boldsymbol{\alpha},\boldsymbol{\beta}}}\frac{G_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)}{F_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)}\right),$$

Proposition 2 ensures that the integrality property (1.3) holds true if and only if, for all primes p, we have

$$\sum_{\substack{a=1\\ \gcd(a,d_{\boldsymbol{\alpha},\boldsymbol{\beta}})=1}}^{d_{\boldsymbol{\alpha},\boldsymbol{\beta}}}\frac{G_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z^p)}{F_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z^p)} - p\sum_{\substack{a=1\\ \gcd(a,d_{\boldsymbol{\alpha},\boldsymbol{\beta}})=1}}^{d_{\boldsymbol{\alpha},\boldsymbol{\beta}}}\frac{G_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)}{F_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)} \in \mathfrak{n}_{\boldsymbol{\alpha},\boldsymbol{\beta}}pz\mathbb{Z}_p[[z]]. \tag{1.4}$$

The very basic strategy for proving such a congruence (for a fixed prime p) is to construct a permutation $a\mapsto a'$ of $\{a\in\{1,\dots,d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\} \,:\, \gcd(a,d_{\boldsymbol{\alpha},\boldsymbol{\beta}})=1\}$ such that

$$\frac{G_{\langle a'\boldsymbol{\alpha}\rangle,\langle a'\boldsymbol{\beta}\rangle}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z^p)}{F_{\langle a'\boldsymbol{\alpha}\rangle,\langle a'\boldsymbol{\beta}\rangle}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z^p)} - p\frac{G_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)}{F_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)} \tag{1.5}$$

satisfies "nice congruences". The meaning of "nice congruences" and the explicit construction of a' are too technical for this introduction and we refer to Section 2.2, and especially Theorem 6, for details. We shall just mention the fact that congruences for (1.5) were first derived by Dwork when $\boldsymbol{\alpha}\in\mathbb{Z}_p^r$, $\boldsymbol{\beta}\in(\mathbb{Z}_p^\times)^s$ and $\mathfrak{n}_{\boldsymbol{\alpha},\boldsymbol{\beta}}=1$. In this paper, it is fundamental to get rid of these hypotheses, and to study how the congruence (1.5) depends on a.

With these "nice congruences" in hands, the proof of the congruence (1.4) is (with simplifications) a consequence of the equality

$$\sum_{\substack{a=1\\ \gcd(a,d_{\boldsymbol{\alpha},\boldsymbol{\beta}})=1}}^{d_{\boldsymbol{\alpha},\boldsymbol{\beta}}} \frac{G_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(Cz^p)}{F_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(Cz^p)} - p\sum_{\substack{a=1\\ \gcd(a,d_{\boldsymbol{\alpha},\boldsymbol{\beta}})=1}}^{d_{\boldsymbol{\alpha},\boldsymbol{\beta}}} \frac{G_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(Cz)}{F_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(Cz)}$$
$$= \sum_{\substack{a=1\\ \gcd(a,d_{\boldsymbol{\alpha},\boldsymbol{\beta}})=1}}^{d_{\boldsymbol{\alpha},\boldsymbol{\beta}}} \left(\frac{G_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(Cz^p)}{F_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(Cz^p)} - p\frac{G_{\langle a'\boldsymbol{\alpha}\rangle,\langle a'\boldsymbol{\beta}\rangle}(Cz)}{F_{\langle a'\boldsymbol{\alpha}\rangle,\langle a'\boldsymbol{\beta}\rangle}(Cz)}\right). \quad (1.6)$$

It is now clear that the study of congruences for (1.5) is central in this paper: they are the main ingredient of the proofs of our N-integrality results for $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$ and $\widetilde{q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$.

2. Statements of the main results.

In this section, we consider tuples $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_r)$ and $\boldsymbol{\beta}=(\beta_1,\dots,\beta_s)$ of parameters in $\mathbb{Q}\setminus\mathbb{Z}_{\leq 0}$. We write $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ for the least common multiple of the exact denominators of the elements of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

2.1. N-integrality of $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. We first state a criterion for $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ to be N-integral, which is due to Chritsol. We will use the following notations:

- for all $x\in\mathbb{Q}$, we write $\langle x\rangle$ for the unique element in $(0,1]$ such that $x-\langle x\rangle\in\mathbb{Z}$. In other words, we have $\langle x\rangle = 1-\{1-x\} = x+\lfloor 1-x\rfloor$, where $\{\cdot\}$ is the fractional part function and where $\lfloor\cdot\rfloor$ is the floor function;
- we write $\preceq$ for the total order on $\mathbb{R}$ defined by
$$x\preceq y \Longleftrightarrow \Big(\langle x\rangle<\langle y\rangle \quad\text{or}\quad \big(\langle x\rangle=\langle y\rangle \quad\text{and}\quad x\geq y\big)\Big);$$
- for all $a\in\{1,\dots,d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ coprime to $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and all $x\in\mathbb{R}$, we set
$$\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a,x) := \#\{1\leq i\leq r\,:\,a\alpha_i\preceq x\} - \#\{1\leq j\leq s\,:\,a\beta_j\preceq x\}.$$

Theorem 3 (Christol, [7]). *The following assertions are equivalent:*

(i) *$F_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is N-integral.*
(ii) *For all $a\in\{1,\dots,d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ coprime to $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and all $x\in\mathbb{R}$, we have $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a,x)\geq 0$.*

If $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is N-integral, then we denote by $C_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ the minimal constant in $\mathbb{Q}^+\setminus\{0\}$ such that

$$F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)\in\mathbb{Z}[[z]].$$

(Actually, it is easily seen that the set of all $C\in\mathbb{Q}$ satisfying $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(Cz)\in\mathbb{Z}[[z]]$ is equal to $C_{\boldsymbol{\alpha},\boldsymbol{\beta}}\mathbb{Z}$.) Our first result, Theorem 4 below, gives some arithmetical properties of $C_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and even a formula for $C_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ when $r=s$, $\boldsymbol{\alpha}\in(0,1]^r$ and $\boldsymbol{\beta}\in(0,1]^s$. We will use the following notations:

- for all primes p, we define
$$\lambda_p = \lambda_p(\boldsymbol{\alpha}, \boldsymbol{\beta}) := \#\{1 \leq i \leq r \,:\, \alpha_i \in \mathbb{Z}_p\} - \#\{1 \leq j \leq s \,:\, \beta_j \in \mathbb{Z}_p\},$$
where $\mathbb{Z}_p$ is the ring of p-adic integers;
- we write $\mathcal{P}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ for the set of all primes p such that p divides $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ or $p \leq r - s + 1$;
- for all $a \in \mathbb{Q} \setminus \{0\}$, we write $d(a)$ for the exact denominator of a.

THEOREM 4. *Assume that $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is N-integral. Then, there exists $C \in \mathbb{N} \setminus \{0\}$ such that*
$$C_{\boldsymbol{\alpha},\boldsymbol{\beta}} = C \frac{\prod_{i=1}^{r} d(\alpha_i)}{\prod_{j=1}^{s} d(\beta_j)} \prod_{p \in \mathcal{P}_{\boldsymbol{\alpha},\boldsymbol{\beta}}} p^{-\left\lfloor \frac{\lambda_p}{p-1} \right\rfloor}. \tag{2.1}$$
Furthermore, if $r = s$, $\boldsymbol{\alpha} \in (0,1]^r$ and $\boldsymbol{\beta} \in (0,1]^s$, then we have $C = 1$.

2.2. Generalizations of Dwork's congruence. As explained at the end of the introduction, Theorem 6 below is the cornerstone of this paper, on which the proofs of the N-integrality results stated in Sections 2.3 and 2.4 below rely. The reader interested in our N-integrality statements for canonical coordinates, but not in the proofs, can skip this section. We will use the following notations:

- For all primes p and all positive integers n, we write $\mathfrak{A}_{p,n}$, respectively $\mathfrak{A}_{p,n}^*$, for the $\mathbb{Z}_p$-algebra of the functions $f : (\mathbb{Z}_p^\times)^n \to \mathbb{Z}_p$ such that, for all positive integers m, all $\mathbf{x} \in (\mathbb{Z}_p^\times)^n$ and all $\mathbf{a} \in \mathbb{Z}_p^n$, we have
$$f(\mathbf{x} + \mathbf{a}p^m) \equiv f(\mathbf{x}) \mod p^m\mathbb{Z}_p,$$
respectively
$$f(\mathbf{x} + \mathbf{a}p^m) \equiv f(\mathbf{x}) \mod p^{m-1}\mathbb{Z}_p.$$
- If D is a positive integer coprime to p, then, for all $\nu \in \mathbb{N}$, and all $b \in \{1, \dots, D\}$ coprime to D, we write $\Omega_b(p^\nu, D)$ for the set of all $t \in \{1, \dots, p^\nu D\}$ coprime to $p^\nu D$ satisfying $t \equiv b \mod D$.
- We write $\mathcal{A}_b(p^\nu, D)$, respectively $\mathcal{A}_b(p^\nu, D)^*$, for the $\mathbb{Z}_p$-algebra of the functions $f : \Omega_b(p^\nu, D) \to \mathbb{Z}_p$ such that, for all positive integers m and all $t_1, t_2 \in \Omega_b(p^\nu, D)$, we have
$$t_1 \equiv t_2 \mod p^m \Rightarrow f(t_1) \equiv f(t_2) \mod p^m\mathbb{Z}_p,$$
respectively
$$t_1 \equiv t_2 \mod p^m \Rightarrow f(t_1) \equiv f(t_2) \mod p^{m-1}\mathbb{Z}_p.$$
- For all $t \in \Omega_b(p^\nu, D)$ and all $r \in \mathbb{N}$, we write $t^{(r)}$ for the unique element of $\{1, \dots, p^\nu D\}$ satisfying
$$t^{(r)} \equiv t \mod p^\nu \text{ and } p^r t^{(r)} \equiv t \mod D.$$
- If $\boldsymbol{\beta} \notin \mathbb{Z}^s$, then we write $\mathfrak{m}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ for the number of elements of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ with exact denominator divisible by 4.
- We write $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}^*$ for the integer obtained by dividing $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ by the product of its prime divisors.

- We set $C'_{\boldsymbol{\alpha},\boldsymbol{\beta}} = 2C_{\langle\boldsymbol{\alpha}\rangle,\langle\boldsymbol{\beta}\rangle}$ and $d'_{\boldsymbol{\alpha},\boldsymbol{\beta}} = 2d^*_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ if $\boldsymbol{\beta} \notin \mathbb{Z}^s$ and if $\mathfrak{m}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is odd, and we set $C'_{\boldsymbol{\alpha},\boldsymbol{\beta}} = C_{\langle\boldsymbol{\alpha}\rangle,\langle\boldsymbol{\beta}\rangle}$ and $d'_{\boldsymbol{\alpha},\boldsymbol{\beta}} = d^*_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ otherwise.
- Throughout this article, when $\boldsymbol{x} = (x_1, \dots, x_m)$ and f is a map defined on $\{x_1, \dots, x_m\}$, we write $f(\boldsymbol{x})$ for $\big(f(x_1), \dots, f(x_m)\big)$. For instance, $\langle\boldsymbol{\alpha}\rangle = (\langle\alpha_1\rangle, ..., \langle\alpha_r\rangle)$.

According to Theorem 3, the N-integrality of $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ depends on the graphs of Christol's functions $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a, \cdot)$. The N-integrality of $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ also strongly depends on them. The following definition involving Christol's functions will play a central role.

Definition 5 (Hypothesis $H_{\boldsymbol{\alpha},\boldsymbol{\beta}}$). *Let $\min_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a)$ denote the smallest element in the ordered set $\big(\{a\alpha_1, \dots, a\alpha_r, a\beta_1, \dots, a\beta_s\}, \preceq\big)$. We denote by $H_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ the following assertion:*

$H_{\boldsymbol{\alpha},\boldsymbol{\beta}}$: "For all $a \in \{1, \dots, d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ coprime to $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and all $x \in \mathbb{R}$ satisfying $\min_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a) \preceq x \prec a$, we have $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a, x) \geq 1$."

We are now in a position to state our generalization of Dwork's congruences.

Theorem 6. *Assume that $r = s$, that $\langle\boldsymbol{\alpha}\rangle$ and $\langle\boldsymbol{\beta}\rangle$ are disjoint and that $H_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ holds. Let p be a fixed prime and write $d_{\boldsymbol{\alpha},\boldsymbol{\beta}} = p^\nu D$ with $\nu, D \in \mathbb{N}$ and D coprime to p. Let $b \in \{1, \dots, D\}$ be coprime to D. Then, there exists a sequence $(R_{k,b})_{k\geq 0}$ of elements in $\mathcal{A}_b(p^\nu, D)^*$ such that, for all $t \in \Omega_b(p^\nu, D)$, we have*

$$\frac{G_{\langle t^{(1)}\boldsymbol{\alpha}\rangle,\langle t^{(1)}\boldsymbol{\beta}\rangle}}{F_{\langle t^{(1)}\boldsymbol{\alpha}\rangle,\langle t^{(1)}\boldsymbol{\beta}\rangle}}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}} z^p) - p\frac{G_{\langle t\boldsymbol{\alpha}\rangle,\langle t\boldsymbol{\beta}\rangle}}{F_{\langle t\boldsymbol{\alpha}\rangle,\langle t\boldsymbol{\beta}\rangle}}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}} z) = p\sum_{k=0}^{\infty} R_{k,b}(t) z^k.$$

Furthermore, if p is a prime divisor of $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, then, for all $k \in \mathbb{N}$,

- *if $\boldsymbol{\beta} \in \mathbb{Z}^r$, then we have $R_{k,b} \in p^{-1-\lfloor\lambda_p/(p-1)\rfloor}\mathcal{A}_b(p^\nu, D)$;*
- *if $\boldsymbol{\beta} \notin \mathbb{Z}^r$ and $p - 1 \nmid \lambda_p$, then we have $R_{k,b} \in \mathcal{A}_b(p^\nu, D)$;*
- *if $\boldsymbol{\beta} \notin \mathbb{Z}^r$, $\mathfrak{m}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is odd and $p = 2$, then we have $R_{k,b} \in \mathcal{A}_b(p^\nu, D)$.*

Remark 7. *Let us make some remarks on the previous result.*

- *The tuples $\langle\boldsymbol{\alpha}\rangle$ and $\langle\boldsymbol{\beta}\rangle$ are disjoint if, and only if, for all $(i, j) \in \{1, \dots, r\} \times \{1, \dots, s\}$, we have $\alpha_i - \beta_j \notin \mathbb{Z}$.*
- *If the hypotheses of Theorem 6 are satisfied, then $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is N-integral (direct consequence of Theorem 3). Indeed, if $x \prec \min_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a)$ then, by definition, we have $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a, x) = 0$. Furthermore, if $\alpha \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$, then $a\alpha \preceq a$. Hence, for all $x \in \mathbb{R}$ satisfying $a \prec x$, we have $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a, x) = r - s = 0$.*
- *Assume that $\boldsymbol{\beta} \in \mathbb{Z}^r$, and that p is a prime divisor of $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. Then, we have $\lambda_p \leq -1$ and $-1 - \lfloor\lambda_p/(p-1)\rfloor \geq 0$ so that $p^{-1-\lfloor\lambda_p/(p-1)\rfloor}\mathcal{A}_b(p^\nu, D) \subset \mathcal{A}_b(p^\nu, D) \subset \mathcal{A}_b(p^\nu, D)^*$.*

2.3. N-Integrality of $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. Our first main result concerning the N-integrality of $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$ can be stated as follows (the constant $C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ involved below was defined in Section 2.2).

Theorem 8. *Assume that $\langle\boldsymbol{\alpha}\rangle$ and $\langle\boldsymbol{\beta}\rangle$ are disjoint and that $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is N-integral. Then, the following assertions are equivalent:*

(*i*) $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$ *is* N*-integral;*
(*ii*) $(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)^{-1}q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z) \in \mathbb{Z}[[z]]$*;*
(*iii*) *assertion* $H_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ *holds, we have* $r = s$ *and, for all* $a \in \{1, \dots, d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ *coprime to* $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$*, we have* $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z) = q_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(z)$*.*

Moreover, if one of the above equivalent properties holds, then we have either $\boldsymbol{\alpha} = (1/2)$ *and* $\boldsymbol{\beta} = (1)$*, or* $s \geq 2$ *and there are at least two* 1*'s in* $\langle\boldsymbol{\beta}\rangle$*.*

Once we know that $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$ is N-integral, it is natural to ask for the signs of its Taylor coefficients.

THEOREM 9. *Under the assumptions of Theorem 8, if* $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$ *is* N*-integral, then all the Taylor coefficients at* $z = 0$ *of* $(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)^{-1}q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)$ *are positive integers.*

The following result improves the implication $(i) \Rightarrow (ii)$ of Theorem 8 when $\boldsymbol{\beta} \in \mathbb{Z}^s$.

THEOREM 10. *Assume that* $\langle\boldsymbol{\alpha}\rangle$ *and* $\langle\boldsymbol{\beta}\rangle$ *are disjoint and that* $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ *and* $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ *are* N*-integral. Assume moreover that* $\boldsymbol{\beta}$ *is a tuple of positive integers. Then, we have*

$$\big((C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)^{-1}q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)\big)^{\frac{1}{\mathfrak{n}'_{\boldsymbol{\alpha},\boldsymbol{\beta}}}} \in \mathbb{Z}[[z]],$$

where

$$\mathfrak{n}'_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \prod_{p \mid d_{\boldsymbol{\alpha},\boldsymbol{\beta}}} p^{-1-\lfloor\frac{\lambda_p}{p-1}\rfloor}.$$

REMARK 11. *Let us note the following facts.*

- *Assume that* $\boldsymbol{\beta} \in \mathbb{Z}^r$*, and that* p *is a prime divisor of* $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$*. Then, we have* $\lambda_p \leq -1$ *and* $-1 - \lfloor\lambda_p/(p-1)\rfloor \geq 0$*. It follows that* $\mathfrak{n}'_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ *is a nonnegative integer.*
- *According to* [**14**, Lemma 5]*, if* $f(z) \in \mathbb{Z}[[z]]$ *and if* V *is the greatest positive integer satisfying* $f(z)^{1/V} \in \mathbb{Z}[[z]]$*, then the positive integers* U *satisfying* $f(z)^{1/U} \in \mathbb{Z}[[z]]$ *are exactely the positive divisors of* V*. Furthermore, by* [**24**, Introduction]*, for all positive integers* v *and all* $C \in \mathbb{Q}$*, we have* $\big((Cq)^{-1}z_{\boldsymbol{\alpha},\boldsymbol{\beta}}(Cq)\big)^{1/v} \in \mathbb{Z}[[q]]$ *if and only if* $\big((Cz)^{-1}q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(Cz)\big)^{1/v} \in \mathbb{Z}[[z]]$*. So, what precedes can be rephrased in terms of* N*-integrality properties of mirror maps.*

2.4. N**-Integrality of** $\widetilde{q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$**.** Instead of considering $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$, which is not N-integral in general, we now focus our attention on

$$\widetilde{q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z) = z \prod_{\substack{a=1\\ \gcd(a,d_{\boldsymbol{\alpha},\boldsymbol{\beta}})=1}}^{d_{\boldsymbol{\alpha},\boldsymbol{\beta}}} z^{-1}q_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(z).$$

Note that

$$\widetilde{q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z) = z\exp\big(S_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)\big) \text{ with } S_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z) := \sum_{\substack{a=1\\ \gcd(a,d_{\boldsymbol{\alpha},\boldsymbol{\beta}})=1}}^{d_{\boldsymbol{\alpha},\boldsymbol{\beta}}} \frac{G_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(z)}{F_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(z)}.$$

THEOREM 12. *Assume that $r = s$, that $\langle \boldsymbol{\alpha} \rangle$ and $\langle \boldsymbol{\beta} \rangle$ are disjoint and that $H_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ holds. Then, $\widetilde{q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$ is N-integral and we have*

$$\big((C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)^{-1}\widetilde{q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)\big)^{\frac{1}{\mathfrak{n}_{\boldsymbol{\alpha},\boldsymbol{\beta}}}} \in \mathbb{Z}[[z]], \tag{2.2}$$

where $\mathfrak{n}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is the integer defined by

$$\mathfrak{n}_{\boldsymbol{\alpha},\boldsymbol{\beta}} := d_{\boldsymbol{\alpha},\boldsymbol{\beta}} \prod_{p \mid d_{\boldsymbol{\alpha},\boldsymbol{\beta}}} p^{-2-\left\lfloor \frac{\lambda_p}{p-1} \right\rfloor} \text{ if } \boldsymbol{\beta} \in \mathbb{Z}^s, \quad \text{and} \quad \mathfrak{n}_{\boldsymbol{\alpha},\boldsymbol{\beta}} := d'_{\boldsymbol{\alpha},\boldsymbol{\beta}} \prod_{\substack{p \mid d'_{\boldsymbol{\alpha},\boldsymbol{\beta}} \\ p-1 \mid \lambda_p}} p^{-1} \text{ otherwise.}$$

REMARK 13. *It is tempting to try to improve (2.2) by replacing $\mathfrak{n}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ by $\varphi(d_{\boldsymbol{\alpha},\boldsymbol{\beta}})$, which is the number of terms in the product defining $\widetilde{q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. But this is not possible in general. Indeed, a counterexample is given by $\boldsymbol{\alpha} = (1/7, 1/4, 3/7, 6/7)$ and $\boldsymbol{\beta} = (1, 1, 1, 1)$, where we have $d_{\boldsymbol{\alpha},\boldsymbol{\beta}} = 28$, $C'_{\boldsymbol{\alpha},\boldsymbol{\beta}} = C_{\boldsymbol{\alpha},\boldsymbol{\beta}} = 2^3 7^2$, $\varphi(28) = 12$, $\mathfrak{n}_{\boldsymbol{\alpha},\boldsymbol{\beta}} = 2$,*

$$\big((C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)^{-1}\widetilde{q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)\big)^{\frac{1}{\varphi(d_{\boldsymbol{\alpha},\boldsymbol{\beta}})}} \in 1 + 4802z + \frac{81541341}{2}z^2 + \frac{1328534273395}{3}z^3 + z^4\mathbb{Q}[[z]].$$

This example also shows that one cannot replace $\mathfrak{n}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ by $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, since

$$\big((C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)^{-1}\widetilde{q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)\big)^{\frac{1}{d_{\boldsymbol{\alpha},\boldsymbol{\beta}}}} \in 1 + 2058z + \frac{29299137}{2}z^2 + z^3\mathbb{Q}[[z]].$$

As a consequence of Theorem 12, we obtain the following result.

COROLLARY 14. *Assume that $\langle \boldsymbol{\alpha} \rangle$ and $\langle \boldsymbol{\beta} \rangle$ are disjoint and that $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ are N-integral. Then, we have*

$$\big((C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)^{-1}q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)\big)^{\varphi(d_{\boldsymbol{\alpha},\boldsymbol{\beta}})/\mathfrak{n}_{\boldsymbol{\alpha},\boldsymbol{\beta}}} \in \mathbb{Z}[[z]], \tag{2.3}$$

where φ denotes Euler's totient function.

REMARK 15. *If $\boldsymbol{\beta} \in \mathbb{Z}^r$, then Theorem 10 is stronger than Corollary 14 because $\mathfrak{n}_{\boldsymbol{\alpha},\boldsymbol{\beta}}/\mathfrak{n}'_{\boldsymbol{\alpha},\boldsymbol{\beta}} = d^*_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ divides $\varphi(d_{\boldsymbol{\alpha},\boldsymbol{\beta}})$.*

3. Structure of the paper

In Section 4, we make comments on our main results (those stated in Section 2) and we compare these results with previous ones on the N-integrality of mirror maps associated with generalized hypergeometric functions. Then, we formulate some open questions and we give a corrected version of a lemma of Lang on Mojita's p-adic Gamma function ([4]).

Section 5 contains a detailed study of the p-adic valuation of the Pochhammer symbols. In particular, we define and study step functions $\Delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ associated with tuples $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ which play a central role in the rest of the paper.

[4]Indeed, while working on this article, we found an error in a lemma in Lang's book [**23**, Lemma 1.1, Section 1, Chapter 14] about the arithmetic properties of Mojita's p-adic Gamma function. This lemma has been used in several articles on the integrality of the Taylor coefficients of mirror maps including papers of the authors. Even if we do not use this lemma in this article, we give in Section 4.4 a corrected version and we explain why the initial error does not change the validity of our previous results.

Section 6 is devoted to the proof of Theorem 4.

Section 7 is devoted to the statement and the proof of Theorem 30, which is a vast generalization of Dwork's theorem [**12**, Theorem 1.1] on formal congruences. We also compare Theorem 30 with previous generalizations of Dwork's formal congruences.

Section 8 is devoted to the proof of Theorem 6. This proof relies on Theorem 30 and constitute (by far) the longest and the most technical part of this article.

Sections 9, 10, 11, 12 and 13 are dedicated to the proofs of Theorem 9, Theorem 12, Theorem 8, Theorem 10 and Corollary 14 respectively. These proofs rely on Theorem 6, via Dieudonné-Dwork's lemma.

We warmly thank the referee for his very careful reading of the paper and for this comments that helped to improve the presentation.

4. Comments on the main results, comparison with previous results and open questions

This section contains a detailed study of certain consequences of our main results (stated in Section 2). We also compare our theorems with previous results on the N-integrality of generalized hypergeometric series and of their associated mirror maps. This section also contains some results that we use throughout this article.

4.1. Comments on Theorem 3, Theorem 4 and on the hypothesis $H_{\boldsymbol{\alpha},\boldsymbol{\beta}}$.

4.1.1. *An example of application of Theorem 4.* We illustrate Theorem 3 and Theorem 4 with an example. Let $\boldsymbol{\alpha} := (1/6, 1/2, 2/3)$ and $\boldsymbol{\beta} := (1/3, 1, 1)$ so that we have $d_{\boldsymbol{\alpha},\boldsymbol{\beta}} = 6$. According to Theorem 3, $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is N-integral if and only if, for all $a \in \{1, 5\}$ and all $x \in \mathbb{R}$, we have $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a, x) \geq 0$.

We have $1/6 \prec 1/3 \prec 1/2 \prec 2/3 \prec 1$ thus, for all $x \in \mathbb{R}$, we get $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(1, x) \geq 0$. Furthermore, we have $1/3 + 3 = 10/3 \prec 5/2 \prec 5/3 \prec 5/6 \prec 5$ and thus, for all $x \in \mathbb{R}$, we get $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(5, x) \geq 0$. This shows that $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is N-integral.

Moreover, we have $r = s$, all elements of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ lie in $(0, 1]$, $\lambda_2(\boldsymbol{\alpha},\boldsymbol{\beta}) = 1 - 3 = -2$ and $\lambda_3(\boldsymbol{\alpha},\boldsymbol{\beta}) = 1 - 2 = -1$ thus, according to Theorem 4, we get

$$C_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \frac{6 \cdot 2 \cdot 3}{3} 2^{-\lfloor -2 \rfloor} 3^{-\lfloor -1/2 \rfloor} = 2^4 3^2.$$

4.1.2. *N-integrality of $F_{\langle\boldsymbol{\alpha}\rangle,\langle\boldsymbol{\beta}\rangle}$.* We show that if $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is N-integral then $F_{\langle\boldsymbol{\alpha}\rangle,\langle\boldsymbol{\beta}\rangle}$ is also N-integral. The converse is false in general, a counterexample being given by $\boldsymbol{\alpha} = (1/2, 1/2)$ and $\boldsymbol{\beta} = (3/2, 1)$ since we have $3/2 \prec 1/2 \prec 1$ and $\langle\boldsymbol{\alpha}\rangle = (1/2, 1/2)$, $\langle\boldsymbol{\beta}\rangle = (1/2, 1)$. But, if $\langle\boldsymbol{\alpha}\rangle$ and $\langle\boldsymbol{\beta}\rangle$ are disjoint, then, for all $a \in \{1, \dots, d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ coprime to $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, $\langle a\boldsymbol{\alpha}\rangle$ and $\langle a\boldsymbol{\beta}\rangle$ are disjoint. Hence, applying Theorem 3, we obtain that ($F_{\langle\boldsymbol{\alpha}\rangle,\langle\boldsymbol{\beta}\rangle}$ is N-integral)$\Rightarrow$($F_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is N-integral). More precisely, we shall prove the following proposition that we use several times in this article.

In Proposition 16 and throughout this article, if f is a function defined on $\mathcal{D} \subset \mathbb{R}$ and $x \in \mathcal{D}$, then we adopt the notations

$$f(x+) := \lim_{\substack{y \to x \\ y \in \mathcal{D}, y > x}} f(y) \quad \text{and} \quad f(x-) := \lim_{\substack{y \to x \\ y \in \mathcal{D}, y < x}} f(y).$$

Proposition 16. *Let* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$ *be tuples of parameters in* $\mathbb{Q}\setminus\mathbb{Z}_{\leq 0}$ *and* $a\in\{1,\dots,d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ *coprime to* $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$*. Then we have* $d_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}=d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$*. Let* $c\in\{1,\dots,d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ *coprime to* $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ *and* $x\in\mathbb{R}$ *be fixed and let* $b\in\{1,\dots,d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ *be such that* $b\equiv ca\mod d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$*. Then we have*

$$\xi_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(c,x)=\begin{cases}\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(b,\langle x\rangle-) \text{ if } x>c;\\ r-s \text{ if } x\leq c \text{ and } x\in\mathbb{Z};\\ \xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(b,\langle x\rangle-) \text{ or } \xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(b,\langle x\rangle+) \text{ otherwise,}\end{cases}.$$

where r*, respectively* s*, is the number of elements of* $\boldsymbol{\alpha}$*, respectively of* $\boldsymbol{\beta}$*.*

Remark 17. *For all* $a\in\{1,\dots,d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ *coprime to* $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$*,* $r-s$ *is the limit of* $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a,n)$ *when* $n\in\mathbb{Z}$ *tends to* $-\infty$*.*

Proof. For all elements α and β of $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$, we have $\big\langle c\langle a\alpha\rangle\big\rangle=\langle ca\alpha\rangle=\langle b\alpha\rangle$ and $\langle b\alpha\rangle=\langle b\beta\rangle$ if and only if $\langle\alpha\rangle=\langle\beta\rangle$. If $\langle b\alpha\rangle=\langle x\rangle$, then we have $c\langle a\alpha\rangle\preceq x\Leftrightarrow c\langle a\alpha\rangle\geq x$. It follows that if $x>c$, then we have

$$\begin{aligned}\xi_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(c,x)&=\#\big\{1\leq i\leq r\,:\,\langle b\alpha_i\rangle<\langle x\rangle\big\}-\#\big\{1\leq j\leq s\,:\,\langle b\beta_j\rangle<\langle x\rangle\big\}\\&=\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(b,\langle x\rangle-).\end{aligned}$$

If $x\in\mathbb{Z}$ and $x\leq c$, then we have $\langle x\rangle=1$ and $\xi_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(c,x)=r-s$. Now we assume that $x\leq c$ and $x\notin\mathbb{Z}$. If α and β are elements of $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ satisfying $\langle x\rangle=\langle b\alpha\rangle=\langle b\beta\rangle$, then $\langle\alpha\rangle=\langle\beta\rangle$ so $\langle a\alpha\rangle=\langle a\beta\rangle$ and we obtain that $c\langle a\alpha\rangle\preceq x\Leftrightarrow c\langle a\beta\rangle\preceq x$. Thus we have

$$\begin{aligned}\xi_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(c,x)&=\begin{cases}\#\big\{1\leq i\leq r\,:\,\langle b\alpha_i\rangle<\langle x\rangle\big\}-\#\big\{1\leq j\leq s\,:\,\langle b\beta_j\rangle<\langle x\rangle\big\}\\ \text{or}\\ \#\big\{1\leq i\leq r\,:\,\langle b\alpha_i\rangle\leq\langle x\rangle\big\}-\#\big\{1\leq j\leq s\,:\,\langle b\beta_j\rangle\leq\langle x\rangle\big\}\end{cases}\\&=\begin{cases}\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(b,\langle x\rangle-)\\ \text{or}\\ \xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(b,\langle x\rangle+)\end{cases}\end{aligned}$$

because $\langle x\rangle<1$. □

By Proposition 16 with $a=1$ together with Theorem 3, we obtain that, if $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is N-integral, then $F_{\langle\boldsymbol{\alpha}\rangle,\langle\boldsymbol{\beta}\rangle}$ is also N-integral. Similarly, if $H_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ holds then $H_{\langle\boldsymbol{\alpha}\rangle,\langle\boldsymbol{\beta}\rangle}$ also holds. More precisely, we have the following result, used several times in the proofs of our N-integrality results.

Lemma 18. *Let* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$ *be two disjoint tuples of parameters in* $\mathbb{Q}\setminus\mathbb{Z}_{\leq 0}$ *with the same number of elements and such that* $H_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ *holds. Then, for all* $a\in\{1,\dots,d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ *coprime to* $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$*, Assertion* $H_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}$ *holds.*

Proof. Let $c\in\{1,\dots,d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ be coprime to $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and $x\in\mathbb{R}$ be such that $\min_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(c)\preceq x\prec c$. We shall prove that $\xi_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(c,x)\geq 1$ by applying Proposition 16.

Let $b\in\{1,\dots,d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ be such that $b\equiv ac\mod d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. First, note that there exists an element α of $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ such that $c\langle a\alpha\rangle\preceq x$, that is $\langle x\rangle>\langle b\alpha\rangle$ or $\big(\langle x\rangle=\langle b\alpha\rangle$ and $c\langle a\alpha\rangle\geq x\big)$. We distinguish three cases.

• If $x > c$ then we have $\langle x\rangle > \langle b\alpha\rangle$ and $\xi_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(c,x) = \xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(b,\langle x\rangle -)$. Thus there exists $y\in\mathbb{R}$, $\min_{\boldsymbol{\alpha},\boldsymbol{\beta}}(b) \preceq y \prec b$ such that $\xi_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(c,x) = \xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(b,y) \geq 1$.

• If $x \leq c$ and $x \notin \mathbb{Z}$, then we have $\langle x\rangle < 1$ and $\xi_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(c,x) = \xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(b,\langle x\rangle -)$ or $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(b,\langle x\rangle +)$. Since $\langle x\rangle \geq \langle b\alpha\rangle$, there exists $y\in\mathbb{R}$, $\min_{\boldsymbol{\alpha},\boldsymbol{\beta}}(b) \preceq y \prec b$ such that $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(b,\langle x\rangle +) = \xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(b,y) \geq 1$. Furthermore, if $\langle x\rangle > \langle b\alpha\rangle$ then we have $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(b,\langle x\rangle -) \geq 1$ as in the case $x > c$. Now we assume that, for all elements β of $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$, we have $\langle x\rangle \leq \langle b\beta\rangle$. Hence we have $\langle x\rangle = \langle b\alpha\rangle$ and, as explained in the proof of Proposition 16, we have

$$\begin{aligned}\xi_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(c,x) &= \#\big\{1\leq i\leq r\,:\,\langle b\alpha_i\rangle \leq \langle x\rangle\big\} - \#\big\{1\leq j\leq s\,:\,\langle b\beta_j\rangle \leq \langle x\rangle\big\}\\ &= \xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(b,\langle x\rangle +) \geq 1.\end{aligned}$$

• It remains to consider the case $x \leq c$ and $x \in \mathbb{Z}$. But in this case we do not have $x \prec c$ thus $H_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}$ is proved. □

4.1.3. *Numerators of the elements of* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$. Let $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_r)$ and $\boldsymbol{\beta} = (\beta_1,\dots,\beta_r)$ be tuples of parameters in $\mathbb{Q}\setminus\mathbb{Z}_{\leq 0}$. Then, Theorem 4 gives a necessary condition on the numerators of elements of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ for $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ to be N-integral. Indeed, let us assume that $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is N-integral. Then, according to Section 4.1.2, $F_{\langle\boldsymbol{\alpha}\rangle,\langle\boldsymbol{\beta}\rangle}$ is also N-integral. We write n_i, respectively n'_j, for the exact numerator of $\langle\alpha_i\rangle$, respectively of $\langle\beta_j\rangle$. Then, by Theorem 4, the first-order Taylor coefficient at the origin of $F_{\langle\boldsymbol{\alpha}\rangle,\langle\boldsymbol{\beta}\rangle}\big(C_{\langle\boldsymbol{\alpha}\rangle,\langle\boldsymbol{\beta}\rangle}z\big)$ is ([5])

$$\frac{\prod_{i=1}^r n_i}{\prod_{j=1}^r n'_j}\prod_{p\mid d_{\boldsymbol{\alpha},\boldsymbol{\beta}}} p^{-\left\lfloor\frac{\lambda_p(\boldsymbol{\alpha},\boldsymbol{\beta})}{p-1}\right\rfloor}\in\mathbb{Z},$$

so that, for all primes p, we have

$$v_p\left(\frac{\prod_{i=1}^r n_i}{\prod_{j=1}^r n'_j}\right) \geq \left\lfloor\frac{\lambda_p(\boldsymbol{\alpha},\boldsymbol{\beta})}{p-1}\right\rfloor.$$

For instance, the last inequality is not satisfied with $p=2$, $\boldsymbol{\alpha} = (1/5,1/3,3/5)$ and $\boldsymbol{\beta} = (1/2,1,1)$, or with $p=3$, $\boldsymbol{\alpha} = (1/7,2/7,4/7,5/7)$ and $\boldsymbol{\beta} = (3/4,1,1,1)$. Thus in both cases the associated generalized hypergeometric series $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is not N-integral.

4.1.4. *The Eisenstein constant of algebraic generalized hypergeometric series.* Let $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_r)$ and $\boldsymbol{\beta} = (\beta_1,\dots,\beta_r)$ be tuples of parameters in $\mathbb{Q}\setminus\mathbb{Z}_{\leq 0}$. If $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$ is algebraic over $\mathbb{Q}(z)$ then $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is N-integral (Eisenstein's theorem) and one can apply Theorem 4 to get arithmetical properties of the Eisenstein constant of $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. For the sake of completeness, let us remind the reader of a result of Beukers and Heckman [**3**, Theorem 1.5] proved in [**4**] on algebraic hypergeometric functions:

> "Assume that $\beta_r = 1$ and that $\mathcal{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is irreducible. Then the set of solutions of the hypergeometric equation associated with $\mathcal{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ consists of algebraic functions (over $\mathbb{C}(z)$) if and only if the sets $\{a\alpha_i\,:\,1\leq 1\leq r\}$ and

[5]Note that, for all primes p, we have $\lambda_p(\boldsymbol{\alpha},\boldsymbol{\beta}) = \lambda_p(\langle\boldsymbol{\alpha}\rangle,\langle\boldsymbol{\beta}\rangle)$.

$\{a\beta_i : 1 \leq i \leq r\}$ interlace modulo 1 for every integer a with $1 \leq a \leq d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and $\gcd(a, d_{\boldsymbol{\alpha},\boldsymbol{\beta}}) = 1$."

The sets $\{\alpha_i : 1 \leq i \leq r\}$ and $\{\beta_i : 1 \leq i \leq r\}$ interlace modulo 1 if the points of the sets $\{e^{2\pi i\alpha_j} : 1 \leq j \leq r\}$ and $\{e^{2\pi i\beta_j} : 1 \leq j \leq r\}$ occur alternatively when running along the unit circle.

The Beukers-Heckman criterion can be reformulated in terms of Christol's functions as follows.

> "Assume that $\beta_r = 1$ and that $\mathcal{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is irreducible. Then the solution set of the hypergeometric equation associated with $\mathcal{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ consists of algebraic functions (over $\mathbb{C}(z)$) if and only if, for every integer a with $1 \leq a \leq d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and $\gcd(a, d_{\boldsymbol{\alpha},\boldsymbol{\beta}}) = 1$, we have $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a, \mathbb{R}) = \{0, 1\}$."

4.2. Comparison with previous results.

4.2.1. *Theorem 6 and previous results.* The first result on p-adic integrality of $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is due to Dwork [**12**, Theorem 4.1]. This result enables us to prove that, for particular tuples $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, we have $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z) \in \mathbb{Z}_p[[z]]$ for almost all primes p. It follows without much trouble that $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is N-integral. Thus we know that there exists $C \in \mathbb{N}$, $C \geq 1$, such that $(Cz)^{-1}q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(Cz) \in \mathbb{Z}[[z]]$ but the only information on C given by Dwork's result is that we can choose C with prime divisors in an explicit finite set associated with $(\boldsymbol{\alpha}, \boldsymbol{\beta})$. Hence, improvements of Dwork's method consist in finding explicit formulas for C and we discuss such previous improvements in the next section. But Theorem 6 is more general and, in order to compare this theorem with Dwork's result [**12**, Theorem 4.1], we introduce some notations that we use throughout this article. Until the end of this section, we restrict ourself to the case where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ have the same numbers of elements.

- For all primes p and all p-adic integers α in $\mathbb{Q}$, we write $\mathfrak{D}_p(\alpha)$ for the unique p-adic integer in $\mathbb{Q}$ satisfying $p\mathfrak{D}_p(\alpha) - \alpha \in \{0, \dots, p-1\}$. The operator $\alpha \mapsto \mathfrak{D}_p(\alpha)$ has been used by Dwork in [**12**] and denoted by $\alpha \mapsto \alpha'$ ([6]).
- For all primes p, all $x \in \mathbb{Q} \cap \mathbb{Z}_p$ and all $a \in [0, p)$ we define
$$\rho_p(a, x) := \begin{cases} 0 \text{ if } a \leq p\mathfrak{D}_p(x) - x; \\ 1 \text{ if } a > p\mathfrak{D}_p(x) - x. \end{cases}$$
- We write $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r)$ and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_r)$. Let r' be the number of elements β_i of $\boldsymbol{\beta}$ such that $\beta_i \neq 1$. We rearrange the subscripts so that $\beta_i \neq 1$ for $i \leq r'$. For all $a \in [0, p)$ and all $k \in \mathbb{N}$, we set
$$N^k_{p,\boldsymbol{\alpha}}(a) = \sum_{i=1}^{r} \rho_p\big(a, \mathfrak{D}^k_p(\alpha_i)\big) \quad \text{and} \quad N^k_{p,\boldsymbol{\beta}}(a) = \sum_{i=1}^{r'} \rho_p\big(a, \mathfrak{D}^k_p(\beta_i)\big).$$
- For a given prime p not dividing $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, we define two assertions:
$(v)_p$ for all $i \in \{1, \dots, r'\}$ and all $k \in \mathbb{N}$, we have $\mathfrak{D}^k_p(\beta_i) \in \mathbb{Z}_p^\times$;

[6]See Section 5 for a detailed study of Dwork's map $\mathfrak{D}_p$.

$(vi)_p$ for all $a \in [0, p)$ and all $k \in \mathbb{N}$, we have either $N^k_{p,\boldsymbol{\alpha}}(a) = N^k_{p,\boldsymbol{\beta}}(a+) = 0$ or $N^k_{p,\boldsymbol{\alpha}}(a) - N^k_{p,\boldsymbol{\beta}}(a+) \geq 1$.

Dwork's result [**12**, Theorem 4.1] restricted to the case where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ have the same number of elements is the following.

THEOREM 19 (Dwork). *Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be two tuples of parameters in $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ with the same number of elements. Let p be a prime not dividing $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ such that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ satisfy $(v)_p$ and $(vi)_p$. Then we have*

$$\frac{G_{\mathfrak{D}_p(\boldsymbol{\alpha}),\mathfrak{D}_p(\boldsymbol{\beta})}}{F_{\mathfrak{D}_p(\boldsymbol{\alpha}),\mathfrak{D}_p(\boldsymbol{\beta})}}(z^p) - p\frac{G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)}{F_{\boldsymbol{\alpha},\boldsymbol{\beta}}}(z) \in pz\mathbb{Z}_p[[z]].$$

Now let us assume that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are disjoint with elements in $(0, 1]$ and that $H_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ holds. For all primes p not dividing $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, we have $\mathfrak{D}_p(\boldsymbol{\alpha}) = \langle \omega\boldsymbol{\alpha} \rangle$ and $\mathfrak{D}_p(\boldsymbol{\beta}) = \langle \omega\boldsymbol{\beta} \rangle$ where $\omega \in \mathbb{Z}$ satisfies $\omega p \equiv 1 \mod d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ (see Section 5.2 below). Then, by Theorem 6 for a fixed prime p and $b = t = 1$, we obtain that

$$\frac{G_{\mathfrak{D}_p(\boldsymbol{\alpha}),\mathfrak{D}_p(\boldsymbol{\beta})}}{F_{\mathfrak{D}_p(\boldsymbol{\alpha}),\mathfrak{D}_p(\boldsymbol{\beta})}}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}} z^p) - p\frac{G_{\boldsymbol{\alpha},\boldsymbol{\beta}}}{F_{\boldsymbol{\alpha},\boldsymbol{\beta}}}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}} z) \in pz\mathbb{Z}_p[[z]]. \tag{4.1}$$

Thus, contrary to Theorem 19, there is no restriction on the primes p because of the constant $C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. Furthermore, in the proof of Lemma 48 in Section 11.1.3, we show that if $H_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ holds then $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ satisfy Assertions $(v)_p$ and $(vi)_p$ for almost all primes p. By Theorem 8, the converse holds when $\langle \boldsymbol{\alpha} \rangle$ and $\langle \boldsymbol{\beta} \rangle$ are disjoint, $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$ is N-integral and, for all $a \in \{1, \dots, d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ coprime to $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, we have

$$\frac{G_{\langle a\boldsymbol{\alpha} \rangle,\langle a\boldsymbol{\beta} \rangle}}{F_{\langle a\boldsymbol{\alpha} \rangle,\langle a\boldsymbol{\beta} \rangle}}(z) = \frac{G_{\boldsymbol{\alpha},\boldsymbol{\beta}}}{F_{\boldsymbol{\alpha},\boldsymbol{\beta}}}(z).$$

Indeed, in this case, Theorem 19 in combination with Proposition 2 implies that, for almost all primes p, we have $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z) \in \mathbb{Z}_p[[z]]$. Then it is a simple exercise to show that $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$ is N-integral and, by Theorem 8, we obtain that $H_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ holds.

The main improvement in Theorem 6 is the use of algebras of $\mathbb{Z}_p$-valued functions instead of $\mathbb{Z}_p$. This is precisely this generalization which enables us to prove the integrality of the Taylor coefficients of certain roots of $(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}} z)^{-1}\widetilde{q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}} z)$.

4.2.2. *Theorem 8 and previous results.* The constants $C \in \mathbb{Q}^\times$ such that an N-integral canonical coordinate $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ satisfies $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(Cz) \in \mathbb{Z}[[z]]$ was first studied when there exist some disjoint tuples of positive integers $\mathbf{e} = (e_1, \dots, e_u)$, $\mathbf{f} = (f_1, \dots, f_v)$ and a constant $C_0 \in \mathbb{Q}^\times$ such that

$$F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C_0 z) = \sum_{n=0}^{\infty} \frac{(e_1 n)! \cdots (e_u n)!}{(f_1 n)! \cdots (f_v n)!} z^n \tag{4.2}$$

and

$$F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C_0 z) \in \mathbb{Z}[[z]]. \tag{4.3}$$

We now assume that such a constant C_0 exists. According to [**8**, Proposition 2], the condition (4.2) ensures that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are R-partitioned, *i. e.* $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r)$, respectively

$\boldsymbol{\beta} = (\beta_1, \ldots, \beta_s)$, is the concatenation of tuples $(b/N_i)_{b\in\{1,\ldots,N_i\},\gcd(b,N_i)=1}$, $1 \leq i \leq r'$, respectively of tuples $(b/N'_j)_{b\in\{1,\ldots,N'_j\},\gcd(b,N'_j)=1}$, $1 \leq j \leq s'$. Furthermore, by [8, Proposition 2], if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are R-partioned, then one can take

$$C_0 = \frac{\prod_{i=1}^{r'} N_i^{\varphi(N_i)} \prod_{p\mid N_i} p^{\frac{\varphi(N_i)}{p-1}}}{\prod_{j=1}^{s'} N_j'^{\varphi(N'_j)} \prod_{p\mid N'_j} p^{\frac{\varphi(N'_j)}{p-1}}} \quad \text{and} \quad \sum_{i=1}^{u} e_i - \sum_{j=1}^{v} f_j = r - s. \tag{4.4}$$

Moreover, Landau's criterion [22] asserts that the condition (4.3) is equivalent to the nonnegativity on $[0,1]$ of the function of Landau

$$\boldsymbol{\Delta}_{\mathbf{e},\mathbf{f}}(x) := \sum_{i=1}^{u} \lfloor e_i x \rfloor - \sum_{j=1}^{v} \lfloor f_j x \rfloor,$$

which can be checked easily because, by [8, Proposition 3], for all $x \in [0,1]$, we have

$$\boldsymbol{\Delta}_{\mathbf{e},\mathbf{f}}(x) = \#\{i \,:\, x \geq \alpha_i\} - \#\{j \,:\, x \geq \beta_j\}. \tag{4.5}$$

The results obtained by Lian and Yau [26], Zudilin [34], Krattenthaler and Rivoal [18] and Delaygue [8] led to an effective criterion [8, Theorem 1] for the N-integrality of $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$.

By combining and reformulating this criterion and [8, Theorem 3], we obtain the following result.

Theorem 20 (Delaygue). *If (4.2) and (4.3) hold, then the following assertions are equivalent:*

1. *$q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$ is N-integral;*
2. *$(C_0 z)^{-1} q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C_0 z) \in \mathbb{Z}[[z]]$;*
3. *we have $\sum_{i=1}^{u} e_i = \sum_{j=1}^{v} f_j$ and, for all $x \in [1/M_{\mathbf{e},\mathbf{f}}, 1[$, we have $\boldsymbol{\Delta}_{\mathbf{e},\mathbf{f}}(x) \geq 1$, where $M_{\mathbf{e},\mathbf{f}}$ is the largest element of $\mathbf{e}$ and $\mathbf{f}$.*

Let us show that Theorem 8 implies Theorem 20. Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be disjoint tuples of parameters in $\mathbb{Q}\setminus\mathbb{Z}_{\leq 0}$ such that (4.2) and (4.3) hold. Then $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are R-partitioned and their elements lie in $(0,1]$ so that $\langle\boldsymbol{\alpha}\rangle$ and $\langle\boldsymbol{\beta}\rangle$ are disjoint and $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is N-integral. First we prove that if $r = s$, then we have $C'_{\boldsymbol{\alpha},\boldsymbol{\beta}} = C_{\boldsymbol{\alpha},\boldsymbol{\beta}} = C_0$. We write λ_p for $\lambda_p(\boldsymbol{\alpha},\boldsymbol{\beta})$. Since $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are R-partitioned, the number of elements of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ with exact denominator divisible by 4 is a sum of multiples of integers of the form $\varphi(2^k)$ with $k \in \mathbb{N}$, $k \geq 2$, so this number is even. Thus, we have $C'_{\boldsymbol{\alpha},\boldsymbol{\beta}} = C_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. Furthermore, for all primes p, we have

$$\lambda_p = r - \sum_{\substack{i=1\\ p\mid N_i}}^{r'} \varphi(N_i) - s + \sum_{\substack{j=1\\ p\mid N'_j}}^{s'} \varphi(N'_j) = -\sum_{\substack{i=1\\ p\mid N_i}}^{r'} \varphi(N_i) + \sum_{\substack{j=1\\ p\mid N'_j}}^{s'} \varphi(N'_j).$$

If p divides N_i then $p-1$ divides $\varphi(N_i)$ so that $-\lfloor \lambda_p/(p-1) \rfloor = -\lambda_p/(p-1)$ and $C_{\boldsymbol{\alpha},\boldsymbol{\beta}} = C_0$ as expected. Now we assume that (4.2) and Theorem 8 hold and we prove that Assertions (1), (2) and (3) of Theorem 20 are equivalent.

• (1) ⇒ (2): If $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$ is N-integral, then we obtain that $(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)^{-1}q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z) \in \mathbb{Z}[[z]]$ and $r = s$ so that $C'_{\boldsymbol{\alpha},\boldsymbol{\beta}} = C_0$ and Assertion (2) of Theorem 20 holds.

• (2) ⇒ (3): If $(C_0z)^{-1}q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C_0z) \in \mathbb{Z}[[z]]$ then $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$ is N-integral and, according to Theorem 8, we have $r = s$ and $H_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is true. We deduce that we have $\sum_{i=1}^{u} e_i = \sum_{j=1}^{v} f_j$. Now, since $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are disjoint tuples with elements in $(0,1]$, Equation (4.5) ensures that the assertions "for all $x \in [1/M_{\mathbf{e},\mathbf{f}}, 1[$, we have $\boldsymbol{\Delta}_{\mathbf{e},\mathbf{f}}(x) \geq 1$" and "for all $x \in \mathbb{R}$, $\min_{\boldsymbol{\alpha},\boldsymbol{\beta}}(1) \preceq x \prec 1$, we have $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(1,x) \geq 1$" are equivalent. Thus Assertion (3) of Theorem 20 holds.

• (3) ⇒ (1): We assume that $\sum_{i=1}^{u} e_i = \sum_{j=1}^{v} f_j$, that is $r = s$, and that, for all $x \in [1/M_{\mathbf{e},\mathbf{f}}, 1[$, we have $\boldsymbol{\Delta}_{\mathbf{e},\mathbf{f}}(x) \geq 1$. Since $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are R-partitioned, for all $a \in \{1, \dots, d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ coprime to $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ we have $\langle a\boldsymbol{\alpha}\rangle = \boldsymbol{\alpha}$ and $\langle a\boldsymbol{\beta}\rangle = \boldsymbol{\beta}$, and these tuples are disjoint. We deduce that, for all $a \in \{1, \dots, d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ coprime to $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and all $x \in \mathbb{R}$, $\min_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a) \preceq x \prec a$, Equation (4.5) gives us that $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a,x) \geq 1$, so that $H_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ holds. Thus Assertion (iii) of Theorem 8 holds and $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$ is N-integral as expected. This finishes the proof that Theorem 8 implies Theorem 20.

Furthermore, when (4.2) holds, Delaygue [**11**, Theorem 8] generalized some of the results of Krattenthaler and Rivoal [**21**] and proved that all Taylor coefficients at the origin of $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C_0z)$ are positive but its constant term, which is 0. Proposition 42 generalizes this result since it does not use the assumption that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are R-partitioned.

Later, Roques studied (see [**31**] and [**32**]) the integrality of the Taylor coefficients of canonical coordinates $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ without assuming that (4.2) holds, in the case $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ have the same number of elements $r \geq 2$, all the elements of $\boldsymbol{\beta}$ are equal to 1 and all the elements of $\boldsymbol{\alpha}$ lie in $(0,1] \cap \mathbb{Q}$. In this case, we have $r = s$ and it is easy to prove that $H_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ holds but $\boldsymbol{\alpha}$ is not necessarily R-partitioned. Roques proved that $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$ is N-integral if and only if, for all $a \in \{1, \dots, d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ coprime to $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, we have $q_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(z) = q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$ in accordance with Theorem 8. Furthermore, when $r = 2$, he found the exact finite set ([7]) of tuples $\boldsymbol{\alpha}$ such that $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$ is N-integral (see [**31**, Theorem 3]) and, when $r \geq 3$, he proved (see [**32**]) that $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$ is N-integral if and only if $\boldsymbol{\alpha}$ is R-partitioned (the "if part" is proved by Krattenthaler and Rivoal in [**18**]). Note that if $\boldsymbol{\beta} = (1, \dots, 1)$, then it is easy to prove that $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$ is N-integral.

The integrality of Taylor coefficients of roots of a rescaling of $z^{-1}q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$ has been studied in case (4.2) holds by Lian and Yau [**24**], Krattenthaler and Rivoal [**19**], and by Delaygue [**9**]. For a detailed survey of these results, we refer the reader to [**9**, Section 1.2].

• In [**24**], Lian and Yau studied the case $\mathbf{e} = (p)$ and $\mathbf{f} = (1, \dots, 1)$ with p 1's in $\mathbf{f}$ and where p is a prime. In this case, we have $\boldsymbol{\beta} = (1, \dots, 1)$ and $\mathfrak{n}'_{\boldsymbol{\alpha},\boldsymbol{\beta}} = 1$, thus we do not obtain a root with Theorem 10.

• In [**19**], Krattenthaler and Rivoal studied the case $\mathbf{e} = (N, \dots, N)$ with k N's in $\mathbf{e}$ and $\mathbf{f} = (1, \dots, 1)$ with kN 1's in $\mathbf{f}$. In this case, we also have $\boldsymbol{\beta} = (1, \dots, 1)$. For all prime divisors p of N, we write $N = p^{\eta_p}N_p$ with $\eta_p, N_p \in \mathbb{N}$ and N_p not divisible by p. A simple

[7]This set contains 28 elements amongst which 4 are R-partitioned.

computation of the associated tuples $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ shows that $d_{\boldsymbol{\alpha},\boldsymbol{\beta}} = N$ and $\lambda_p = k(N_p - N)$. Thus, for all prime divisors p of N, $p-1$ divides λ_p and we have

$$\mathfrak{n}'_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \prod_{p \mid N} p^{-1+k\frac{N-N_p}{p-1}}.$$

It seems that the integrality properties of roots of mirror maps found by Krattenthaler and Rivoal are always stronger in these cases.

• However, in a lot of cases, our root $\mathfrak{n}'_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ improves the one found by Delaygue in [9]. For example, if $\mathbf{e} = (4,2)$ and $\mathbf{f} = (1,1,1,1,1,1)$, then [9, Corollary 1.1] gives us the root 4 while $\boldsymbol{\beta} = (1,\dots,1)$ and $\mathfrak{n}'_{\boldsymbol{\alpha},\boldsymbol{\beta}} = 32$.

4.3. Open questions. We formulate some open questions directly related to our main results.

• Does the equivalence of Theorem 8 still hold if we do not assume that $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$ is N-integral?

• Theorem 9 leads to a natural question: do the coefficients of $(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)^{-1}q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)$ count any object?

• One of the conditions for $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$ to be N-integral is that, for all $a \in \{1,\dots,d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ coprime to $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, we have $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z) = q_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(z)$. According to [31] and [32], we know that, when $\boldsymbol{\beta} = (1,\dots,1)$ and all elements of $\boldsymbol{\alpha}$ belong to $(0,1]$, this condition implies a stronger characterization related to the exact forms of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Is it possible to deduce a similar characterization in the general case?

4.4. A corrected version of a lemma of Lang. While working on this article, we noticed an error in a lemma stated by Lang [23, Lemma 1.1, Section 1, Chapter 14] about arithmetic properties of Mojita's p-adic Gamma function. This lemma has been used in several articles on the integrality of the Taylor coefficients of mirror maps including papers of the authors. First we give a corrected version of Lang's lemma, then we explain why this error does not change the validity of our previous results.

Let p be a fixed prime. For all $n \in \mathbb{N}$, we define the p-adic Gamma function Γ_p by

$$\Gamma_p(n) := (-1)^n \prod_{\substack{k=1\\ \gcd(k,p)=1}}^{n-1} k.$$

In particular, $\Gamma_p(0) = 1$, $\Gamma_p(1) = -1$ and Γ_p can be extended to $\mathbb{Z}_p$.

Proposition 21. *For all $k, m, s \in \mathbb{N}$, we have*

$$\Gamma_p(k + mp^s) \equiv \begin{cases} \Gamma_p(k) \mod p^s \text{ if } p^s \neq 4; \\ (-1)^m \Gamma_p(k) \mod p^s \text{ if } p^s = 4. \end{cases}$$

The case $p^s \neq 4$ in Proposition 21 is proved by Morita in [28]. We provide a complete proof of the proposition.

PROOF. If $s = 0$ or if $m = 0$ this is trivial. We assume in the sequel that $s \geq 1$ and $m \geq 1$. Then

$$\begin{aligned}\frac{\Gamma_p(k+mp^s)}{\Gamma_p(k)} &= (-1)^{mp^s} \prod_{\substack{i=k \\ \gcd(i,p)=1}}^{k+mp^s-1} i = (-1)^{mp^s} \prod_{\substack{i=0 \\ \gcd(k+i,p)=1}}^{p^s-1} \prod_{j=0}^{m-1} (k+i+jp^s) \\ &\equiv (-1)^{mp^s} \prod_{\substack{i=0 \\ \gcd(k+i,p)=1}}^{p^s-1} (k+i)^m \mod p^s \\ &\equiv (-1)^{mp^s} \prod_{\substack{j=0 \\ \gcd(j,p)=1}}^{p^s-1} j^m \mod p^s, \end{aligned} \tag{4.6}$$

because, for all $j \in \{0, \ldots, p^s - 1\}$, there exists a unique $i \in \{0, \ldots, p^s - 1\}$ such that $k + i \equiv j \mod p^s$.

We first assume that $p \geq 3$. In this case, the group $(\mathbb{Z}/p^s\mathbb{Z})^\times$ is cyclic and contains just one element of order 2. Collecting each element of $(\mathbb{Z}/p^s\mathbb{Z})^\times$ of order ≥ 3 with its inverse, we obtain

$$\prod_{\substack{j=0 \\ \gcd(j,p)=1}}^{p^s-1} j \equiv -1 \mod p^s.$$

Together, with (4.6), we get

$$\frac{\Gamma_p(k+mp^s)}{\Gamma_p(k)} \equiv 1 \mod p^s,$$

because p is odd.

Let us now assume that $p = 2$. If $s = 1$, then

$$\prod_{\substack{j=0 \\ \gcd(j,p)=1}}^{p^s-1} j = 1$$

and by (4.6) this yields $\Gamma_p(k + mp^s) \equiv \Gamma_p(k) \mod p^s$. If $s = 2$, then

$$\prod_{\substack{j=0 \\ \gcd(j,p)=1}}^{p^s-1} j = 3 \equiv -1 \mod p^s,$$

and by (4.6), this yields $\Gamma_p(k+mp^s) \equiv (-1)^m \Gamma_p(k) \mod p^s$. It remains to deal with the case $s \geq 3$. The group $(\mathbb{Z}/2^s\mathbb{Z})^\times$ is isomorphic to $\mathbb{Z}/2^{s-2}\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Moreover,

$$\sum_{k=0}^{2^{s-2}-1} \sum_{j=0}^{1} (k,j) = \left(2 \sum_{k=0}^{2^{s-2}-1} k, 2^{s-2}\right) = \left(2^{s-2}(2^{s-2}-1), 2^{s-2}\right) \in 2^{s-2}\mathbb{Z} \times 2\mathbb{Z},$$

because $s \geq 3$. Hence,

$$\prod_{\substack{j=0 \\ \gcd(j,p)=1}}^{p^s-1} j \equiv 1 \mod p^s$$

and by (4.6), this yields

$$\frac{\Gamma_p(k+mp^s)}{\Gamma_p(k)} \equiv 1 \mod p^s,$$

which completes the proof of the proposition. □

The error in Lang's version is that he wrote $\Gamma_p(k+mp^s) \equiv \Gamma_p(k) \mod p^s$ if $p^s = 4$, forgetting the factor $(-1)^m$. He gives the proof only for $p \geq 3$ and he claims that the proof goes through similarly when $p = 2$, overlooking the subtility. Delaygue and Krattenthaler and Rivoal used Lang's version in [**8**, Lemma 11], [**10**, Lemma 8] and [**18**]. Fortunately, the resulting mistakes in these papers are purely local and can be fixed. Indeed, the factor $(-1)^\ell$ (that should have been added when $p^s = 4$) would have occurred for an *even* value of ℓ and thus would have immediately disappeared without changing the rest of the proof.

5. The p-adic valuation of Pochhammer symbols

We introduce certain step functions, defined over $\mathbb{R}$, that enable us to compute the p-adic valuation of Pochhammer symbols. We will then provide a connection between the values of these functions and the functions $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a,\cdot)$. This construction is inspired by various articles of Christol [**7**], Dwork [**12**] and Katz [**16**].

5.1. Christol's criterion for the N-integrality of $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. We shall first state and prove the following preliminary result.

PROPOSITION 22. *Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be tuples of parameters in $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$. Then, $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is N-integral if and only if, for almost all primes p, we have $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z) \in \mathbb{Z}_p[[z]]$.*

PROOF. Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be two sequences taking their values in $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$. If there exists $C \in \mathbb{Q}^*$ such that $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(Cz) \in \mathbb{Z}[[z]]$, then for all primes p such that $v_p(C) \leq 0$, we have $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z) \in \mathbb{Z}_p[[z]]$. Hence, there exists only a finite number of primes p such that $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z) \notin \mathbb{Z}_p[[z]]$.

Conversely, let us assume there exists only a finite number of primes p such that $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z) \notin \mathbb{Z}_p[[z]]$. To prove Proposition 22, it is enough to prove that, for all primes p, there exists $m \in \mathbb{Z}_{\leq 0}$ such that, for all $n \in \mathbb{N}$, we have

$$v_p\left(\frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n}\right) \geq mn. \tag{5.1}$$

Let $x \in \mathbb{Q}$, $x = a/b$ with $a, b \in \mathbb{Z}$, $b \geq 1$, and a and b coprime. If b is not divisible by p, then for all $n \in \mathbb{N}$, we have $v_p\big((x)_n\big) \geq 0$. On the other hand, if p divides b, then $v_p\big((x)_n\big) = v_p(x)n$.

Let us now assume that $x \notin \mathbb{Z}_{\leq 0}$. Then, for all $n \in \mathbb{N}$, $n \geq 1$,

$$v_p\left(\frac{1}{(x)_n}\right) = v_p\left(\frac{b^n}{a(a+b)\cdots\big(a+b(n-1)\big)}\right) \geq v_p\left(\frac{b^n}{|a|!(|a|+bn)!}\right)$$
$$\geq \left(v_p(b) - \frac{b}{p-1}\right)n - 2\frac{|a|}{p-1},$$

because

$$v_p\big((|a|+bn)!\big) = \sum_{\ell=1}^{\infty}\left\lfloor\frac{|a|+bn}{p^\ell}\right\rfloor < \sum_{\ell=1}^{\infty}\frac{|a|+bn}{p^\ell} = \frac{|a|}{p-1} + \frac{b}{p-1}n.$$

Hence, (5.1) holds and Proposition 22 is proved. □

We shall now come to Theorem 3 stated in Section 2.1.

THEOREM 3. *The following assertions are equivalent:*

(i) $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ *is* N*-integral.*

(ii) For all $a \in \{1, \dots, d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ *coprime to* $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ *and all* $x \in \mathbb{R}$*, we have* $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a, x) \geq 0$.

PROOF. According to Proposition 22, $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is N-integral if and only if, for almost all ([8]) primes p, we have $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z) \in \mathbb{Z}_p[[z]]$. Then, the proof is a consequence of Christol's Proposition 1 in [7]. Note that Christol assumes that $r = s$, that there is $j \in \{1, \dots, s\}$ such that $\beta_j \in \mathbb{N}$ and that all elements $\alpha \in \mathbb{N}$ of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ satisfies $\alpha \geq \beta_j$. But, his proof does not use these assumptions. □

5.2. Dwork's map $\mathfrak{D}_p$. Given a prime p and some $\alpha \in \mathbb{Z}_p \cap \mathbb{Q}$, we recall that $\mathfrak{D}_p(\alpha)$ denotes the unique element in $\mathbb{Z}_p \cap \mathbb{Q}$ such that

$$p\mathfrak{D}_p(\alpha) - \alpha \in \{0, \dots, p-1\}.$$

The map $\alpha \mapsto \mathfrak{D}_p(\alpha)$ was used by Dwork in [12] (denoted there as $\alpha \mapsto \alpha'$). We observe that the unique element $k \in \{0, \dots, p-1\}$ such that $k + \alpha \in p\mathbb{Z}_p$ is $k = p\mathfrak{D}_p(\alpha) - \alpha$. More precisely, the p-adic expansion of $-\alpha$ in $\mathbb{Z}_p$ is

$$-\alpha = \sum_{\ell=0}^{\infty}\big(p\mathfrak{D}_p^{\ell+1}(\alpha) - \mathfrak{D}_p^{\ell}(\alpha)\big)p^\ell,$$

where $\mathfrak{D}_p^\ell$ is the ℓ-th iteration of $\mathfrak{D}_p$. In particular, for all $\ell \in \mathbb{N}$, $\ell \geq 1$, $\mathfrak{D}_p^\ell(\alpha)$ is the unique element in $\mathbb{Z}_p \cap \mathbb{Q}$ such that $p^\ell\mathfrak{D}_p^\ell(\alpha) - \alpha \in \{0, \dots, p^\ell - 1\}$.

For all primes p, we have $\mathfrak{D}_p(1) = 1$. Let us now assume that α is in $\mathbb{Z}_p \cap \mathbb{Q} \cap (0, 1)$. Set $N \in \mathbb{N}$, $N \geq 2$ and $r \in \{1, \dots, N-1\}$, $\gcd(r, N) = 1$, such that $\alpha = r/N$. Let s_N

[8]"For almost all" means "for all but finitely many".

be the unique right inverse of the canonical morphism $\pi_N : \mathbb{Z} \to \mathbb{Z}/N\mathbb{Z}$ with values in $\{0, \dots, N-1\}$. Then (see [31] for details)

$$\mathfrak{D}_p(\alpha) = \frac{s_N\big(\pi_N(p)^{-1}\pi_N(r)\big)}{N}.$$

Hence, for all $\ell \in \mathbb{N}$, $\ell \geq 1$, we obtain

$$\mathfrak{D}_p^\ell(\alpha) = \frac{s_N\big(\pi_N(p)^{-\ell}\pi_N(r)\big)}{N}. \tag{5.2}$$

In particular, if $\alpha \in (0,1)$, then $\mathfrak{D}_p(\alpha)$ depends only on the congruence class of p modulo N. If $a \in \mathbb{Z}$ satisfies $ap \equiv 1 \mod N$, then $\mathfrak{D}_p^\ell(\alpha) = \{a^\ell\alpha\} = \langle a^\ell\alpha\rangle$ because a is coprime to N, hence $a^\ell\alpha \notin \mathbb{Z}$. This formula is still valid when $\alpha = 1$ and a is any integer.

Lemma 23. *Let $\alpha \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$. Then for any prime p such that $\alpha \in \mathbb{Z}_p$ and all $\ell \in \mathbb{N}$, $\ell \geq 1$, such that $p^\ell \geq d(\alpha)\big(|\lfloor 1-\alpha\rfloor| + \langle\alpha\rangle\big)$, we have $\mathfrak{D}_p^\ell(\alpha) = \mathfrak{D}_p^\ell(\langle\alpha\rangle) = \langle\omega\alpha\rangle$, where $\omega \in \mathbb{Z}$ satisfies $\omega p^\ell \equiv 1 \mod d(\alpha)$.*

Proof. Let $\alpha \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ and p be such that $\alpha \in \mathbb{Z}_p$ and $\ell \in \mathbb{N}$, $\ell \geq 1$ be such that $p^\ell \geq d(\alpha)(|\lfloor 1-\alpha\rfloor| + \langle\alpha\rangle)$. By definition, $\mathfrak{D}_p^\ell(\alpha)$ is the unique rational number in $\mathbb{Z}_p$ such that $p^\ell\mathfrak{D}_p^\ell(\alpha) - \alpha \in \{0, \dots, p^\ell - 1\}$. We set $\alpha = \langle\alpha\rangle + k$, $k \in \mathbb{Z}$ and $r := \mathfrak{D}_p^\ell(\langle\alpha\rangle) + \lfloor k/p^\ell\rfloor + a$, with $a = 0$ if $k - p^\ell\lfloor k/p^\ell\rfloor \leq p^\ell\mathfrak{D}_p^\ell(\langle\alpha\rangle) - \langle\alpha\rangle$ and $a = 1$ otherwise. We obtain

$$p^\ell r - \alpha = p^\ell\mathfrak{D}_p^\ell(\langle\alpha\rangle) - \langle\alpha\rangle + p^\ell\left\lfloor\frac{k}{p^\ell}\right\rfloor - k + p^\ell a \in \{0, \dots, p^\ell - 1\},$$

because $p^\ell\mathfrak{D}_p^\ell(\langle\alpha\rangle) - \langle\alpha\rangle$ and $k - p^\ell\lfloor k/p^\ell\rfloor$ are in $\{0, \dots, p^\ell - 1\}$. Since $r \in \mathbb{Z}_p$, we get $\mathfrak{D}_p^\ell(\alpha) = r$. We have $d(\alpha)(|k| + \langle\alpha\rangle) > |k|$ thus $\lfloor k/p^\ell\rfloor \in \{-1, 0\}$.

If $\lfloor k/p^\ell\rfloor = 0$, then, since $\mathfrak{D}_p^\ell(\langle\alpha\rangle) \geq 1/d(\alpha)$, we get $p^\ell\mathfrak{D}_p^\ell(\langle\alpha\rangle) - \langle\alpha\rangle \geq |k|$ and thus $a = 0$. In this case, we have $\mathfrak{D}_p^\ell(\alpha) = \mathfrak{D}_p^\ell(\langle\alpha\rangle)$.

Let us now assume that $\lfloor k/p^\ell\rfloor = -1$, *i. e.* $k \leq -1$. We have $\langle\alpha\rangle < 1$ because $\alpha \notin \mathbb{Z}_{\leq 0}$, hence $d(\alpha) \geq 2$. We have

$$\begin{aligned} p^\ell\mathfrak{D}_p^\ell(\langle\alpha\rangle) - \langle\alpha\rangle - (k + p^\ell) &\leq p^\ell\left(\frac{d(\alpha)-1}{d(\alpha)} - 1\right) - \langle\alpha\rangle - k \leq -\frac{p^\ell}{d(\alpha)} - \langle\alpha\rangle - k \\ &\leq -|k| - 2\langle\alpha\rangle - k \leq -2\langle\alpha\rangle < 0, \end{aligned}$$

thus $a = 1$ and $\mathfrak{D}_p^\ell(\alpha) = \mathfrak{D}_p^\ell(\langle\alpha\rangle)$. □

5.3. Analogues of Landau functions. We now define the step functions that will enable us to compute the p-adic valuation of the Taylor coefficients at $z = 0$ of $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$. For all primes p, all $\alpha \in \mathbb{Q} \cap \mathbb{Z}_p$ and all $\ell \in \mathbb{N}$, $\ell \geq 1$, we denote by $\delta_{p,\ell}(\alpha, \cdot)$ the step function defined, for all $x \in \mathbb{R}$, by

$$\left(\delta_{p,\ell}(\alpha, x) = k \Longleftrightarrow x - \mathfrak{D}_p^\ell(\alpha) - \frac{\lfloor 1-\alpha\rfloor}{p^\ell} \in [k-1, k)\right),\ k \in \mathbb{Z}.$$

In particular, if $\alpha \in (0,1]$, then for all $k \in \mathbb{Z}$, we have

$$\delta_{p,\ell}(\alpha, x) = k \Longleftrightarrow x - \mathfrak{D}_p^\ell(\alpha) \in [k-1, k).$$

Let $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_r)$ and $\boldsymbol{\beta} := (\beta_1, \dots, \beta_s)$ be two sequences taking their values in $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$. For any p that does not divide $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, and all $\ell \in \mathbb{N}$, $\ell \geq 1$, we denote by $\Delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{p,\ell}$ the step function defined, for all $x \in \mathbb{R}$, by

$$\Delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{p,\ell}(x) := \sum_{i=1}^{r} \delta_{p,\ell}(\alpha_i, x) - \sum_{j=1}^{s} \delta_{p,\ell}(\beta_j, x).$$

The motivation behind the functions $\Delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{p,\ell}$ is given by the following result.

Proposition 24. *Let $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_r)$ and $\boldsymbol{\beta} := (\beta_1, \dots, \beta_s)$ be two sequences taking their values in $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$. Let p be a prime such that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are in $\mathbb{Z}_p$. Then, for all $n \in \mathbb{N}$, we have*

$$v_p\left(\frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n}\right) = \sum_{\ell=1}^{\infty} \Delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{p,\ell}\left(\frac{n}{p^\ell}\right) = \sum_{\ell=1}^{\infty} \Delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{p,\ell}\left(\left\{\frac{n}{p^\ell}\right\}\right) + (r-s)v_p(n!).$$

Remark 25. *This proposition is a reformulation of results in Section III of [7], proved by Christol in order to compute the p-adic valuation of the Pochhammer symbol $(x)_n$ for $x \in \mathbb{Z}_p$.*

Proof. For any p, any $n := \sum_{k=0}^{\infty} n_k p^k \in \mathbb{Z}_p$ with $n_k \in \{0, \dots, p-1\}$, and any $\ell \in \mathbb{N}$, $\ell \geq 1$, we set $T_p(n, \ell) := \sum_{k=0}^{\ell-1} n_k p^k$. For all $\ell \in \mathbb{N}$, $\ell \geq 1$, we have

$$T_p(-\alpha, \ell) = p^\ell \mathfrak{D}_p^\ell(\alpha) - \alpha.$$

We fix a p-adic integer $\alpha \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$. For all $k \in \mathbb{Z}$ and all $\ell \in \mathbb{N}$, $\ell \geq 1$, we have

$$\begin{aligned}
\delta_{p,\ell}\left(\alpha, \frac{n}{p^\ell}\right) = k &\Longleftrightarrow \mathfrak{D}_p^\ell(\alpha) + \frac{\lfloor 1-\alpha \rfloor}{p^\ell} + k - 1 \leq \frac{n}{p^\ell} < \mathfrak{D}_p^\ell(\alpha) + \frac{\lfloor 1-\alpha \rfloor}{p^\ell} + k \\
&\Longleftrightarrow p^\ell \mathfrak{D}_p^\ell(\alpha) + \lfloor 1-\alpha \rfloor + (k-1)p^\ell \leq n < p^\ell \mathfrak{D}_p^\ell(\alpha) + \lfloor 1-\alpha \rfloor + kp^\ell \\
&\Longleftrightarrow p^\ell \mathfrak{D}_p^\ell(\alpha) - \alpha + (k-1)p^\ell < n \leq p^\ell \mathfrak{D}_p^\ell(\alpha) - \alpha + kp^\ell \qquad (5.3) \\
&\Longleftrightarrow \left\lceil \frac{n - T_p(-\alpha, \ell)}{p^\ell} \right\rceil = k,
\end{aligned}$$

where, for all $x \in \mathbb{R}$, $\lceil x \rceil$ is the smallest integer larger than x. We have used in (5.3) the fact that $-\alpha = -\langle \alpha \rangle + \lfloor 1-\alpha \rfloor$, $-1 \leq -\langle \alpha \rangle < 0$ and $p^\ell \mathfrak{D}_p^\ell(\alpha) - \alpha \in \mathbb{N}$. We then obtain

$$\delta_{p,\ell}\left(\alpha, \frac{n}{p^\ell}\right) = \left\lceil \frac{n - T_p(-\alpha, \ell)}{p^\ell} \right\rceil. \qquad (5.4)$$

Christol proved in [7] that, for all $\alpha \in \mathbb{Z}_p \setminus \mathbb{Z}_{\leq 0}$ and all $n \in \mathbb{N}$, we have

$$v_p\big((\alpha)_n\big) = \sum_{\ell=1}^{\infty} \left\lfloor \frac{n + p^\ell - 1 - T_p(-\alpha, \ell)}{p^\ell} \right\rfloor. \tag{5.5}$$

For all $\ell \in \mathbb{N}$, $\ell \geq 1$, we have

$$\frac{n + p^\ell - 1 - T_p(-\alpha, \ell)}{p^\ell} \in \frac{1}{p^\ell}\mathbb{Z},$$

so that if $k \in \mathbb{Z}$ is such that

$$k \leq \frac{n + p^\ell - 1 - T_p(-\alpha, \ell)}{p^\ell} < k + 1,$$

then

$$k - 1 < \frac{n - T_p(-\alpha, \ell)}{p^\ell} \leq k.$$

Hence, we get

$$\left\lfloor \frac{n + p^\ell - 1 - T_p(-\alpha, \ell)}{p^\ell} \right\rfloor = \left\lceil \frac{n - T_p(-\alpha, \ell)}{p^\ell} \right\rceil.$$

By (5.4) and (5.5), it follows that

$$v_p\big((\alpha)_n\big) = \sum_{\ell=1}^{\infty} \delta_{p,\ell}\left(\alpha, \frac{n}{p^\ell}\right) = \sum_{\ell=1}^{\infty} \delta_{p,\ell}\left(\alpha, \left\{\frac{n}{p^\ell}\right\}\right) + v_p(n!),$$

because $\delta_{p,\ell}(\alpha, n/p^\ell) = \delta_{p,\ell}(\alpha, \{n/p^\ell\}) + \lfloor n/p^\ell \rfloor$ and $v_p(n!) = \sum_{\ell=1}^{\infty} \lfloor n/p^\ell \rfloor$. □

The following lemma provides an upper bound for the abscissae of the jumps of the functions $\Delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{p,\ell}$.

LEMMA 26. *Let $\alpha \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$. There exists a constant $M(\alpha) > 0$ such that, for all p such that $\alpha \in \mathbb{Z}_p$, and all $\ell \in \mathbb{N}$, $\ell \geq 1$, we have*

$$\frac{1}{M(\alpha)} \leq \mathfrak{D}_p^\ell(\alpha) + \frac{\lfloor 1 - \alpha \rfloor}{p^\ell} \leq 1.$$

REMARK 27. *In particular, if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are two sequences taking their values in $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$, there exists a constant $M(\boldsymbol{\alpha}, \boldsymbol{\beta}) > 0$ such that for all p that do not divide $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, all $\ell \in \mathbb{N}$, $\ell \geq 1$, and all $x \in [0, 1/M(\boldsymbol{\alpha}, \boldsymbol{\beta}))$, we have $\Delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{p,\ell}(x) = 0$.*

PROOF. Set $a := p^\ell \mathfrak{D}_p^\ell(\alpha) - \alpha \in \{0, \dots, p^\ell - 1\}$. We have

$$\mathfrak{D}_p^\ell(\alpha) + \frac{\lfloor 1 - \alpha \rfloor}{p^\ell} = \frac{a}{p^\ell} + \frac{\langle \alpha \rangle}{p^\ell} \in (0, 1],$$

because $0 < \langle\alpha\rangle \leq 1$. By Lemma 23, if $p^\ell \geq d(\alpha)\big(|\lfloor 1-\alpha\rfloor|+\langle\alpha\rangle\big)$, then $\mathfrak{D}_p^\ell(\alpha) = \mathfrak{D}_p^\ell(\langle\alpha\rangle) \geq 1/d(\langle\alpha\rangle)$ and hence

$$\mathfrak{D}_p^\ell(\alpha) + \frac{\lfloor 1-\alpha\rfloor}{p^\ell} \geq \frac{1}{d(\alpha)} - \frac{|\lfloor 1-\alpha\rfloor|}{p^\ell}$$
$$\geq \frac{1}{d(\alpha)}\left(\frac{\langle\alpha\rangle}{|\lfloor 1-\alpha\rfloor| + \langle\alpha\rangle}\right).$$

This completes the proof of Lemma 26 because there exists only a finite number of pairs (p,ℓ) such that $p^\ell < d(\alpha)\big(|\lfloor 1-\alpha\rfloor| + \langle\alpha\rangle\big)$. □

Finally, our next lemma enables us to connect the functions $\Delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{p,\ell}$ to the values of the functions $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a,\cdot)$. This is useful to decide if $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is N-integral.

Lemma 28. *Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be two sequences taking their values in $\mathbb{Q}\setminus\mathbb{Z}_{\leq 0}$. There exists a constant $\mathcal{N}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ such that, for all elements α and β of the sequence $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$, for all p that do not divide $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and all $\ell\in\mathbb{N}$, $\ell\geq 1$ such that $p^\ell \geq \mathcal{N}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, we have*

$$a\alpha \preceq a\beta \Longleftrightarrow \mathfrak{D}_p^\ell(\alpha) + \frac{\lfloor 1-\alpha\rfloor}{p^\ell} \leq \mathfrak{D}_p^\ell(\beta) + \frac{\lfloor 1-\beta\rfloor}{p^\ell},$$

where $a\in\{1,\dots,d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ satisfies $p^\ell a \equiv 1 \mod d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. Moreover, if the sequences $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ take their values in $(0,1]$, then we can take $\mathcal{N}_{\boldsymbol{\alpha},\boldsymbol{\beta}} = 1$.

Proof. Let p be such that the sequences $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ take their values in $\mathbb{Z}_p$. By Lemma 23, there exists a constant $\mathcal{N}_1$ such that, for all $\ell\in\mathbb{N}$, $\ell\geq 1$ such that $p^\ell\geq\mathcal{N}_1$, and all elements α of $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$, we have $\mathfrak{D}_p^\ell(\alpha) = \mathfrak{D}_p^\ell(\langle\alpha\rangle)$. Moreover, if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ take their values in $(0,1]$, we can take $\mathcal{N}_1 = 1$ because $\alpha = \langle\alpha\rangle$. We set

$$\mathcal{N}_2 := \max\big\{d_{\boldsymbol{\alpha},\boldsymbol{\beta}}|\lfloor 1-\alpha\rfloor - \lfloor 1-\beta\rfloor| \,:\, \alpha,\,\beta \text{ in } \boldsymbol{\alpha} \text{ or } \boldsymbol{\beta}\big\} + 1$$

and $\mathcal{N}_{\boldsymbol{\alpha},\boldsymbol{\beta}} := \max(\mathcal{N}_1,\mathcal{N}_2)$. In particular, if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ take their values in $(0,1]$, then $\mathcal{N}_{\boldsymbol{\alpha},\boldsymbol{\beta}} = 1$. Let $\ell\in\mathbb{N}$, $\ell\geq 1$ be such that $p^\ell\geq\mathcal{N}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and $a\in\{1,\dots,d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ coprime to $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ such that $p^\ell a\equiv 1 \mod d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$.

Let α and β be elements of $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$. We set $k_1 := \lfloor 1-\alpha\rfloor$ and $k_2 := \lfloor 1-\beta\rfloor$. By (5.2), we have $a\langle\alpha\rangle - \mathfrak{D}_p^\ell(\langle\alpha\rangle)\in\mathbb{Z}$. Hence,

$$a\alpha = a\langle\alpha\rangle - ak_1 = \mathfrak{D}_p^\ell(\langle\alpha\rangle) + a\langle\alpha\rangle - \mathfrak{D}_p^\ell(\langle\alpha\rangle) - ak_1,$$

with $\mathfrak{D}_p^\ell(\langle\alpha\rangle)\in(0,1]$ and $a\langle\alpha\rangle-\mathfrak{D}_p^\ell(\langle\alpha\rangle)-ak_1\in\mathbb{Z}$. Moreover, if $\mathfrak{D}_p^\ell(\langle\alpha\rangle)=\mathfrak{D}_p^\ell(\langle\beta\rangle)$, then still by (5.2), we have $\langle\alpha\rangle=\langle\beta\rangle$. By definition of the total order $\prec$, we obtain

$$\begin{aligned} a\alpha\preceq a\beta &\Longleftrightarrow \mathfrak{D}_p^\ell(\langle\alpha\rangle)<\mathfrak{D}_p^\ell(\langle\beta\rangle)\quad\text{or}\quad\big(\mathfrak{D}_p^\ell(\langle\alpha\rangle)=\mathfrak{D}_p^\ell(\langle\beta\rangle)\quad\text{and}\quad a\alpha\geq a\beta\big)\\ &\Longleftrightarrow \mathfrak{D}_p^\ell(\langle\alpha\rangle)<\mathfrak{D}_p^\ell(\langle\beta\rangle)\quad\text{or}\quad\big(\mathfrak{D}_p^\ell(\langle\alpha\rangle)=\mathfrak{D}_p^\ell(\langle\beta\rangle)\quad\text{and}\quad k_2\geq k_1\big)\\ &\Longleftrightarrow \mathfrak{D}_p^\ell(\langle\alpha\rangle)-\mathfrak{D}_p^\ell(\langle\beta\rangle)\leq\frac{k_2-k_1}{p^\ell} \qquad (5.6)\\ &\Longleftrightarrow \mathfrak{D}_p^\ell(\langle\alpha\rangle)+\frac{k_1}{p^\ell}\leq\mathfrak{D}_p^\ell(\langle\beta\rangle)+\frac{k_2}{p^\ell}\\ &\Longleftrightarrow \mathfrak{D}_p^\ell(\alpha)+\frac{k_1}{p^\ell}\leq\mathfrak{D}_p^\ell(\beta)+\frac{k_2}{p^\ell}, \qquad (5.7)\end{aligned}$$

where in (5.6) we have used the fact that if $\mathfrak{D}_p^\ell(\langle\alpha\rangle)\neq\mathfrak{D}_p^\ell(\langle\beta\rangle)$, then $|\mathfrak{D}_p^\ell(\langle\alpha\rangle)-\mathfrak{D}_p^\ell(\langle\beta\rangle)|\geq 1/d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. The equivalence (5.7) finishes the proof of Lemma 28. □

Proposition 24 shows that the functions $\Delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{p,\ell}$ allow to compute the p-adic valuation of $(\boldsymbol{\alpha})_n/(\boldsymbol{\beta})_n$ ([9]) when p does not divide $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. If $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ have the same number of parameters and if these parameters are in $(0,1]$, the constant $C_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ enables us to get a very convenient formula for the computation of the p-adic valuation of $C_{\boldsymbol{\alpha},\boldsymbol{\beta}}^n(\boldsymbol{\alpha})_n/(\boldsymbol{\beta})_n$ when p divides $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. This formula, stated in the next proposition, is the key to the proof of Theorem 4 and is also used many times in the proof of Theorem 6.

Proposition 29. *Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be two tuples of r parameters in $\mathbb{Q}\cap(0,1]$ such that $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is N-integral. Let p be a prime divisor of $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. We set $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}=p^fD$, $f\geq 1$, with $D\in\mathbb{N}$, D not divisible by p. For all $a\in\{1,\dots,p^f\}$ not divisible by p, and all $\ell\in\mathbb{N}$, $\ell\geq 1$, we choose a prime $p_{a,\ell}$ such that*

$$p_{a,\ell}\equiv p^\ell\mod D\quad\text{and}\quad p_{a,\ell}\equiv a\mod p^f. \qquad (5.8)$$

Then, for all $n\in\mathbb{N}$, we have

$$v_p\left(C_0^n\frac{(\alpha_1)_n\cdots(\alpha_r)_n}{(\beta_1)_n\cdots(\beta_s)_n}\right)=\frac{1}{\varphi(p^f)}\sum_{\substack{a=1\\ \gcd(a,p)=1}}^{p^f}\sum_{\ell=1}^{\infty}\Delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{p_{a,\ell},1}\left(\left\{\frac{n}{p^\ell}\right\}\right)+n\left\{\frac{\lambda_p(\boldsymbol{\alpha},\boldsymbol{\beta})}{p-1}\right\}, \qquad (5.9)$$

where

$$C_0=\frac{\prod_{i=1}^r d(\alpha_i)}{\prod_{j=1}^r d(\beta_j)}\prod_{p\mid d_{\boldsymbol{\alpha},\boldsymbol{\beta}}}p^{-\left\lfloor\frac{\lambda_p(\boldsymbol{\alpha},\boldsymbol{\beta})}{p-1}\right\rfloor}.$$

Proof. We denote by $\widetilde{\boldsymbol{\alpha}}$, respectively $\widetilde{\boldsymbol{\beta}}$, the (possibly empty) sequence of elements of $\boldsymbol{\alpha}$, respectively of $\boldsymbol{\beta}$, whose denominator is not divisible by p. We also set $\lambda_p:=\lambda_p(\boldsymbol{\alpha},\boldsymbol{\beta})$.

[9] For all $\boldsymbol{x}=(x_1,\dots,x_r)\in\mathbb{R}^r$ and all $n\in\mathbb{N}$, we set $(\boldsymbol{x})_n:=(x_1)_n\cdots(x_r)_n$.

For all $n\in\mathbb{N}$, we have

$$v_p\left(C_0^n\frac{(\alpha_1)_n\cdots(\alpha_r)_n}{(\beta_1)_n\cdots(\beta_s)_n}\right)=\sum_{\ell=1}^{\infty}\Delta_{\tilde{\boldsymbol{\alpha}},\tilde{\boldsymbol{\beta}}}^{p,\ell}\left(\left\{\frac{n}{p^\ell}\right\}\right)+\lambda_p v_p(n!)-n\left\lfloor\frac{\lambda_p}{p-1}\right\rfloor. \tag{5.10}$$

Let α be an element of $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$. Let N be the denominator of α. If p does not divide N, then N divides D and, for all $a\in\{1,\dots,p^f\}$, $\gcd(a,p)=1$, and all $\ell\in\mathbb{N}$, $\ell\geq 1$, we have $p_{a,\ell}\equiv p^\ell \mod N$. Hence, $\mathfrak{D}_p^\ell(\alpha)=\mathfrak{D}_{p_{a,\ell}}(\alpha)$ because $\alpha\in(0,1]$.

On the other hand, if p divides N, then for all $n,\ell\in\mathbb{N}$, $\ell\geq 1$, we define $\omega_\ell(\alpha,n)$ as the number of elements $a\in\{1,\dots,p^f\}$, $\gcd(a,p)=1$, such that $\{n/p^\ell\}\geq\mathfrak{D}_{p_{a,\ell}}(\alpha)$. Thus for all $n,\ell\in\mathbb{N}$, $\ell\geq 1$, we get

$$\sum_{\substack{a=1\\ \gcd(a,p)=1}}^{p^f}\Delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{p_{a,\ell},1}\left(\left\{\frac{n}{p^\ell}\right\}\right)=\varphi\big(p^f\big)\Delta_{\tilde{\boldsymbol{\alpha}},\tilde{\boldsymbol{\beta}}}^{p,\ell}\left(\left\{\frac{n}{p^\ell}\right\}\right)+\sum_{\substack{i=1\\ \alpha_i\notin\mathbb{Z}_p}}^{r}\omega_\ell(\alpha_i,n)-\sum_{\substack{j=1\\ \beta_j\notin\mathbb{Z}_p}}^{r}\omega_\ell(\beta_j,n). \tag{5.11}$$

Let α be an element of $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ such that p divides $d(\alpha)$. We now compute $\sum_{\ell=1}^{\infty}\omega_\ell(\alpha,n)$. Let $\alpha=r/(p^eN)$ where $1\leq e\leq f$, N divides D, $1\leq r\leq p^eN$ and r is coprime to p^eN. Given $\ell\in\mathbb{N}$, $\ell\geq 1$, there exists $r_{a,\ell}\in\{1,\dots,p^eN\}$ coprime to p^eN such that $\mathfrak{D}_{p_{a,\ell}}(\alpha)=r_{a,\ell}/(p^eN)$ and $p_{a,\ell}r_{a,\ell}-r\equiv 0 \mod p^eN$. In particular, by (5.8), we have

$$p^\ell r_{a,\ell}-r\equiv 0 \mod N \quad\text{and}\quad ar_{a,\ell}-r\equiv 0 \mod p^e,$$

i. e.

$$r_{a,\ell}\equiv s_N\left(\frac{\pi_N(r)}{\pi_N(p^{\ell+e})}\right)p^e+s_{p^e}\left(\frac{\pi_{p^e}(r)}{\pi_{p^e}(aN)}\right)N \mod p^eN.$$

In the rest of the proof, if a/b is a rational number written in reduced form and the integer $c\geq 1$ is coprime to b, we set

$$\varpi_c\left(\frac{a}{b}\right):=s_c\left(\frac{\pi_c(a)}{\pi_c(b)}\right).$$

Then,

$$\frac{r_{a,\ell}}{p^eN}\equiv\frac{\varpi_N(r/p^{\ell+e})}{N}+\frac{\varpi_{p^e}\big(r/(aN)\big)}{p^e} \mod 1. \tag{5.12}$$

For all $\ell\in\mathbb{N}$, we have $p^{\ell+1}\varpi_N(r/p^{\ell+1})-p^\ell\varpi_N(r/p^\ell)\equiv 0 \mod N$, hence, since p and N are coprime, we obtain $p\varpi_N(r/p^{\ell+1})-\varpi_N(r/p^\ell)\equiv 0 \mod N$, *i. e.*

$$\mathfrak{D}_p\left(\frac{\varpi_N(r/p^\ell)}{N}\right)=\frac{\varpi_N(r/p^{\ell+1})}{N},$$

yielding

$$\frac{\varpi_N(r/p^{\ell+1})}{N}=\mathfrak{D}_p^{\ell+1}\left(\frac{r}{N}\right).$$

Let $-r/N = \sum_{k=0}^{\infty} a_k p^k$ be the p-adic expansion of $-r/N$. For all $\ell \in \mathbb{N}$, we have

$$p^{\ell+1}\mathfrak{D}_p^{\ell+1}\left(\frac{r}{N}\right) - \frac{r}{N} = \sum_{k=0}^{\ell} a_k p^k$$

and thus

$$\frac{\varpi_N(r/p^{\ell+e})}{N} = \frac{r}{p^{\ell+e}N} + \frac{\sum_{k=0}^{\ell+e-1} a_k p^k}{p^{\ell+e}} = \frac{r}{p^{\ell+e}N} + \frac{\sum_{k=0}^{\ell-1} a_k p^k}{p^{\ell+e}} + \frac{\sum_{k=0}^{e-1} a_{\ell+k} p^k}{p^e}. \tag{5.13}$$

Moreover, $p\varpi_N(r/p) \equiv r \mod N$ but $p\varpi_N(r/p) \neq r$ because r is not divisible by p. Hence, $p\varpi_N(r/p) - r \geq N$ and $a_0 \geq 1$.

The elements of the multiset ([10])

$$\left\{\left\{\varpi_{p^e}\left(\frac{r}{aN}\right) \,:\, 1 \leq a \leq p^f,\, \gcd(a,p) = 1\right\}\right\}$$

are those $b \in \{1, \dots, p^e\}$ not divisible by p, where each b is repeated exactly p^{f-e} times. We fix $\ell \in \mathbb{N}$, $\ell \geq 1$. We have

$$0 < \frac{r}{p^{\ell+e}N} + \frac{\sum_{k=0}^{\ell-1} a_k p^k}{p^{\ell+e}} \leq \frac{1}{p^{\ell+e}} + \frac{p^\ell - 1}{p^{\ell+e}} \leq \frac{1}{p^e} \quad \text{and} \quad \frac{r_{a,\ell}}{p^e N} \in (0,1].$$

By (5.12) et (5.13), the multiset

$$\Phi_\ell(\alpha) := \left\{\left\{\frac{r_{a,\ell}}{p^e N} \,:\, 1 \leq a \leq p^f,\, \gcd(a,p) = 1\right\}\right\}$$

has the elements

$$\eta_{\ell,b} := \frac{r}{p^{\ell+e}N} + \frac{\sum_{k=0}^{\ell-1} a_k p^k}{p^{\ell+e}} + \frac{b}{p^e},$$

where $b = \sum_{k=0}^{e-1} b_k p^k$, $b_k \in \{0, \dots, p-1\}$, $b_0 \neq a_\ell$, and each $\eta_{\ell,b}$ is repeated exactly p^{f-e} times. In the sequel, we fix $n = \sum_{k=0}^{\infty} n_k p^k$ with $n_k \in \{0, \dots, p-1\}$ and, for all $k \geq K$, $n_k = 0$, where $K \in \mathbb{N}$. For all $\ell \in \mathbb{N}$, we let $\Lambda_\ell(\alpha, n) = 1$ if

$$\sum_{k=0}^{\ell-e-1} n_k p^k > \sum_{k=e}^{\ell-1} a_k p^{k-e},$$

and $\Lambda_\ell(\alpha, n) = 0$ otherwise. Let us compute the number $\omega_\ell(\alpha, n)$ of elements in $\Phi_\ell(\alpha)$ which are $\leq \{n/p^\ell\}$.

[10] A multiset is a set where the repetition of elements is permitted. We use $\{\{\cdots\}\}$ to denote multisets.

If $\ell \leq e-1$, then

$$\left\{\frac{n}{p^\ell}\right\} \geq \eta_{\ell,b} \iff \frac{\sum_{k=0}^{\ell-1} n_k p^k}{p^\ell} \geq \frac{r}{p^{\ell+e}N} + \frac{\sum_{k=0}^{\ell-1} a_k p^k}{p^{\ell+e}} + \frac{\sum_{k=0}^{e-1} b_k p^k}{p^e}$$

$$\iff \sum_{k=0}^{\ell-1} n_k p^k \geq \frac{r}{p^e N} + \frac{\sum_{k=0}^{\ell-1} a_k p^k}{p^e} + \frac{\sum_{k=0}^{e-1} b_k p^{k+\ell}}{p^e}$$

$$\iff \sum_{k=0}^{\ell-1} n_k p^k > \sum_{k=0}^{\ell-1} b_{e-\ell+k} p^k,$$

because

$$0 < \frac{r}{p^e N} + \frac{\sum_{k=0}^{\ell-1} a_k p^k}{p^e} + \frac{\sum_{k=0}^{e-\ell-1} b_k p^{k+\ell}}{p^e} \leq \frac{1}{p^e} + \frac{p^\ell - 1}{p^e} + \frac{p^\ell(p^{e-\ell}-1)}{p^e} \leq 1.$$

Thus

$$\omega_\ell(\alpha, n) = \left((p-1)p^{e-\ell-1} \sum_{k=0}^{\ell-1} n_k p^k\right) p^{f-e}.$$

If $\ell \geq e$, then

$$\left\{\frac{n}{p^\ell}\right\} \geq \eta_{\ell,b} \iff \frac{\sum_{k=0}^{\ell-1} n_k p^k}{p^\ell} \geq \frac{r}{p^{\ell+e}N} + \frac{\sum_{k=0}^{\ell-1} a_k p^k}{p^{\ell+e}} + \frac{\sum_{k=0}^{e-1} b_k p^k}{p^e}$$

$$\iff \sum_{k=0}^{\ell-1} n_k p^k \geq \frac{r}{p^e N} + \frac{\sum_{k=0}^{\ell-1} a_k p^k}{p^e} + \sum_{k=0}^{e-1} b_k p^{k+\ell-e}$$

$$\iff \sum_{k=0}^{\ell-1} n_k p^k > \sum_{k=e}^{\ell-1} a_k p^{k-e} + \sum_{k=0}^{e-1} b_k p^{k+\ell-e}, \tag{5.14}$$

because

$$0 < \frac{r}{p^e N} + \frac{\sum_{k=0}^{e-1} a_k p^k}{p^e} \leq \frac{1}{p^e} + \frac{p^e - 1}{p^e} \leq 1.$$

If we have

$$\sum_{k=\ell-e+1}^{\ell-1} n_k p^k > \sum_{k=1}^{e-1} b_k p^{k+\ell-e},$$

then (5.14) holds and we obtain

$$(p-1)\frac{\sum_{k=\ell-e+1}^{\ell-1} n_k p^k}{p^{\ell-e+1}}$$

numbers b satisfying the above inequality. Let us now assume that

$$\sum_{k=\ell-e+1}^{\ell-1} n_k p^k = \sum_{k=1}^{e-1} b_k p^{k+\ell-e}.$$

Then (5.14) is the same thing as

$$\sum_{k=0}^{\ell-e} n_k p^k > \sum_{k=e}^{\ell-1} a_k p^{k-e} + b_0 p^{\ell-e}. \tag{5.15}$$

If $n_{\ell-e} \geq a_\ell + 1$, then there are $n_{\ell-e} - 1$ elements $b_0 \in \{0, \ldots, p-1\} \setminus \{a_\ell\}$ such that $n_{\ell-e} > b_0$, and, for $b_0 = n_{\ell-e}$, we have (5.15) if and only if $\Lambda_\ell(\alpha, n) = 1$. Moreover, when $n_{\ell-e} \geq a_\ell + 1$, we have $\Lambda_{\ell+1}(\alpha, n) = 1$. Hence, if $n_{\ell-e} \geq a_\ell + 1$, we have $n_{\ell-e} + \Lambda_\ell(\alpha, n) - \Lambda_{\ell+1}(\alpha, n)$ numbers b_0 such that (5.15) holds.

If $n_{\ell-e} = a_\ell$, then there are $n_{\ell-e}$ numbers b_0 such that (5.15) holds. Furthermore, we have $\Lambda_\ell(\alpha, n) = \Lambda_{\ell+1}(\alpha, n)$ and in this case we also have $n_{\ell-e} + \Lambda_\ell(\alpha, n) - \Lambda_{\ell+1}(\alpha, n)$ numbers b_0 such that (5.15) holds.

If $n_{\ell-e} \leq a_\ell - 1$, then there are $n_{\ell-e}$ numbers b_0 such that $b_0 < n_{\ell-e}$, and for $b_0 = n_{\ell-e}$, we have (5.15) if and only if $\Lambda_\ell(\alpha, n) = 1$. Moreover, if $n_{\ell-e} \leq a_\ell - 1$, then $\Lambda_{\ell+1}(\alpha, n) = 0$ and again there are $n_{\ell-e} + \Lambda_\ell(\alpha, n) - \Lambda_{\ell+1}(\alpha, n)$ numbers b_0 satisfying (5.15).

It follows that, if $\ell \geq e$, then,

$$\omega_\ell(\alpha, n) = \left(n_{\ell-e} + \Lambda_\ell(\alpha, n) - \Lambda_{\ell+1}(\alpha, n) + (p-1) \sum_{k=\ell-e+1}^{\ell-1} n_k p^{k-\ell+e-1} \right) p^{f-e}.$$

Hence, for all $m \in \mathbb{N}$, $m \geq K + e$, we get

$$\begin{aligned} p^{e-f} \sum_{\ell=1}^{m} \omega_\ell(\alpha, n) &= (p-1) \sum_{\ell=1}^{e-1} p^{e-\ell-1} \sum_{k=0}^{\ell-1} n_k p^k \\ &\quad + \sum_{\ell=e}^{m} \big(n_{\ell-e} + \Lambda_\ell(\alpha, n) - \Lambda_{\ell+1}(\alpha, n) \big) + (p-1) \sum_{\ell=e}^{m} \sum_{k=\ell-e+1}^{\ell-1} n_k p^{k-\ell+e-1}. \end{aligned} \tag{5.16}$$

Let us compute the coefficients h_k of n_k, $0 \leq k \leq K$, on the right hand side of (5.16), so that

$$p^{e-f} \sum_{\ell=1}^{m} \omega_\ell(\alpha, n) = \Lambda_e(\alpha, n) - \Lambda_{m+1}(\alpha, n) + \sum_{k=0}^{K} h_k n_k. \tag{5.17}$$

If $e = 1$, then for all $k \in \{0, \ldots, K\}$, we have $h_k = 1 = p^{e-1}$. Let us assume that $e \geq 2$. We have

$$h_0 = (p-1) \sum_{\ell=1}^{e-1} p^{e-\ell-1} + 1 = p^{e-1}.$$

If $1 \leq k \leq e-2$, then

$$h_k = (p-1) \sum_{\ell=k+1}^{e-1} p^{k-\ell+e-1} + 1 + (p-1) \sum_{\ell=e}^{k+e-1} p^{k-\ell+e-1} = p^{e-1} - p^k + 1 + p^k - 1 = p^{e-1}.$$

Finally, if $k \geq e-1$, then

$$h_k = 1 + (p-1) \sum_{\ell=k+1}^{k+e-1} p^{k-\ell+e-1} = 1 + p^{e-1} - 1 = p^{e-1}.$$

Hence, we obtain

$$p^{e-f} \sum_{\ell=1}^{m} \omega_\ell(\alpha, n) = \Lambda_e(\alpha, n) - \Lambda_{m+1}(\alpha, n) + p^{e-1}\mathfrak{s}_p(n),$$

where $\mathfrak{s}_p(n) := \sum_{k=0}^{\infty} n_k = \sum_{k=0}^{K} n_k$.

Moreover, we have $\Lambda_e(\alpha, n) = 0$ and there exists $K' \geq K + e$ such that, for all $m \geq K'$, we have $\Lambda_{m+1}(\alpha, n) = 0$. Indeed, $\sum_{k=0}^{\infty} a_k p^k$ is the p-adic expansion of $-r/N \notin \mathbb{N}$. Thus, there exists $K' \geq K + e$ such that $a_{K'} \neq 0$ and hence, for all $m \geq K'$, we have $\Lambda_{m+1}(\alpha, n) = 0$. Consequently, for all large enough ℓ, we have $\omega_\ell(\alpha, n) = 0$ and

$$\sum_{\ell=1}^{\infty} \omega_\ell(\alpha, n) = \varphi\big(p^f\big) \frac{\mathfrak{s}_p(n)}{p-1}. \tag{5.18}$$

By (5.11) and (5.18), we obtain, for all $n \in \mathbb{N}$,

$$\sum_{\ell=1}^{\infty} \sum_{\substack{a=1 \\ \gcd(a,p)=1}}^{p^f} \Delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{p_{a,\ell},1}\left(\left\{\frac{n}{p^\ell}\right\}\right) = \varphi\big(p^f\big) \sum_{\ell=1}^{\infty} \Delta_{\tilde{\boldsymbol{\alpha}},\tilde{\boldsymbol{\beta}}}^{p,\ell}\left(\left\{\frac{n}{p^\ell}\right\}\right) + (r - s - \lambda_p)\varphi\big(p^f\big)\frac{\mathfrak{s}_p(n)}{p-1}.$$

Together with (5.10), this implies that

$$v_p\left(C_0^n \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n}\right) = \frac{1}{\varphi\big(p^f\big)} \sum_{\ell=1}^{\infty} \sum_{\substack{a=1 \\ \gcd(a,p)=1}}^{p^f} \Delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{p_{a,\ell},1}\left(\left\{\frac{n}{p^\ell}\right\}\right) + \lambda_p\left(\frac{\mathfrak{s}_p(n)}{p-1} + v_p(n!)\right) - n\left\lfloor\frac{\lambda_p}{p-1}\right\rfloor. \tag{5.19}$$

But for all $n \in \mathbb{N}$, we have

$$v_p(n!) = \frac{n - \mathfrak{s}_p(n)}{p-1},$$

so that, for all $n \in \mathbb{N}$,

$$\lambda_p\left(\frac{\mathfrak{s}_p(n)}{p-1} + v_p(n!)\right) - n\left\lfloor\frac{\lambda_p}{p-1}\right\rfloor = n\left\{\frac{\lambda_p}{p-1}\right\}. \tag{5.20}$$

Hence, using (5.20) in (5.19), we get equation (5.9), which completes the proof of Proposition 29. □

6. Proof of Theorem 4

Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be two sequences taking their values in $\mathbb{Q}\setminus\mathbb{Z}_{\leq 0}$. Let us assume that $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is N-integral. We first prove (2.1).

We fix a prime p. We denote by $\widetilde{\boldsymbol{\alpha}}$, respectively $\widetilde{\boldsymbol{\beta}}$, the (possibly empty) sequence $(\widetilde{\alpha}_1,\dots,\widetilde{\alpha}_u)$, respectively $(\widetilde{\beta}_1,\dots,\widetilde{\beta}_v)$, made from the elements of $\boldsymbol{\alpha}$, respectively of $\boldsymbol{\beta}$, and whose denominator is not divisible by p. In particular, we have $\lambda_p(\boldsymbol{\alpha},\boldsymbol{\beta})=u-v$. By Proposition 24, for all $n\in\mathbb{N}$, we thus have

$$\begin{aligned}v_p\left(\frac{(\alpha_1)_n\cdots(\alpha_r)_n}{(\beta_1)_n\cdots(\beta_s)_n}\right)&=-nv_p\left(\frac{\prod_{i=1}^r d(\alpha_i)}{\prod_{j=1}^s d(\beta_j)}\right)+v_p\left(\frac{(\widetilde{\alpha}_1)_n\cdots(\widetilde{\alpha}_u)_n}{(\widetilde{\beta}_1)_n\cdots(\widetilde{\beta}_v)_n}\right)\\&=-nv_p\left(\frac{\prod_{i=1}^r d(\alpha_i)}{\prod_{j=1}^s d(\beta_j)}\right)+\sum_{\ell=1}^{\infty}\Delta_{\widetilde{\boldsymbol{\alpha}},\widetilde{\boldsymbol{\beta}}}^{p,\ell}\left(\left\{\frac{n}{p^\ell}\right\}\right)+\lambda_p(\boldsymbol{\alpha},\boldsymbol{\beta})v_p(n!).\end{aligned}\tag{6.1}$$

By Lemma 26, there exists a constant $M>0$ such that, for any prime p that does not divide $d_{\widetilde{\boldsymbol{\alpha}},\widetilde{\boldsymbol{\beta}}}$, for any $\ell\in\mathbb{N}$, $\ell\geq 1$, and any $x\in[0,1/M)$, we have $\Delta_{\widetilde{\boldsymbol{\alpha}},\widetilde{\boldsymbol{\beta}}}^{p,\ell}(x)=0$. Hence, for all $n\in\mathbb{N}$, we have

$$-v\big\lfloor\log_p(nM)\big\rfloor\leq\sum_{\ell=1}^{\infty}\Delta_{\widetilde{\boldsymbol{\alpha}},\widetilde{\boldsymbol{\beta}}}^{p,\ell}\left(\left\{\frac{n}{p^\ell}\right\}\right)\leq u\big\lfloor\log_p(nM)\big\rfloor,$$

so that

$$\frac{1}{n}\sum_{\ell=1}^{\infty}\Delta_{\widetilde{\boldsymbol{\alpha}},\widetilde{\boldsymbol{\beta}}}^{p,\ell}\left(\left\{\frac{n}{p^\ell}\right\}\right)\underset{n\to+\infty}{\longrightarrow}0.\tag{6.2}$$

Moreover, for all $n\in\mathbb{N}$, we have $v_p(n!)=\sum_{\ell=1}^{\infty}\lfloor n/p^\ell\rfloor$, hence

$$\sum_{\ell=1}^{\lfloor\log_p(n)\rfloor}\frac{n}{p^\ell}-\big\lfloor\log_p(n)\big\rfloor\leq v_p(n!)\leq\sum_{\ell=1}^{\lfloor\log_p(n)\rfloor}\frac{n}{p^\ell}$$

and

$$\frac{1}{n}v_p(n!)\underset{n\to+\infty}{\longrightarrow}\frac{1}{p-1}.\tag{6.3}$$

We now use (6.2) and (6.3) in (6.1), and we obtain

$$\frac{1}{n}v_p\left(\frac{(\alpha_1)_n\cdots(\alpha_r)_n}{(\beta_1)_n\cdots(\beta_s)_n}\right)\underset{n\to+\infty}{\longrightarrow}-v_p\left(\frac{\prod_{i=1}^r d(\alpha_i)}{\prod_{j=1}^s d(\beta_j)}\right)+\frac{\lambda_p(\boldsymbol{\alpha},\boldsymbol{\beta})}{p-1}.$$

But for all $n\in\mathbb{N}$,

$$C_{\boldsymbol{\alpha},\boldsymbol{\beta}}^n\frac{(\alpha_1)_n\cdots(\alpha_r)_n}{(\beta_1)_n\cdots(\beta_s)_n}\in\mathbb{Z}_p.$$

It follows that, for all $n \in \mathbb{N}$, $n \geq 1$,

$$v_p\big(C_{\boldsymbol{\alpha},\boldsymbol{\beta}}\big) \geq -\frac{1}{n} v_p\left(\frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n}\right) \underset{n \to +\infty}{\longrightarrow} v_p\left(\frac{\prod_{i=1}^r d(\alpha_i)}{\prod_{j=1}^s d(\beta_j)}\right) - \frac{\lambda_p(\boldsymbol{\alpha},\boldsymbol{\beta})}{p-1}$$

and thus

$$v_p\big(C_{\boldsymbol{\alpha},\boldsymbol{\beta}}\big) \geq v_p\left(\frac{\prod_{i=1}^r d(\alpha_i)}{\prod_{j=1}^s d(\beta_j)}\right) - \left\lfloor \frac{\lambda_p(\boldsymbol{\alpha},\boldsymbol{\beta})}{p-1} \right\rfloor,$$

because $v_p\big(C_{\boldsymbol{\alpha},\boldsymbol{\beta}}\big) \in \mathbb{Z}$. Furthermore, if p does not divide $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and if $p \geq r-s+2$, then $\lambda_p(\boldsymbol{\alpha},\boldsymbol{\beta}) = r-s$ and $\lfloor \lambda_p(\boldsymbol{\alpha},\boldsymbol{\beta})/(p-1) \rfloor = 0$. This proves the existence of $C \in \mathbb{N}^*$ such that

$$C_{\boldsymbol{\alpha},\boldsymbol{\beta}} = C \frac{\prod_{i=1}^r d(\alpha_i)}{\prod_{j=1}^s d(\beta_j)} \prod_{p \in \mathcal{P}_{\boldsymbol{\alpha},\boldsymbol{\beta}}} p^{-\left\lfloor \frac{\lambda_p(\boldsymbol{\alpha},\boldsymbol{\beta})}{p-1} \right\rfloor}. \tag{6.4}$$

We now define

$$C_0 := \frac{\prod_{i=1}^r d(\alpha_i)}{\prod_{j=1}^s d(\beta_j)} \prod_{p \mid d_{\boldsymbol{\alpha},\boldsymbol{\beta}}} p^{-\left\lfloor \frac{\lambda_p(\boldsymbol{\alpha},\boldsymbol{\beta})}{p-1} \right\rfloor}.$$

In the sequel, we assume that both sequences $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ take their values in $(0,1]$ and that $r = s$. We show that in this case $C = 1$ and for this it is enough to prove that $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C_0 z) \in \mathbb{Z}[[z]]$.

Consider a prime p that does not divide $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, so that $\lambda_p(\boldsymbol{\alpha},\boldsymbol{\beta}) = r-s = 0$. Together with (6.1), this yields

$$v_p\left(C_0^n \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n}\right) = \sum_{\ell=1}^{\infty} \Delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{p,\ell}\left(\left\{\frac{n}{p^\ell}\right\}\right).$$

By Lemma 28 and Theorem 3, for all $\ell \in \mathbb{N}$, $\ell \geq 1$, we have

$$\Delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{p,\ell}([0,1]) = \xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a,\mathbb{R}) \subset \mathbb{N},$$

where $a \in \{1,\dots,d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ satisfies $p^\ell a \equiv 1 \mod d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. Hence, we obtain that $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C_0 z) \in \mathbb{Z}_p[[z]]$. It remains to show that for any prime p that divides $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, we also have that $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C_0 z) \in \mathbb{Z}_p[[z]]$.

Consider a prime p that divides $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. With the notations of Proposition 29, for all $n \in \mathbb{N}$, we have

$$v_p\left(C_0^n \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n}\right) = \frac{1}{\varphi\big(p^f\big)} \sum_{\substack{a=1 \\ \gcd(a,p)=1}}^{p^f} \sum_{\ell=1}^{\infty} \Delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{p_{a,\ell},1}\left(\left\{\frac{n}{p^\ell}\right\}\right) + n\left\{\frac{\lambda_p(\boldsymbol{\alpha},\boldsymbol{\beta})}{p-1}\right\}.$$

Since none of the primes $p_{a,\ell}$ divides $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, we have $\Delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{p_{a,\ell},1}([0,1]) \subset \mathbb{N}$ so that $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C_0 z) \in \mathbb{Z}_p[[z]]$. This completes the proof of Theorem 4. ∎

7. Formal congruences

To prove Theorem 6, we need a "formal congruences" result, stated in Theorem 30 below that we prove in this section.

We fix a prime p and denote by Ω the completion of the algebraic closure of $\mathbb{Q}_p$, and by $\mathcal{O}$ the ring of integers of Ω.

To state the main result of this section, we introduce some notations. If $\mathcal{N} := (\mathcal{N}_r)_{r\geq 0}$ is a sequence of subsets of $\bigcup_{t\geq 1}\big(\{0,\dots,p^t-1\}\times\{t\}\big)$, then for all $r\in\mathbb{Z}$, $r\geq -1$ and all $s\in\mathbb{N}$, we denote by $\Psi_{\mathcal{N}}(r,s)$ the set of the $u\in\{0,\dots,p^s-1\}$ such that, for all $(n,t)\in\mathcal{N}_{r+s-t+1}$, with $t\leq s$, and all $j\in\{0,\dots,p^{s-t}-1\}$, we have $u\neq j+p^{s-t}n$. In particular, for all $r\geq -1$, we have $\Psi_{\mathcal{N}}(r,0)=\{0\}$.

For completeness, let us recall some basic notions. Let $\mathcal{A}$ be a commutative algebra (with a unit) over a commutative ring (with a unit) $\mathcal{Z}$. An element $a\in\mathcal{A}$ is *regular* if, for all $b\in\mathcal{A}$, we have $(ab=0\Rightarrow b=0)$. We define $\mathcal{S}$ as the set of the regular elements of $\mathcal{A}$. Hence $\mathcal{S}$ is a multiplicative set of $\mathcal{A}$ and the ring $\mathcal{S}^{-1}\mathcal{A}$ with the map

$$\begin{array}{ccc}\mathcal{Z}\times\mathcal{S}^{-1}\mathcal{A} & \longrightarrow & \mathcal{S}^{-1}\mathcal{A}\\ (\lambda\,,\,a/s) & \mapsto & (\lambda\cdot a)/s\end{array}$$

is a $\mathcal{Z}$-algebra. Moreover, the algebra homomorphism $a\in\mathcal{A}\mapsto a/1\in\mathcal{S}^{-1}\mathcal{A}$ is injective and enables us to identify $\mathcal{A}$ with a sub-algebra of $\mathcal{S}^{-1}\mathcal{A}$. This is what we do in the statement of Theorem 30.

THEOREM 30. *Let $\mathcal{Z}$ denote a sub-ring of $\mathcal{O}$ and $\mathcal{A}$ a $\mathcal{Z}$-algebra (commutative with a unit) such that 2 is a regular element of $\mathcal{A}$. We consider a sequence of maps $(\mathbf{A}_r)_{r\geq 0}$ from $\mathbb{N}$ into $\mathcal{S}$, and a sequence of maps $(\mathbf{g}_r)_{r\geq 0}$ from $\mathbb{N}$ into $\mathcal{Z}\setminus\{0\}$. We assume there exists a sequence $\mathcal{N}:=(\mathcal{N}_r)_{r\geq 0}$ of subsets of $\bigcup_{t\geq 1}\big(\{0,\dots,p^t-1\}\times\{t\}\big)$ such that, for all $r\geq 0$, we have the following properties:*

(*i*) *$\mathbf{A}_r(0)$ is invertible in $\mathcal{A}$;*
(*ii*) *for all $m\in\mathbb{N}$, we have $\mathbf{A}_r(m)\in\mathbf{g}_r(m)\mathcal{A}$;*
(*iii*) *for all $s,m\in\mathbb{N}$, we have:*
(*a*) *for all $u\in\Psi_{\mathcal{N}}(r,s)$ and all $v\in\{0,\dots,p-1\}$, we have*

$$\frac{\mathbf{A}_r(v+up+mp^{s+1})}{\mathbf{A}_r(v+up)}-\frac{\mathbf{A}_{r+1}(u+mp^s)}{\mathbf{A}_{r+1}(u)}\in p^{s+1}\frac{\mathbf{g}_{r+s+1}(m)}{\mathbf{A}_r(v+up)}\mathcal{A};$$

(a_1) *moreover, if $v+up\in\Psi_{\mathcal{N}}(r-1,s+1)$, then*

$$\mathbf{g}_r(v+up)\left(\frac{\mathbf{A}_r(v+up+mp^{s+1})}{\mathbf{A}_r(v+up)}-\frac{\mathbf{A}_{r+1}(u+mp^s)}{\mathbf{A}_{r+1}(u)}\right)\in p^{s+1}\mathbf{g}_{r+s+1}(m)\mathcal{A};$$

(a_2) *however, if $v+up\notin\Psi_{\mathcal{N}}(r-1,s+1)$, then*

$$\mathbf{g}_r(v+up)\frac{\mathbf{A}_{r+1}(u+mp^s)}{\mathbf{A}_{r+1}(u)}\in p^{s+1}\mathbf{g}_{r+s+1}(m)\mathcal{A};$$

(*b*) *for all $(n,t)\in\mathcal{N}_r$, we have $\mathbf{g}_r(n+mp^t)\in p^t\mathbf{g}_{r+t}(m)\mathcal{Z}$.*

Then, for all $a \in \{0, \ldots, p-1\}$ and all $m, s, r, K \in \mathbb{N}$, we have

$$S_r(a, K, s, p, m) := \sum_{j=mp^s}^{(m+1)p^s-1} \Big(\mathbf{A}_r\big(a + (K-j)p\big)\mathbf{A}_{r+1}(j) - \mathbf{A}_{r+1}(K-j)\mathbf{A}_r(a + jp)\Big) \in p^{s+1}\mathbf{g}_{r+s+1}(m)\mathcal{A}, \tag{7.1}$$

where $\mathbf{A}_r(n) = 0$ if $n < 0$.

Theorem 30 is a generalization of a result due to Dwork [**12**, Theorem 1.1], first used (in a weaker version [**13**]) to obtain the analytic continuation of certain p-adic functions. Dwork then developed in [**12**] a method to prove the p-adic integrality of the Taylor coefficients of canonical coordinates. This method is the basis of the proofs of the N-integrality of $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$. In the literature, one finds many generalizations of Dwork's formal congruences used to prove the integrality of Taylor coefficients of canonical coordinates with increasing generality (see [**20**], [**8**] and [**10**]).

If we consider only the univariate case, then Theorem 30 encompasses all the analogous results in [**20**] and [**10**]. Its interest is due to the two following improvements.

• Theorem 30 can be applied to $\mathbb{Z}_p$-algebras more "abstract" than $\mathcal{O}$. We use this possibility in this paper, where we consider algebras of functions taking values in $\mathbb{Z}_p$. This improvement enables us to consider the integer $\mathfrak{n}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ in Theorem 12.

• Beside this difference, Theorem 30 is a univariate version of Theorem 4 in [**10**] that allows to consider a set $\mathcal{N}$ that depends on r. This property is crucial when we deal with the case of non R-partitioned tuples $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

There also exist in the literature other types of generalizations of Dwork's formal congruences, such as the truncated version of Ota [**29**] and the recent version of Mellit and Vlasenko [**27**] (applied to constant terms of powers of Laurent polynomials).

7.1. Proof of Theorem 30. For all $s \in \mathbb{N}$, $s \geq 1$, we denote by α_s the following assertion: "For all $a \in \{0, \ldots, p-1\}$, all $u \in \{0, \ldots, s-1\}$, all $m, r \in \mathbb{N}$ and all $K \in \mathbb{Z}$, we have

$$\mathbf{S}_r(a, K, u, p, m) \in p^{u+1}\mathbf{g}_{r+u+1}(m)\mathcal{A}.''$$

For all $s \in \mathbb{N}$, $s \geq 1$, and all $t \in \{0, \ldots, s\}$, we denote by $\beta_{t,s}$ the following assertion: "For all $a \in \{0, \ldots, p-1\}$, all $m, r \in \mathbb{N}$ and all $K \in \mathbb{Z}$, we have

$$\mathbf{S}_r(a, K + mp^s, s, p, m) \equiv \sum_{j \in \Psi_{\mathcal{N}}(r+t, s-t)} \frac{\mathbf{A}_{r+t+1}(j + mp^{s-t})}{\mathbf{A}_{r+t+1}(j)} \mathbf{S}_r(a, K, t, p, j) \mod p^{s+1}\mathbf{g}_{r+s+1}(m)\mathcal{A}.''$$

For all $a \in \{0, \ldots, p-1\}$, all $K \in \mathbb{Z}$ and all $r, j \in \mathbb{N}$, we define

$$\mathbf{U}_r(a, K, p, j) := \mathbf{A}_r\big(a + (K-j)p\big)\mathbf{A}_{r+1}(j) - \mathbf{A}_{r+1}(K-j)\mathbf{A}_r(a + jp).$$

Then, we have

$$\mathbf{S}_r(a,K,s,p,m) = \sum_{j=0}^{p^s-1} \mathbf{U}_r(a,K,p,j+mp^s).$$

We now state four lemmas that will be needed to prove (7.1).

Lemma 31. *Assertion α_1 holds.*

Lemma 32. *For all $s,r,m \in \mathbb{N}$, all $a \in \{0,\dots,p-1\}$, all $j \in \Psi_{\mathcal{N}}(r,s)$ and all $K \in \mathbb{Z}$, we have*

$$\mathbf{U}_r(a,K+mp^s,p,j+mp^s) \equiv \frac{\mathbf{A}_{r+1}(j+mp^s)}{\mathbf{A}_{r+1}(j)} \mathbf{U}_r(a,K,p,j) \mod p^{s+1}\mathbf{g}_{r+s+1}(m)\mathcal{A}.$$

Lemma 33. *For all $s \in \mathbb{N}$, $s \geq 1$, if α_s holds, then, for all $a \in \{0,\dots,p-1\}$, all $K \in \mathbb{Z}$ and all $r,m \in \mathbb{N}$, we have*

$$\mathbf{S}_r(a,K,s,p,m) \equiv \sum_{j\in\Psi_{\mathcal{N}}(r,s)} \mathbf{U}_r(a,K,p,j+mp^s) \mod p^{s+1}\mathbf{g}_{r+s+1}(m)\mathcal{A};$$

Lemma 34. *For all $s \in \mathbb{N}$, $s \geq 1$, all $t \in \{0,\dots,s-1\}$, Assertions α_s and $\beta_{t,s}$ imply Assertion $\beta_{t+1,s}$.*

Before we prove these lemmas, let us check that they imply (7.1). We show that α_s holds for all $s \geq 1$ by induction on s, which gives the conclusion of Theorem 30. By Lemma 31, α_1 holds. Let us assume that α_s holds for some $s \geq 1$. We observe that $\beta_{0,s}$ is the assertion

$$\beta_{0,s} : \mathbf{S}_r(a,K+mp^s,s,p,m) \equiv \sum_{j\in\Psi_{\mathcal{N}}(r,s)} \frac{\mathbf{A}_{r+1}(j+mp^s)}{\mathbf{A}_{r+1}(j)} \mathbf{S}_r(a,K,0,p,j) \mod p^{s+1}\mathbf{g}_{r+s+1}(m)\mathcal{A}.$$

Since $\mathbf{S}_r(a,K,0,p,j) = \mathbf{U}_r(a,K,p,j)$, we have

$$\sum_{j\in\Psi_{\mathcal{N}}(r,s)} \frac{\mathbf{A}_{r+1}(j+mp^s)}{\mathbf{A}_{r+1}(j)} \mathbf{S}_r(a,K,0,p,j) = \sum_{j\in\Psi_{\mathcal{N}}(r,s)} \frac{\mathbf{A}_{r+1}(j+mp^s)}{\mathbf{A}_{r+1}(j)} \mathbf{U}_r(a,K,p,j)$$

and, by Lemma 32, we obtain, modulo $p^{s+1}\mathbf{g}_{r+s+1}(m)\mathcal{A}$, that

$$\begin{aligned}\sum_{j\in\Psi_{\mathcal{N}}(r,s)} \frac{\mathbf{A}_{r+1}(j+mp^s)}{\mathbf{A}_{r+1}(j)} \mathbf{U}_r(a,K,p,j) &\equiv \sum_{j\in\Psi_{\mathcal{N}}(r,s)} \mathbf{U}_r(a,K+mp^s,p,j+mp^s) \\ &\equiv \mathbf{S}_r(a,K+mp^s,s,p,m), \qquad (7.2)\end{aligned}$$

where (7.2) is obtained *via* Lemma 33.

Consequently, Assertion $\beta_{0,s}$ holds. We then obtain the validity of $\beta_{1,s}$ by means of Lemma 34. Iterating Lemma 34, we finally obtain $\beta_{s,s}$ which, modulo $p^{s+1}\mathbf{g}_{r+s+1}(m)\mathcal{A}$, can be written

$$\mathbf{S}_r(a, K+mp^s, s, p, m) \equiv \sum_{j\in\Psi_{\mathcal{N}}(r+s,0)} \frac{\mathbf{A}_{r+s+1}(j+m)}{\mathbf{A}_{r+s+1}(j)} \mathbf{S}_r(a, K, s, p, j)$$
$$\equiv \frac{\mathbf{A}_{r+s+1}(m)}{\mathbf{A}_{r+s+1}(0)} \mathbf{S}_r(a, K, s, p, 0), \tag{7.3}$$

where we have used in (7.3) the fact that $\Psi_{\mathcal{N}}(r+s, 0) = \{0\}$.

Let us now prove that, for all $a \in \{0, \dots, p-1\}$, all $r \in \mathbb{N}$ and all $K \in \mathbb{Z}$, we have $\mathbf{S}_r(a, K, s, p, 0) \in p^{s+1}\mathcal{A}$. For all $N \in \mathbb{Z}$, we denote by P_N the assertion: "For all $a \in \{0, \dots, p-1\}$ and all $r \in \mathbb{N}$, we have $\mathbf{S}_r(a, N, s, p, 0) \in p^{s+1}\mathcal{A}$".

If $N < 0$, then for all $j \in \{0, \dots, p^s - 1\}$, we have $\mathbf{A}_r\big(a + (N-j)p\big) = 0$ and $\mathbf{A}_{r+1}(N-j) = 0$, so that $\mathbf{S}_r(a, N, s, p, 0) = 0 \in p^{s+1}\mathcal{A}$. To find a contradiction, let us assume the existence of a minimal element $N \in \mathbb{N}$ such that P_N does not hold. Consider $m \in \mathbb{N}$, $m \geq 1$, and set $N' := N - mp^s$. Using (7.3) with N' instead of K, we obtain

$$\mathbf{S}_r(a, N, s, p, m) \equiv \frac{\mathbf{A}_{r+s+1}(m)}{\mathbf{A}_{r+s+1}(0)} \mathbf{S}_r(a, N', s, p, 0) \mod p^{s+1}\mathbf{g}_{r+s+1}(m)\mathcal{A}.$$

Since $m \geq 1$, we have $N' < N$, which, by definition of N, yields that $\mathbf{S}_r(a, N', s, p, 0) \in p^{s+1}\mathcal{A}$. By Condition (i), $\mathbf{A}_{r+s+1}(0)$ is an invertible element of $\mathcal{A}$ and thus

$$\mathbf{S}_r(a, N, s, p, m) \in p^{s+1}\mathcal{A}.$$

Hence, for all $m \in \mathbb{N}$, $m \geq 1$, we have $\mathbf{S}_r(a, N, s, p, m) \in p^{s+1}\mathcal{A}$. Consider $T \in \mathbb{N}$ such that $(T+1)p^s > N$. Then,

$$\sum_{m=0}^{T} \mathbf{S}_r(a, N, s, p, m)$$
$$= \sum_{m=0}^{T} \sum_{j=mp^s}^{(m+1)p^s-1} \Big(\mathbf{A}_r\big(a + (N-j)p\big)\mathbf{A}_{r+1}(j) - \mathbf{A}_{r+1}(N-j)\mathbf{A}_r(a+jp)\Big)$$
$$= \sum_{j=0}^{N} \Big(\mathbf{A}_r\big(a + (N-j)p\big)\mathbf{A}_{r+1}(j) - \mathbf{A}_{r+1}(N-j)\mathbf{A}_r(a+jp)\Big) \tag{7.4}$$
$$= 0, \tag{7.5}$$

where we have used in (7.4) the fact that $\mathbf{A}_r(n) = 0$ if $n < 0$. Equation (7.5) holds because 2 is a regular element of $\mathcal{A}$ and the sign of the term of the sum (7.4) changes when we change the index j to $N - j$. It follows that we have

$$\mathbf{S}_r(a, N, s, p, 0) = -\sum_{m=1}^{T} \mathbf{S}_r(a, N, s, p, m) \in p^{s+1}\mathcal{A}.$$

This contradicts the definition of N. Hence, for all $N \in \mathbb{Z}$, P_N holds.

Moreover, Conditions (i) and (ii) respectively imply that $\mathbf{A}_{r+s+1}(0)$ is an invertible element of $\mathcal{A}$ and that $\mathbf{A}_{r+s+1}(m) \in \mathbf{g}_{r+s+1}(m)\mathcal{A}$. By (7.3), we deduce that

$$\mathbf{S}_r(a, K + mp^s, s, p, m) \in p^{s+1}\mathbf{g}_{r+s+1}(m)\mathcal{A}.$$

The latter congruence holds for all $a \in \{0, \ldots, p-1\}$, all $K \in \mathbb{Z}$ and all $m, r \in \mathbb{N}$, which proves that Assertion α_{s+1} holds, and finishes the induction on s. It remains to prove Lemmas 31, 32, 33 and 34.

7.1.1. *Proof of Lemma 31.* Let $a \in \{0, \ldots, p-1\}$, $K \in \mathbb{Z}$ and $m, r \in \mathbb{N}$. We have

$$\mathbf{S}_r(a, K, 0, p, m) = \mathbf{A}_r\big(a + (K-m)p\big)\mathbf{A}_{r+1}(m) - \mathbf{A}_{r+1}(K-m)\mathbf{A}_r(a+mp). \tag{7.6}$$

If $K - m \notin \mathbb{N}$, then $\mathbf{A}_r\big(a+(K-m)p\big) = 0$ and $\mathbf{A}_{r+1}(K-m) = 0$ so that $\mathbf{S}_r(a, K, 0, p, m) = 0 \in p\mathbf{g}_{r+1}(m)\mathcal{A}$, as stated. We may thus assume that $K - m \in \mathbb{N}$. We write (7.6) as follows:

$$\begin{aligned}\mathbf{S}_r(a, K, 0, p, m) = \mathbf{A}_r(a)\Bigg(\mathbf{A}_{r+1}(m)&\left(\frac{\mathbf{A}_r\big(a+(K-m)p\big)}{\mathbf{A}_r(a)} - \frac{\mathbf{A}_{r+1}(K-m)}{\mathbf{A}_{r+1}(0)}\right)\\ &- \mathbf{A}_{r+1}(K-m)\left(\frac{\mathbf{A}_r(a+mp)}{\mathbf{A}_r(a)} - \frac{\mathbf{A}_{r+1}(m)}{\mathbf{A}_{r+1}(0)}\right)\Bigg).\end{aligned} \tag{7.7}$$

Since $\Psi_{\mathcal{N}}(r, 0) = \{0\}$, we can use Hypothesis (a) of Theorem 30 with 0 instead of u, and a instead of v. In this way, we get

$$\frac{\mathbf{A}_r\big(a+(K-m)p\big)}{\mathbf{A}_r(a)} - \frac{\mathbf{A}_{r+1}(K-m)}{\mathbf{A}_{r+1}(0)} \in p\frac{\mathbf{g}_{r+1}(K-m)}{\mathbf{A}_r(a)}\mathcal{A}$$

and

$$\frac{\mathbf{A}_r(a+mp)}{\mathbf{A}_r(a)} - \frac{\mathbf{A}_{r+1}(m)}{\mathbf{A}_{r+1}(0)} \in p\frac{\mathbf{g}_{r+1}(m)}{\mathbf{A}_r(a)}\mathcal{A}.$$

Therefore,

$$\begin{aligned}\mathbf{A}_r(a)\mathbf{A}_{r+1}(m)\left(\frac{\mathbf{A}_r\big(a+(K-m)p\big)}{\mathbf{A}_r(a)} - \frac{\mathbf{A}_{r+1}(K-m)}{\mathbf{A}_{r+1}(0)}\right) &\in p\mathbf{g}_{r+1}(K-m)\mathbf{A}_{r+1}(m)\mathcal{A}\\ &\in p\mathbf{g}_{r+1}(m)\mathcal{A}\end{aligned} \tag{7.8}$$

and

$$\begin{aligned}\mathbf{A}_r(a)\mathbf{A}_{r+1}(K-m)\left(\frac{\mathbf{A}_r(a+mp)}{\mathbf{A}_r(a)} - \frac{\mathbf{A}_{r+1}(m)}{\mathbf{A}_{r+1}(0)}\right) &\in p\mathbf{g}_{r+1}(m)\mathbf{A}_{r+1}(K-m)\mathcal{A}\\ &\in p\mathbf{g}_{r+1}(m)\mathcal{A},\end{aligned} \tag{7.9}$$

where we have used, in (7.8), Condition (ii) that yields $\mathbf{A}_{r+1}(m) \in \mathbf{g}_{r+1}(m)\mathcal{A}$. Using (7.8) and (7.9) in (7.7), we obtain $\mathbf{S}_r(a, K, 0, p, m) \in p\mathbf{g}_{r+1}(m)\mathcal{A}$, as expected.

7.1.2. *Proof of Lemma 32.* We have

$$\mathbf{U}_r(a,K+mp^s,p,j+mp^s)-\frac{\mathbf{A}_{r+1}(j+mp^s)}{\mathbf{A}_{r+1}(j)}\mathbf{U}_r(a,K,p,j)$$
$$=-\mathbf{A}_{r+1}(K-j)\mathbf{A}_r(a+jp)\left(\frac{\mathbf{A}_r(a+jp+mp^{s+1})}{\mathbf{A}_r(a+jp)}-\frac{\mathbf{A}_{r+1}(j+mp^s)}{\mathbf{A}_{r+1}(j)}\right). \quad (7.10)$$

Since $j\in\Psi_{\mathcal{N}}(r,s)$, Hypothesis (a) implies that the right hand side of (7.10) is in

$$\mathbf{A}_{r+1}(K-j)\mathbf{A}_r(a+jp)p^{s+1}\frac{\mathbf{g}_{r+s+1}(m)}{\mathbf{A}_r(a+jp)}\mathcal{A}.$$

These estimates show that the left hand side of (7.10) is in $p^{s+1}\mathbf{g}_{r+s+1}(m)\mathcal{A}$, which concludes the proof of the lemma.

7.1.3. *Proof of Lemma 33.* We consider $s\in\mathbb{N}$, $s\geq 1$, such that α_s holds. We fix $r\in\mathbb{N}$. If $\Psi_{\mathcal{N}}(r,s)=\{0,\dots,p^s-1\}$, Lemma 33 is trivial. In the sequel, we assume that $\Psi_{\mathcal{N}}(r,s)\neq\{0,\dots,p^s-1\}$.

We have $u\in\{0,\dots,p^s-1\}\setminus\Psi_{\mathcal{N}}(r,s)$ if and only if there exist $(n,t)\in\mathcal{N}_{r+s-t+1}$, $t\leq s$, and $j\in\{0,\dots,p^{s-t}-1\}$ such that $u=j+p^{s-t}n$. We denote by $\mathcal{M}$ the set of the $(n,t)\in\mathcal{N}_{r+s-t+1}$ with $t\leq s$. We thus have

$$\{0,\dots,p^s-1\}\setminus\Psi_{\mathcal{N}}(r,s)=\bigcup_{(n,t)\in\mathcal{M}}\big\{j+p^{s-t}n\,:\,0\leq j\leq p^{s-t}-1\big\}.$$

In particular, the set $\mathcal{M}$ is nonempty.

We will show that there exist $k\in\mathbb{N}$, $k\geq 1$, and $(n_1,t_1),\dots,(n_k,t_k)\in\mathcal{M}$ such that the sets

$$J(n_i,t_i):=\big\{j+p^{s-t_i}n_i\,:\,0\leq j\leq p^{s-t_i}-1\big\}$$

form a partition of $\{0,\dots,p^s-1\}\setminus\Psi_{\mathcal{N}}(r,s)$. We observe that

$$\mathcal{M}\subset\bigcup_{t=1}^{s}\Big(\big\{0,\dots,p^t-1\big\}\times\big\{t\big\}\Big)$$

and thus $\mathcal{M}$ is finite. Hence, it is enough to show that if $(n,t),(n',t')\in\mathcal{M}$, $j\in\{0,\dots,p^{s-t}-1\}$ and $j'\in\{0,\dots,p^{s-t'}-1\}$ satisfy $j+p^{s-t}n=j'+p^{s-t'}n'$, then we have either $J(n,t)\subset J(n',t')$ or $J(n',t')\subset J(n,t)$.

Let us assume, for instance, that $t\leq t'$. Then there exists $j_0\in\{0,\dots,p^{t'-t}-1\}$ such that $j=j'+p^{s-t'}j_0$, so that $p^{s-t'}n'=p^{s-t}n+p^{s-t'}j_0$ and thus $J(n',t')\subset J(n,t)$. Similarly, if $t\geq t'$, then $J(n,t)\subset J(n',t')$. Hence, we obtain

$$\mathbf{S}_r(a,K,s,p,m)=$$
$$\sum_{j\in\Psi_{\mathcal{N}}(r,s)}\mathbf{U}_r(a,K,p,j+mp^s)+\sum_{j\in\{0,\dots,p^s-1\}\setminus\Psi_{\mathcal{N}}(r,s)}\mathbf{U}_r(a,K,p,j+mp^s), \quad (7.11)$$

where

$$\sum_{j\in\{0,\dots,p^s-1\}\setminus\Psi_{\mathcal{N}}(r,s)} \mathbf{U}_r(a,K,p,j+mp^s) = \sum_{i=1}^{k} \sum_{j=0}^{p^{s-t_i}-1} \mathbf{U}_r(a,K,p,j+p^{s-t_i}n_i+mp^s). \quad (7.12)$$

We will now prove that, for all $i \in \{1,\dots,k\}$, we have

$$\sum_{j=0}^{p^{s-t_i}-1} \mathbf{U}_r(a,K,p,j+p^{s-t_i}n_i+mp^s) \in p^{s+1}\mathbf{g}_{r+s+1}(m)\mathcal{A}. \quad (7.13)$$

Let $i \in \{1,\dots,k\}$. By definition of $\mathbf{U}_r$, we have

$$\sum_{j=0}^{p^{s-t_i}-1} \mathbf{U}_r(a,K,p,j+p^{s-t_i}n_i+mp^s) = \mathbf{S}_r(a,K,s-t_i,p,n_i+mp^{t_i}).$$

Since $t_i \geq 1$, we get *via* α_s that

$$\mathbf{S}_r(a,K,s-t_i,p,n_i+mp^{t_i}) \in p^{s-t_i+1}\mathbf{g}_{r+s-t_i+1}(n_i+mp^{t_i})\mathcal{A}.$$

We have $(n_i,t_i) \in \mathcal{N}_{r+s-t_i+1}$ and thus we can apply Hypothesis (b) of Theorem 30 with $r+s-t_i+1$ instead of r:

$$p^{s-t_i+1}\mathbf{g}_{r+s-t_i+1}(n_i+mp^{t_i}) \in p^{s-t_i+1}p^{t_i}\mathbf{g}_{r+s+1}(m)\mathcal{Z} = p^{s+1}\mathbf{g}_{r+s+1}(m)\mathcal{Z}.$$

It follows that, for all $i \in \{1,\dots,k\}$, we have (7.13).

Congruence (7.13), together with (7.12) and (7.11), shows that

$$\mathbf{S}_r(a,K,s,p,m) \equiv \sum_{j\in\Psi_{\mathcal{N}}(r,s)} \mathbf{U}_r(a,K,p,j+mp^s) \mod p^{s+1}\mathbf{g}_{r+s+1}(m)\mathcal{A},$$

which completes the proof of Lemma 33.

7.1.4. *Proof of Lemma 34.* In this proof, i is an element of $\{0,\dots,p-1\}$ and u is an element of $\{0,\dots,p^{s-t-1}-1\}$. For $t<s$, we write $\beta_{t,s}$ as

$$\mathbf{S}_r(a,K+mp^s,s,p,m) \equiv \sum_{i+up\in\Psi_{\mathcal{N}}(r+t,s-t)} \frac{\mathbf{A}_{r+t+1}(i+up+mp^{s-t})}{\mathbf{A}_{r+t+1}(i+up)}\mathbf{S}_r(a,K,t,p,i+up) \mod p^{s+1}\mathbf{g}_{r+s+1}(m)\mathcal{A}. \quad (7.14)$$

We want to prove the congruence $\beta_{t+1,s}$, which can be written

$$\mathbf{S}_r(a,K+mp^s,s,p,m) \equiv \sum_{u\in\Psi_{\mathcal{N}}(r+t+1,s-t-1)} \frac{\mathbf{A}_{r+t+2}(u+mp^{s-t-1})}{\mathbf{A}_{r+t+2}(u)}\mathbf{S}_r(a,K,t+1,p,u) \mod p^{s+1}\mathbf{g}_{r+s+1}(m)\mathcal{A}.$$

We see that $\mathbf{S}_r(a,K,t+1,p,u)=\sum_{i=0}^{p-1}\mathbf{S}_r(a,K,t,p,i+up)$. Hence, with

$$X := \mathbf{S}_r(a,K+mp^s,s,p,m) - \sum_{i=0}^{p-1}\sum_{u\in\Psi_{\mathcal{N}}(r+t+1,s-t-1)}\frac{\mathbf{A}_{r+t+2}(u+mp^{s-t-1})}{\mathbf{A}_{r+t+2}(u)}\mathbf{S}_r(a,K,t,p,i+up),$$

it remains to show that $X\in p^{s+1}\mathbf{g}_{r+s+1}(m)\mathcal{A}$. We have

$$i+up\in\Psi_{\mathcal{N}}(r+t,s-t)\Rightarrow u\in\Psi_{\mathcal{N}}(r+t+1,s-t-1). \tag{7.15}$$

Indeed if $u\notin\Psi_{\mathcal{N}}(r+t+1,s-t-1)$, then there exist $(n,k)\in\mathcal{N}_{r+s-k+1}$, $k\leq s-t-1$, and $j\in\{0,\dots,p^{s-t-1-k}-1\}$ such that $u=j+p^{s-t-1-k}n$. Hence, $i+up=i+jp+p^{s-t-k}n$, so that $i+up\notin\Psi_{\mathcal{N}}(r+t,s-t)$. By $\beta_{t,s}$ in the form (7.14) and modulo $p^{s+1}\mathbf{g}_{r+s+1}(m)\mathcal{A}$, we obtain

$$X\equiv\sum_{i+up\in\Psi_{\mathcal{N}}(r+t,s-t)}\mathbf{S}_r(a,K,t,p,i+up)\left(\frac{\mathbf{A}_{r+t+1}(i+up+mp^{s-t})}{\mathbf{A}_{r+t+1}(i+up)}-\frac{\mathbf{A}_{r+t+2}(u+mp^{s-t-1})}{\mathbf{A}_{r+t+2}(u)}\right) - \sum_{\substack{u\in\Psi_{\mathcal{N}}(r+t+1,s-t-1)\\ i+up\notin\Psi_{\mathcal{N}}(r+t,s-t)}}\frac{\mathbf{A}_{r+t+2}(u+mp^{s-t-1})}{\mathbf{A}_{r+t+2}(u)}\mathbf{S}_r(a,K,t,p,i+up).$$

But, by Hypothesis (a_1) of Theorem 30 applied with $s-t-1$ for s and $r+t+1$ for r, we have

$$\mathbf{g}_{r+t+1}(i+up)\left(\frac{\mathbf{A}_{r+t+1}(i+up+mp^{s-t})}{\mathbf{A}_{r+t+1}(i+up)}-\frac{\mathbf{A}_{r+t+2}(u+mp^{s-t-1})}{\mathbf{A}_{r+t+2}(u)}\right)\in p^{s-t}\mathbf{g}_{r+s+1}(m)\mathcal{A}.$$

Moreover, since $t<s$ and α_s holds, we have

$$\mathbf{S}_r(a,K,t,p,i+up)\in p^{t+1}\mathbf{g}_{r+t+1}(i+up)\mathcal{A} \tag{7.16}$$

and, modulo $p^{s+1}\mathbf{g}_{r+s+1}(m)\mathcal{A}$, we obtain

$$X\equiv-\sum_{\substack{u\in\Psi_{\mathcal{N}}(r+t+1,s-t-1)\\ i+up\notin\Psi_{\mathcal{N}}(r+t,s-t)}}\frac{\mathbf{A}_{r+t+2}(u+mp^{s-t-1})}{\mathbf{A}_{r+t+2}(u)}\mathbf{S}_r(a,K,t,p,i+up). \tag{7.17}$$

Finally, when $i+up\notin\Psi_{\mathcal{N}}(r+t,s-t)$, we can apply Condition (a_2) of Theorem 30 with $s-t-1$ for s, i for v and $r+t+1$ for r, so that

$$\mathbf{g}_{r+t+1}(i+up)\frac{\mathbf{A}_{r+t+2}(u+mp^{s-t-1})}{\mathbf{A}_{r+t+2}(u)}\in p^{s-t}\mathbf{g}_{r+s+1}(m)\mathcal{A}. \tag{7.18}$$

Using (7.16) and (7.18) in (7.17), we thus have $X\in p^{s+1}\mathbf{g}_{r+s+1}(m)\mathcal{A}$. This completes the proof of Lemma 34 and consequently that of Theorem 30. □

8. Proof of Theorem 6

The aim of this section if to prove Theorem 6. We will first prove some elementary properties of the algebras of functions $\mathcal{A}_b$ and $\mathcal{A}_b^*$.

8.1. Algebras of functions taking values into $\mathbb{Z}_p$. In the following lemma, we gather a few properties of the algebras $\mathfrak{A}_{p,n}$ and $\mathfrak{A}_{p,n}^*$.

LEMMA 35. *We fix a prime p and $n \in \mathbb{N}$, $n \geq 1$.*

(1) *An element f of $\mathfrak{A}_{p,n}$, respectively of $\mathfrak{A}_{p,n}^*$, is invertible in $\mathfrak{A}_{p,n}$, respectively in $\mathfrak{A}_{p,n}^*$, if and only if $f\big((\mathbb{Z}_p^\times)^n\big) \subset \mathbb{Z}_p^\times$;*

(2) *the algebra $\mathfrak{A}_{p,n}$ contains the rational functions*

$$f: \begin{array}{ccc} (\mathbb{Z}_p^\times)^n & \to & \mathbb{Z}_p \\ (x_1, \dots, x_n) & \mapsto & \frac{P(x_1,\dots,x_n)}{Q(x_1,\dots,x_n)}, \end{array}$$

where $P, Q \in \mathbb{Z}_p[X_1, \dots, X_n]$ and, for all $x_1, \dots, x_n \in \mathbb{Z}_p^\times$, we have $Q(x_1, \dots, x_n) \in \mathbb{Z}_p^\times$;

(3) *if $f \in \mathfrak{A}_{p,n}^\times$ and if $\mathfrak{E}_s$, $s \geq 1$, is the function Euler quotient defined by*

$$\mathfrak{E}_s: \begin{array}{ccc} \mathbb{Z}_p^\times & \to & \mathbb{Z}_p \\ x & \mapsto & \big(x^{\varphi(p^s)} - 1\big)/p^s, \end{array}$$

then we have $\mathfrak{E}_s \circ f \in \mathfrak{A}_{p,n}^*$.

PROOF. Let $f \in \mathfrak{A}_{p,n}$. For f to be invertible in $\mathfrak{A}_{p,n}$, we clearly need that $f\big((\mathbb{Z}_p^\times)^n\big) \subset \mathbb{Z}_p^\times$ and in this case, for all $\mathbf{x} \in (\mathbb{Z}_p^\times)^n$, all $\mathbf{a} \in \mathbb{Z}_p^n$ and all $m \in \mathbb{N}$, $m \geq 1$, we have

$$\frac{1}{f(\mathbf{x} + \mathbf{a}p^m)} = \frac{1}{f(\mathbf{x}) + \eta p^m} = \frac{1}{f(\mathbf{x})} \frac{1}{1 + \frac{\eta}{f(\mathbf{x})} p^m} \equiv \frac{1}{f(\mathbf{x})} \mod p^m\mathbb{Z}_p,$$

because $f(\mathbf{x}) \in \mathbb{Z}_p^\times$, $\eta \in \mathbb{Z}_p$, and $(1 + p^m\mathbb{Z}_p, \times)$ is a group. The case $f \in \mathfrak{A}_{p,n}^*$ being similar, Assertion (1) is proved.

To prove Assertion (2), we apply Assertion (1) because any polynomial function $f : \mathbf{x} \in (\mathbb{Z}_p^\times)^n \mapsto P(\mathbf{x})$, with $P \in \mathbb{Z}_p[X_1, \dots, X_n]$ is in $\mathfrak{A}_{p,n}$.

Let us now prove Assertion (3). For all $s \in \mathbb{N}$, $s \geq 1$, the cardinal of $(\mathbb{Z}_p/p^s\mathbb{Z}_p)^\times$ is $\varphi(p^s)$ because $\mathbb{Z}_p/p^s\mathbb{Z}_p$ is isomorphic to $\mathbb{Z}/p^s\mathbb{Z}$. Hence, for all $x \in \mathbb{Z}_p^\times$, we have $x^{\varphi(p^s)} \equiv 1 \mod p^s\mathbb{Z}_p$ and the function $\mathfrak{E}_s$ is well defined.

We fix $s \in \mathbb{N}$, $s \geq 1$. To prove Assertion (3), it is enough to prove that, for all $x \in \mathbb{Z}_p^\times$, all $a \in \mathbb{Z}_p$ and all $m \in \mathbb{N}$, $m \geq 1$, we have $\mathfrak{E}_s(x + ap^m) \equiv \mathfrak{E}_s(x) \mod p^{m-1}\mathbb{Z}_p$. We have

$$\begin{aligned} (x + ap^m)^{\varphi(p^s)} &= \sum_{k=0}^{\varphi(p^s)} \binom{\varphi(p^s)}{k} \frac{a^k}{x^k} p^{km} x^{\varphi(p^s)} \\ &\equiv x^{\varphi(p^s)} + \sum_{k=1}^{\varphi(p^s)} \binom{\varphi(p^s)}{k} \frac{a^k}{x^k} p^{km} \mod p^{s+m}\mathbb{Z}_p, \end{aligned}$$

because $x^{\varphi(p^s)} \equiv 1 \mod p^s\mathbb{Z}_p$. By a result of Kummer, the p-adic valuation of $\binom{\varphi(p^s)}{k}$ is the number of carries in the addition of k and $\varphi(p^s)-k$ in base p. Let us show that this number is equal to $s-1-v_p(k)$.

Indeed, if $v_p(k)=0$, then this number is $s-1$ because $\varphi(p^s)=(p-1)p^{s-1}$. If $v_p(k)=\alpha\geq 1$, then we write $k=k'p^\alpha$ and $\varphi(p^s)-k=p^\alpha\big((p-1)p^{s-1-\alpha}-k'\big)$ with $v_p(k')=0$, so that the number of carries of the addition of k and $\varphi(p^s)-k$ in base p is the number of carries in the addition of k' and $\varphi(p^{s-\alpha})-k'$, *i. e.* $s-1-\alpha=s-1-v_p(k)$.

In particular, we obtain that, for all $k\geq 1$,

$$v_p\left(\binom{\varphi(p^s)}{k}\frac{a^k}{x^k}p^{km}\right)\geq s+m+(k-1)m-v_p(k)-1\geq s+m-1,$$

hence $(x+ap^m)^{\varphi(p^s)}\equiv x^{\varphi(p^s)} \mod p^{s+m-1}\mathbb{Z}_p$. Consequently, we have $\mathfrak{E}_s(x+ap^m)\equiv\mathfrak{E}_s(x) \mod p^{m-1}\mathbb{Z}_p$, and the proof of Lemma 35 is complete. □

Lemma 36. *Let $\nu, D\in\mathbb{N}$, $D\geq 1$, and $b\in\{1,\dots,D\}$,* $\gcd(b,D)=1$.

1. *We have $\mathcal{A}_b(p^\nu,D)\subset\mathcal{A}_b(p^\nu,D)^*$ and $p\mathcal{A}_b(p^\nu,D)^*\subset\mathcal{A}_b(p^\nu,D)$;*
2. *An element f of $\mathcal{A}_b(p^\nu,D)$, respectively of $\mathcal{A}_b(p^\nu,D)^*$, is invertible in $\mathcal{A}_b(p^\nu,D)$, respectively in $\mathcal{A}_b(p^\nu,D)^*$, if and only if $f\big(\Omega_b(p^\nu,D)\big)\subset\mathbb{Z}_p^\times$;*
3. *Any constant function from $\Omega_b(p^\nu,D)$ into $\mathbb{Z}_p$ is in $\mathcal{A}_b(p^\nu,D)$;*
4. *If $r\in\mathbb{N}$ and $\alpha\in\mathbb{Q}$ satisfy $d(\alpha)=p^\mu D'$, with $1\leq\mu\leq\nu$ and $D'\mid D$, then the map $t\in\Omega_b(p^\nu,D)\mapsto d(\alpha)\langle t^{(r)}\alpha\rangle$ is in $\mathcal{A}_b(p^\nu,D)^\times$;*
5. *If $\alpha\in\mathbb{Q}\cap\mathbb{Z}_p$ and $k\in\mathbb{N}$, then the map $t\in\Omega_b(p^\nu,D)\mapsto\varpi_{p^k}(t\alpha)$ is in $\mathcal{A}_b(p^\nu,D)$;*
6. *If $n\in\mathbb{N}$, $n\geq 1$, $f_1,\dots,f_n\in\mathcal{A}_b(p^\nu,D)^\times$, $g\in\mathfrak{A}_{p,n}$ and $h\in\mathfrak{A}_{p,n}^*$, then $g':=g\circ(f_1,\dots,f_n)\in\mathcal{A}_b(p^\nu,D)$ and $h':=h\circ(f_1,\dots,f_n)\in\mathcal{A}_b(p^\nu,D)^*$. Furthermore if g is invertible in $\mathfrak{A}_{p,n}$, respectively h is invertible in $\mathfrak{A}_{p,n}^*$, then g' is invertible in $\mathcal{A}_b(p^\nu,D)$, respectively h' is invertible in $\mathcal{A}_b(p^\nu,D)^*$;*
7. *If $f\in\mathcal{A}_b(p^\nu,D)$ and $g\in\mathcal{A}_b(p^\nu,D)^*$, then*

$$\sum_{t\in\Omega_b(p^\nu,D)}f(t)\in p^{\nu-1}\mathbb{Z}_p\quad\text{and}\quad\sum_{t\in\Omega_b(p^\nu,D)}g(t)\in p^{\nu-2}\mathbb{Z}_p.$$

Proof. Assertions (1) and (3) are obvious. The proof of Assertion (2) is similar to that of Assertion (2) of Lemma 35.

Let us prove Assertion (4). For all $t\in\Omega_b(p^\nu,D)$, the number $d(\alpha)\langle t^{(r)}\alpha\rangle$ is the numerator of $\langle t^{(r)}\alpha\rangle$ and thus it is in $\mathbb{Z}_p^\times$ because p divides $d(\alpha)$.

Let $\alpha=\kappa/d(\alpha)$, $t_1,t_2\in\Omega_b(p^\nu,D)$ and $m\in\mathbb{N}$, $m\geq 1$ be such that $t_1\equiv t_2 \mod p^m$. Since $t_1\equiv t_2\equiv b \mod D$, we get $t_1^{(r)}\equiv t_2^{(r)} \mod D$.

If $m\geq\mu$, then $t_1^{(r)}\equiv t_2^{(r)} \mod p^\mu$ and the chinese remainder theorem gives $t_1^{(r)}\equiv t_2^{(r)} \mod p^\mu D$. Since $D'\mid D$, we obtain $t_1^{(r)}\kappa\equiv t_2^{(r)}\kappa \mod d(\alpha)$ and thus $d(\alpha)\langle t_1^{(r)}\alpha\rangle = d(\alpha)\langle t_2^{(r)}\alpha\rangle$, as expected.

On the other hand, if $m<\mu$, then $t_1^{(r)}\equiv t_2^{(r)} \mod p^m$. Since $D'\mid D$ and $d(\alpha)\langle t_i^{(r)}\alpha\rangle\equiv t_i^{(r)}\kappa \mod d(\alpha)$ for $i\in\{1,2\}$, we obtain $d(\alpha)\langle t_1^{(r)}\alpha\rangle\equiv d(\alpha)\langle t_2^{(r)}\alpha\rangle \mod p^m$, which proves Assertion (4).

Assertion (5) is obvious and Assertion (6) is a direct consequence of the definitions and of Assertion (2).

Let us prove Assertion (7) by induction on ν in the case $f \in \mathcal{A}_b(p^\nu, D)$. We denote by A_ν the assertion

$$\sum_{t\in\Omega_b(p^\nu,D)} f(t) \in p^{\nu-1}\mathbb{Z}_p.$$

Assertion A_1 trivially holds. Let $\nu \in \mathbb{N}$, $\nu \geq 1$ be such that A_ν holds.

The set $\Omega_b(p^{\nu+1}, D)$ is the set of the $t_{\ell,\nu+1} \in \{1, \dots, p^{\nu+1}D\}$ such that $t_{\ell,\nu+1} \equiv b \mod D$ and $t_{\ell,\nu+1} \equiv \ell \mod p^{\nu+1}$, with $\ell \in \{1, \dots, p^{\nu+1}\}$, $\gcd(\ell, p) = 1$. Let $\ell := u + vp^\nu$ with $u \in \{1, \dots, p^\nu\}$, $\gcd(u, p) = 1$ and $v \in \{0, \dots, p-1\}$. Then, we have $t_{\ell,\nu+1} \equiv u \mod p^\nu$ and by the chinese remainder theorem, we obtain $t_{\ell,\nu+1} \equiv t_{u,\nu} \mod p^\nu D$, so that

$$\begin{aligned}
\sum_{t\in\Omega_b(p^{\nu+1},D)} f(t) = \sum_{\substack{\ell=1\\ \gcd(\ell,p)=1}}^{p^{\nu+1}} f(t_{\ell,\nu+1}) &= \sum_{\substack{u=1\\ \gcd(u,p)=1}}^{p^\nu} \sum_{v=0}^{p-1} f(t_{u+vp^\nu,\nu+1}) \\
&\equiv p \sum_{\substack{u=1\\ \gcd(u,p)=1}}^{p^\nu} f(t_{u,\nu}) \mod p^\nu\mathbb{Z}_p \\
&\equiv p \sum_{t\in\Omega_b(p^\nu,D)} f(t) \mod p^\nu\mathbb{Z}_p \\
&\equiv 0 \mod p^\nu\mathbb{Z}_p,
\end{aligned}$$

by Assertion A_ν. Hence, Assertion $A_{\nu+1}$ holds, which completes the proof of Asssertion (7) when $f \in \mathcal{A}_b(p^\nu, D)$. The case $f \in \mathcal{A}_b(p^\nu, D)^*$ is similar. □

8.2. Proof of Theorem 6. In this section, we fix two r-tuples $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ of elements of $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$. We assume that $\langle\boldsymbol{\alpha}\rangle$ and $\langle\boldsymbol{\beta}\rangle$ are disjoint and that $H_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ holds.

We set $C = C_{\langle\boldsymbol{\alpha}\rangle,\langle\boldsymbol{\beta}\rangle}$, $C' = C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, $\mathfrak{n} = \mathfrak{n}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, $\mathfrak{m} = \mathfrak{m}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and $\lambda_p = \lambda_p(\boldsymbol{\alpha}, \boldsymbol{\beta})$. We write $d_{\boldsymbol{\alpha},\boldsymbol{\beta}} = p^\nu D$ with $\nu \geq 0$ and $\gcd(D, p) = 1$. For all $t \in \{1, \dots, d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ coprime to $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and all $r \in \mathbb{N}$, we recall that $t^{(r)}$ is the unique element in $\{1, \dots, d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ coprime to $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ such that $t^{(r)} \equiv t \mod p^\nu$ and $p^r t^{(r)} \equiv t \mod D$.

We fix $b \in \{1, \dots, D\}$ coprime to D and set $\Omega_b := \Omega_b(p^\nu, D)$, $\mathcal{A}_b := \mathcal{A}_b(p^\nu, D)$, $\mathcal{A}_b^* := \mathcal{A}_b(p^\nu, D)^*$. We recall that, if $\nu = 0$, then $\Omega_b = \{b\}$ and that $\mathcal{A}_b = \mathcal{A}_b^*$ is the algebra of functions from $\{b\}$ into $\mathbb{Z}_p$.

For all $t \in \Omega_b$ and all $r, n \in \mathbb{N}$, we set

$$\mathcal{Q}_{r,t}(n) := (C')^n \frac{(\langle t^{(r)}\boldsymbol{\alpha}\rangle)_n}{(\langle t^{(r)}\boldsymbol{\beta}\rangle)_n} \quad \text{and} \quad \mathcal{Q}_{r,\cdot}(n) := \big(t \in \Omega_b \mapsto \mathcal{Q}_{r,t}(n)\big).$$

For all $c \in \{1, \dots, p^\nu\}$ not divisible by p and all $\ell \in \mathbb{N}$, we fix a prime $p_{c,\ell}$ such that $p_{c,\ell} \equiv p^\ell \mod D$ and $p_{c,\ell} \equiv c \mod p^\nu$. For all $t \in \Omega_b$ and all $r \in \mathbb{N}$, we set

$$\Delta_{r,t}^{c,\ell} := \Delta^{p_{c,\ell},1}_{\langle t^{(r)}\boldsymbol{\alpha}\rangle,\langle t^{(r)}\boldsymbol{\beta}\rangle}.$$

Let $\widetilde{\boldsymbol{\alpha}}$, respectively $\widetilde{\boldsymbol{\beta}}$, be the sequence of elements of $\langle t^{(r)}\boldsymbol{\alpha}\rangle$, respectively of $\langle t^{(r)}\boldsymbol{\beta}\rangle$, whose denominator is not divisible by p. We set $\widetilde{\Delta}_{r,t}^{p,\ell} := \Delta_{\widetilde{\boldsymbol{\alpha}},\widetilde{\boldsymbol{\beta}}}^{p,\ell}$. We gather in the following lemma a few properties of the sequences $\mathcal{Q}_{r,\cdot}$. We set $\iota = 1$ if $\mathfrak{m}$ is odd and if $\boldsymbol{\beta} \notin \mathbb{Z}^r$, and $\iota = 0$ otherwise.

Lemma 37. *For all $n, r \in \mathbb{N}$, there exists $\Lambda_{b,r}(n) \in \mathbb{Z}_p$ such that $\mathcal{Q}_{r,\cdot}(n) \in 2^{\iota n}\Lambda_{b,r}(n)\mathcal{A}_b^\times$, where*

$$v_p\big(\Lambda_{b,r}(n)\big) = \sum_{\ell=1}^{\infty} \widetilde{\Delta}_{r,t}^{p,\ell}\left(\left\{\frac{n}{p^\ell}\right\}\right) - \lambda_p \frac{\mathfrak{s}_p(n)}{p-1} + n\left\{\frac{\lambda_p}{p-1}\right\}$$
$$= \frac{1}{\varphi(p^\nu)} \sum_{\ell=1}^{\infty} \sum_{\substack{c=1\\ \gcd(c,p)=1}}^{p^\nu} \Delta_{r,t}^{c,\ell}\left(\left\{\frac{n}{p^\ell}\right\}\right) + n\left\{\frac{\lambda_p}{p-1}\right\}.$$

If p divides $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, then for all $n, r \in \mathbb{N}$, $n \geq 1$, we have $v_p\big(\Lambda_{b,r}(n)\big) \geq 1$ and if $\boldsymbol{\beta} \in \mathbb{Z}^r$ then

$$v_p\big(\Lambda_{b,r}(n)\big) \geq -\left\lfloor \frac{\lambda_p}{p-1} \right\rfloor.$$

Proof. For all $t \in \Omega_b$, we have $\mathcal{Q}_{r,t}(n) = 2^{\iota n}\Lambda_{b,r}(n)\mathfrak{R}_r(n,t)$ with

$$\Lambda_{b,r}(n) := \left(C\frac{\prod_{\beta_i \notin \mathbb{Z}_p} d(\beta_i)}{\prod_{\alpha_i \notin \mathbb{Z}_p} d(\alpha_i)}\right)^n \frac{\prod_{\alpha_i \in \mathbb{Z}_p}(\langle t^{(r)}\alpha_i\rangle)_n}{\prod_{\beta_i \in \mathbb{Z}_p}(\langle t^{(r)}\beta_i\rangle)_n}$$

and

$$\mathfrak{R}_r(n,t) := \frac{\prod_{\alpha_i \notin \mathbb{Z}_p} d(\alpha_i)^n(\langle t^{(r)}\alpha_i\rangle)_n}{\prod_{\beta_i \notin \mathbb{Z}_p} d(\beta_i)^n(\langle t^{(r)}\beta_i\rangle)_n} = \frac{\prod_{\alpha_i \notin \mathbb{Z}_p}\prod_{k=0}^{n-1}\big(d(\alpha_i)\langle t^{(r)}\alpha_i\rangle + kd(\alpha_i)\big)}{\prod_{\beta_i \notin \mathbb{Z}_p}\prod_{k=0}^{n-1}\big(d(\beta_i)\langle t^{(r)}\beta_i\rangle + kd(\beta_i)\big)}.$$

By Assertions (2) and (4) of Lemma 36, we have $\mathfrak{R}_r(n,\cdot) \in \mathcal{A}_b^\times$. Moreover, if α is a term of the sequences $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ whose denominator is not divisible by p, then $\langle t^{(r)}\alpha\rangle$ depends only of the class of $t^{(r)}$ in $\mathbb{Z}/D\mathbb{Z}$ which is that of $\varpi_D(p^{-r}b)$ when $t \in \Omega_b$. Indeed, if $\langle\alpha\rangle = 1$, then $\langle t^{(r)}\alpha\rangle = 1$ and if $\langle\alpha\rangle = k/N \neq 1$, where N is a divisor of D, then $N\langle t^{(r)}\alpha\rangle = N\{t^{(r)}\langle\alpha\rangle\} = \varpi_N(t^{(r)}k)$. For all $t \in \Omega_b$ and all $r \in \mathbb{N}$, we have $p^r t^{(r)} \equiv b \mod D$, so that $\varpi_N(t^{(r)}k) = \varpi_N(bp^{-r}k)$. It follows that $\Lambda_{b,r}(n)$ depends only on b, r and n. By Proposition 29 and Equation 5.10, we have

$$v_p\big(\Lambda_{b,r}(n)\big) = v_p\left(C^n\frac{(\langle t^{(r)}\boldsymbol{\alpha}\rangle)_n}{(\langle t^{(r)}\boldsymbol{\beta}\rangle)_n}\right) = \sum_{\ell=1}^{\infty} \widetilde{\Delta}_{r,t}^{p,\ell}\left(\left\{\frac{n}{p^\ell}\right\}\right) - \lambda_p\frac{\mathfrak{s}_p(n)}{p-1} + n\left\{\frac{\lambda_p}{p-1}\right\}$$
$$= \frac{1}{\varphi(p^\nu)} \sum_{\ell=1}^{\infty} \sum_{\substack{c=1\\ \gcd(c,p)=1}}^{p^\nu} \Delta_{r,t}^{c,\ell}\left(\left\{\frac{n}{p^\ell}\right\}\right) + n\left\{\frac{\lambda_p}{p-1}\right\}.$$

In the sequel, we assume that p divides $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. Let us now show that, if $n \geq 1$, then $v_p\big(\Lambda_{b,r}(n)\big) \geq 1$. Let α be a term of the sequences $\langle t^{(r)}\boldsymbol{\alpha}\rangle$ or $\langle t^{(r)}\boldsymbol{\beta}\rangle$ whose denominator is divisible by p. By (5.18), the number of elements $\mathfrak{D}_{p_{c,\ell}}(\alpha)$, $\ell \geq 1$, $c \in \{1, \dots, p^\nu\}$, $\gcd(c,p) = 1$, that satisfy $\{n/p^\ell\} \geq \mathfrak{D}_{p_{c,\ell}}(\alpha)$ is equal to $\varphi(p^\nu)\mathfrak{s}_p(n)/(p-1)$. In particular, if $n \geq 1$, then there exist at least one $\ell \geq 1$ and one $c \in \{1, \dots, p^\nu\}$, $\gcd(c,p) = 1$, such that $\{n/p^\ell\} \geq \mathfrak{D}_{p_{c,\ell}}(\alpha)$. Thus, there exists one term $\alpha' \in (0,1)$ of the sequence $\langle t^{(r)}\boldsymbol{\alpha}\rangle$ or $\langle t^{(r)}\boldsymbol{\beta}\rangle$ such that $\Delta_{r,t}^{c,\ell}\big(\{n/p^\ell\}\big) = \Delta_{r,t}^{c,\ell}\big(\mathfrak{D}_{p_{c,\ell}}(\alpha')\big)$.

By Lemma 28, we obtain $\Delta_{r,t}^{c,\ell}\big(\{n/p^\ell\}\big) = \xi_{\langle t^{(r)}\boldsymbol{\alpha}\rangle,\langle t^{(r)}\boldsymbol{\beta}\rangle}(a, a\alpha')$, where $a \in \{1, \dots, d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ satisfies $p_{c,\ell}a \equiv 1 \mod d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. Since $\alpha' \notin \mathbb{Z}$, we have $\min_{\langle t^{(r)}\boldsymbol{\alpha}\rangle,\langle t^{(r)}\boldsymbol{\beta}\rangle}(a) \preceq a\alpha' \prec a$ and by Lemma 18, Assertion $H_{\langle t^{(r)}\boldsymbol{\alpha}\rangle,\langle t^{(r)}\boldsymbol{\beta}\rangle}$ holds, so that $\Delta_{r,t}^{c,\ell}\big(\{n/p^\ell\}\big) \geq 1$. Hence, $v_p\big(\Lambda_{b,r}(n)\big) \geq 1$.

Moreover, if $\boldsymbol{\beta} \in \mathbb{Z}^r$, then $\lambda_p \leq -1$ and the functions $\widetilde{\Delta}_{r,t}^{p,\ell}$ are nonnegative on $[0,1)$. It follows that

$$v_p\big(\Lambda_{b,r}(n)\big) \geq -\lambda_p \frac{\mathfrak{s}_p(n)}{p-1} + n\left\{\frac{\lambda_p}{p-1}\right\} \geq -\frac{\lambda_p}{p-1} + \left\{\frac{\lambda_p}{p-1}\right\} \geq -\left\lfloor \frac{\lambda_p}{p-1}\right\rfloor.$$

This completes the proof of Lemma 37. □

In the sequel, we set $\mathcal{K}_b := \mathcal{A}_b^*$ if p does not divide $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. If p divides $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, we set

$$\mathcal{K}_b := \begin{cases} p^{-1-\lfloor \lambda_p/(p-1)\rfloor}\mathcal{A}_b \text{ if } \boldsymbol{\beta} \in \mathbb{Z}^r; \\ \mathcal{A}_b \text{ if } \boldsymbol{\beta} \notin \mathbb{Z}^r, \mathfrak{m} \text{ is odd and } p = 2; \\ \mathcal{A}_b \text{ if } \boldsymbol{\beta} \notin \mathbb{Z}^r \text{ and } p-1 \nmid \lambda_p; \\ \mathcal{A}_b^* \text{ otherwise.} \end{cases}$$

By Lemma 37, for all $r \in \mathbb{N}$,

$$\big(t \in \Omega_b \mapsto F_{\langle t^{(r)}\boldsymbol{\alpha}\rangle,\langle t^{(r)}\boldsymbol{\beta}\rangle}(C'z)\big) \in 1 + z\mathcal{K}_b[[z]]$$

is an invertible formal power series in $\mathcal{K}_b[[z]]$. Hence, to prove Theorem 6, it is enough to prove that the function

$$t \in \Omega_b \mapsto G_{\langle t^{(1)}\boldsymbol{\alpha}\rangle,\langle t^{(1)}\boldsymbol{\beta}\rangle}(C'z^p)F_{\langle t\boldsymbol{\alpha}\rangle,\langle t\boldsymbol{\beta}\rangle}(C'z) - pG_{\langle t\boldsymbol{\alpha}\rangle,\langle t\boldsymbol{\beta}\rangle}(C'z)F_{\langle t^{(1)}\boldsymbol{\alpha}\rangle,\langle t^{(1)}\boldsymbol{\beta}\rangle}(C'z^p) \tag{8.1}$$

is in $p\mathcal{K}_b[[z]]$.

For all $a \in \{0, \dots, p-1\}$ and all $K \in \mathbb{N}$, the $(a+Kp)$-th coefficient of the formal power series (8.1) is

$$t \in \Omega_b \mapsto \Phi_t(a+Kp) := \sum_{i=1}^r \big(\Phi_{\alpha_i,t}(a+Kp) - \Phi_{\beta_i,t}(a+Kp)\big),$$

where

$$\Phi_{\alpha,t}(a+Kp) := \sum_{j=0}^K \mathcal{Q}_{0,t}(a+jp)\mathcal{Q}_{1,t}(K-j)\big(H_{\langle t^{(1)}\alpha\rangle}(K-j) - pH_{\langle t\alpha\rangle}(a+jp)\big).$$

It is sufficient to show that, for all terms α of the sequences $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, for all $a \in \{0, \dots, p-1\}$ and all $K \in \mathbb{N}$, we have

$$\Phi_{\alpha,\cdot}(a+Kp) \in p\mathcal{K}_b. \tag{8.2}$$

If $a+Kp=0$, then $\Phi_{\alpha,\cdot}(0)$ is obviously the null map. In the sequel, we assume that $a+Kp\neq 0$, so that, for all $j\in\{0,\dots,K\}$, we have $a+jp\geq 1$ or $K-j\geq 1$.

If p divides $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and if α is a term of the sequences $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ whose denominator is divisible by p, then for all $n,r\in\mathbb{N}$ and all $t\in\Omega_b$, we have

$$H_{\langle t^{(r)}\alpha\rangle}(n)=\sum_{k=0}^{n-1}\frac{d(\alpha)}{d(\alpha)(\langle t^{(r)}\alpha\rangle+k)},$$

yielding $\big(t\in\Omega_b\mapsto H_{\langle t^{(r)}\alpha\rangle}(n)\big)\in p\mathcal{A}_b$. By Lemma 37, for all $n,r\in\mathbb{N}$, $n\geq 1$, we have $\mathcal{Q}_{r,\cdot}(n)\in\Lambda_{b,r}(n)\mathcal{A}_b$ with

$$\Lambda_{b,r}(n)\in\begin{cases}p^{-\lfloor\lambda_p/(p-1)\rfloor}\mathbb{Z}_p\text{ if }\boldsymbol{\beta}\in\mathbb{Z}^r;\\ p\mathbb{Z}_p\text{ otherwise.}\end{cases}$$

Hence, we have $\big(t\in\Omega_b\mapsto\Phi_{\alpha,t}(a+Kp)\big)\in p^2\mathcal{K}_b\subset p\mathcal{K}_b$, as expected.

It remains to deal with the case when the denominator of α is not divisible by p. We fix an element $\alpha\in\mathbb{Z}_p$ of the sequences $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ in the proof of (8.2). We recall that $\langle t\alpha\rangle$ is independent of $t\in\Omega_b$ because $\alpha\in\mathbb{Z}_p$. By [**12**, Lemma 4.1], for all $j\in\{0,\dots,K\}$, we have

$$pH_{\langle t\alpha\rangle}(a+jp)\equiv pH_{\langle t\alpha\rangle}(jp)+\frac{\rho(a,\langle t\alpha\rangle)}{\mathfrak{D}_p(\langle t\alpha\rangle)+j}\mod p\mathbb{Z}_p,$$

where we recall that, for all $x\in\mathbb{Q}\cap\mathbb{Z}_p$, we have

$$\rho(a,x)=\begin{cases}0\text{ if }a\leq p\mathfrak{D}_p(x)-x;\\ 1\text{ if }a>p\mathfrak{D}_p(x)-x.\end{cases}.$$

Moreover,

$$H_{\langle t\alpha\rangle}(jp)=\sum_{k=0}^{jp-1}\frac{1}{\langle t\alpha\rangle+k}=\frac{1}{p}\sum_{k=0}^{j-1}\frac{1}{\mathfrak{D}_p(\langle t\alpha\rangle)+k}+\sum_{\substack{i=0\\ i\neq p\mathfrak{D}_p(\langle t\alpha\rangle)-\langle t\alpha\rangle}}^{p-1}\sum_{k=0}^{j-1}\frac{1}{\langle t\alpha\rangle+i+kp},$$

so that $pH_{\langle t\alpha\rangle}(jp)\equiv H_{\mathfrak{D}_p(\langle t\alpha\rangle)}(j)\mod p\mathbb{Z}_p$. Writing $\langle\alpha\rangle=k/N$ as an irreducible fraction, we obtain

$$\mathfrak{D}_p(\langle t\alpha\rangle)=\frac{\varpi_N(Np^{-1}\langle t\alpha\rangle)}{N}=\frac{\varpi_N\big(p^{-1}\varpi_N(bk)\big)}{N}=\frac{\varpi_N(p^{-1}bk)}{N}=\langle t^{(1)}\alpha\rangle. \tag{8.3}$$

Hence,

$$pH_{\langle t\alpha\rangle}(a+jp)\equiv H_{\langle t^{(1)}\alpha\rangle}(j)+\frac{\rho(a,\langle t\alpha\rangle)}{\mathfrak{D}_p(\langle t\alpha\rangle)+j}\mod p\mathbb{Z}_p. \tag{8.4}$$

We now use the following fact, to be proved in Section 8.2.1: for all $j\in\{0,\dots,K\}$, we have

$$\left(t\in\Omega_b\mapsto\frac{\rho(a,\langle t\alpha\rangle)}{\mathfrak{D}_p(\langle t\alpha\rangle)+j}\mathcal{Q}_{0,t}(a+jp)\mathcal{Q}_{1,t}(K-j)\right)\in p\mathcal{K}_b. \tag{8.5}$$

For any $f : t \in \Omega_b \mapsto f(t) \in \mathbb{Q}_p$, any $g : t \in \Omega_b \mapsto g(t) \in \mathbb{Q}_p$ and any ideal I of $\mathcal{A}_b$, the notation $f(t) \equiv g(t) \mod I$ means that $f - g$ belongs to I. Using (8.4) and (8.5) in the definition of $\Phi_{\alpha,\cdot}(a + Kp)$, we obtain

$$\begin{aligned}\Phi_{\alpha,t}(a+Kp) &\equiv \sum_{j=0}^{K} \mathcal{Q}_{0,t}(a+jp)\mathcal{Q}_{1,t}(K-j)\big(H_{\langle t^{(1)}\alpha\rangle}(K-j) - H_{\langle t^{(1)}\alpha\rangle}(j)\big)\\ &\equiv -\sum_{j=0}^{K} H_{\langle t^{(1)}\alpha\rangle}(j)\Big(\mathcal{Q}_{0,t}(a+jp)\mathcal{Q}_{1,t}(K-j) - \mathcal{Q}_{0,t}\big(a+(K-j)p\big)\mathcal{Q}_{1,t}(j)\Big),\end{aligned}$$

modulo $p\mathcal{K}_b$.

8.2.1. *Proof of Equation* (8.5). For this, we prove several results that will be used again in the proof of Theorem 6.

Lemma 38. *Let $a \in \{0, \dots, p-1\}$, $m \in \mathbb{N}$ and $x \in \mathbb{Z}_p \cap \mathbb{Q} \cap (0, 1]$. If $\rho(a, x) = 1$, then for all $\ell \in \big\{1, \dots, 1 + v_p\big(\mathfrak{D}_p(x) + m\big)\big\}$, we have $\big\{(a + mp)/p^\ell\big\} \geq \mathfrak{D}_p^\ell(x)$.*

Proof. We write $m = \sum_{j=0}^{\infty} m_j p^j$ with $m_j \in \{0, \dots, p-1\}$ and we fix some ℓ in $\big\{1, \dots, 1 + v_p\big(\mathfrak{D}_p(x) + m\big)\big\}$. Then,

$$\left\{\frac{a+mp}{p^\ell}\right\} = \frac{a + p\sum_{j=0}^{\ell-2} m_j p^j}{p^\ell}.$$

We have $\mathfrak{D}_p(x) + m \in p^{\ell-1}\mathbb{Z}_p$ and thus

$$\mathfrak{D}_p(x) + m - \sum_{j=\ell-1}^{\infty} m_j p^j \in p^{\ell-1}\mathbb{Z}_p,$$

so that

$$p\mathfrak{D}_p(x) + p\sum_{j=0}^{\ell-2} m_j p^j - p^\ell \mathfrak{D}_p^\ell(x) \in p^\ell \mathbb{Z},$$

because $p\mathfrak{D}_p(x) - p^\ell \mathfrak{D}_p^\ell(x) \in \mathbb{Z}$. We obtain

$$\frac{p\mathfrak{D}_p(x) + p\sum_{j=0}^{\ell-2} m_j p^j}{p^\ell} - \mathfrak{D}_p^\ell(x) \in \mathbb{Z}.$$

Moreover $\mathfrak{D}_p^\ell(x) \in (0, 1]$ and

$$0 < \frac{p\mathfrak{D}_p(x) + p\sum_{j=0}^{\ell-2} m_j p^j}{p^\ell} \leq \frac{p + p(p^{\ell-1} - 1)}{p^\ell} \leq 1,$$

so that

$$\frac{p\mathfrak{D}_p(x) + p\sum_{j=0}^{\ell-2} m_j p^j}{p^\ell} - \mathfrak{D}_p^\ell(x) = 0.$$

We have $\rho(a,x)=1$ hence $a>p\mathfrak{D}_p(x)-x$ *i. e.* $a\geq p\mathfrak{D}_p(x)-x+1$ and $a\geq p\mathfrak{D}_p(x)$. It follows that

$$\frac{a+p\sum_{j=0}^{\ell-2}m_jp^j}{p^\ell}\geq\mathfrak{D}_p^\ell(x).$$

□

For all $c\in\{1,\dots,p^\nu\}$ not divisible by p and all $\ell,r\in\mathbb{N}$, we define $\tau(r,\ell)$ as the smallest of the numbers $\mathfrak{D}_{p_{c,\ell}}\big(\langle t^{(r)}\alpha\rangle\big)$, where α runs through the elements of the sequences $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ whose denominator is not divisible by p. Since $\langle t^{(r)}\alpha\rangle\in\mathbb{Z}_p$, the number $\mathfrak{D}_{p_{c,\ell}}(\langle t^{(r)}\alpha\rangle)$ does not depend on c and thus $\tau(r,\ell)$ neither. Moreover, since $\alpha\in\mathbb{Z}_p$, the rational number $\langle t^{(r)}\alpha\rangle$ does not depend on $t\in\Omega_b$ and thus $\tau(r,\ell)$ neither. We define $\mathbf{1}_{r,\ell}$ as the characteristic function of the interval $\big[\tau(r,\ell),1\big)$. For all $m,r\in\mathbb{N}$, we set

$$\mu_r(m):=\sum_{\ell=1}^{\infty}\mathbf{1}_{r,\ell}\left(\left\{\frac{m}{p^\ell}\right\}\right)\in\mathbb{N}\quad\text{and}\quad g_r(m):=p^{\mu_r(m)}.$$

Similarly, the function g_r does not depend on $t\in\Omega_b$.

LEMMA 39. *Let $r,\ell,n\in\mathbb{N}$, $\ell\geq 1$, be such that $\{n/p^\ell\}\geq\tau(r,\ell)$. Then for all $t\in\Omega_b$ and all $c\in\{1,\dots,p^\nu\}$ not divisible by p, we have $\Delta_{r,t}^{c,\ell}\big(\{n/p^\ell\}\big)\geq 1$. In particular for all $n\in\mathbb{N}$, we have*

$$v_p\big(\Lambda_{b,r}(n)\big)\geq v_p\big(g_r(n)\big)+n\left\{\frac{\lambda_p}{p-1}\right\}.$$

If $\boldsymbol{\beta}\in\mathbb{Z}^r$, then for all $n\in\mathbb{N}$, $n\geq 1$, we have

$$v_p\big(\Lambda_{b,r}(n)\big)\geq v_p\big(g_r(n)\big)-\left\lfloor\frac{\lambda_p}{p-1}\right\rfloor.$$

PROOF. Let $r,\ell,n\in\mathbb{N}$, $\ell\geq 1$, such that $\{n/p^\ell\}\geq\tau(r,\ell)$. Let $c\in\{1,\dots,p^\nu\}$ not divisible by p. There exists an element α_c of the sequences $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ such that $\Delta_{r,t}^{c,\ell}\big(\{n/p^\ell\}\big)=\Delta_{r,t}^{c,\ell}\big(\mathfrak{D}_{p_{c,\ell}}(\langle t^{(r)}\alpha_c\rangle)\big)$ with $\mathfrak{D}_{p_{c,\ell}}(\langle t^{(r)}\alpha_c\rangle)\leq\{n/p^\ell\}<1$. Hence $\langle t^{(r)}\alpha_c\rangle<1$. By Lemma 28, we obtain

$$\Delta_{r,t}^{c,\ell}\left(\left\{\frac{n}{p^\ell}\right\}\right)=\xi_{\langle t^{(r)}\boldsymbol{\alpha}\rangle,\langle t^{(r)}\boldsymbol{\beta}\rangle}\big(a,a\langle t^{(r)}\alpha_c\rangle\big),$$

where $a\in\{1,\dots,d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ satisfies $p_{c,\ell}a\equiv 1\mod d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. We also have $\min_{\langle t^{(r)}\boldsymbol{\alpha}\rangle,\langle t^{(r)}\boldsymbol{\beta}\rangle}(a)\preceq a\langle t^{(r)}\alpha_c\rangle\prec a$ and by Lemma 18, Assertion $H_{\langle t^{(r)}\boldsymbol{\alpha}\rangle,\langle t^{(r)}\boldsymbol{\beta}\rangle}$ holds. Hence, $\Delta_{r,t}^{c,\ell}\big(\{n/p^\ell\}\big)\geq 1$. By Lemma 37, we have

$$v_p\big(\Lambda_{b,r}(n)\big)=\frac{1}{\varphi(p^\nu)}\sum_{\ell=1}^{\infty}\sum_{\substack{c=1\\ \gcd(c,p)=1}}^{p^\nu}\Delta_{r,t}^{c,\ell}\left(\left\{\frac{n}{p^\ell}\right\}\right)+n\left\{\frac{\lambda_p}{p-1}\right\},$$

so that

$$v_p\big(\Lambda_{b,r}(n)\big)\geq v_p\big(g_r(n)\big)+n\left\{\frac{\lambda_p}{p-1}\right\}.$$

Let us now assume that $\boldsymbol{\beta} \in \mathbb{Z}^r$. If we have $1 > \{n/p^\ell\} \geq \tau(r,\ell)$, then there exists an element α of $\boldsymbol{\alpha}$ whose denominator is not divisible by p and such that $\{n/p^\ell\} \geq \mathfrak{D}_{p_{c,\ell}}(\langle t^{(r)}\alpha\rangle)$ for some $c \in \{1,\dots,p^\nu\}$ not divisible by p. The denominator of $\langle t^{(r)}\alpha\rangle$ divides D and $p_{c,\ell} \equiv p^\ell \mod D$ hence we have $\mathfrak{D}_{p_{c,\ell}}(\langle t^{(r)}\alpha\rangle) = \mathfrak{D}_p^\ell(\langle t^{(r)}\alpha\rangle)$, which yields $\widetilde{\Delta}_{r,t}^{p,\ell}\big(\{n/p^\ell\}\big) \geq 1$. By Lemma 37, for all $n \in \mathbb{N}$, $n \geq 1$, we have

$$\begin{aligned} v_p\big(\Lambda_{b,r}(n)\big) &= \sum_{\ell=1}^{\infty} \widetilde{\Delta}_{r,t}^{p,\ell}\left(\left\{\frac{n}{p^\ell}\right\}\right) - \lambda_p \frac{\mathfrak{s}_p(n)}{p-1} + n\left\{\frac{\lambda_p}{p-1}\right\} \\ &\geq \mu_r(n) - \frac{\lambda_p}{p-1} + \left\{\frac{\lambda_p}{p-1}\right\} \geq v_p\big(g_r(n)\big) - \left\lfloor\frac{\lambda_p}{p-1}\right\rfloor, \end{aligned}$$

because $\lambda_p \leq 0$. This proves Lemma 39. □

We are now in position to prove (8.5).

Proof of (8.5). If $\rho(a,\langle t\alpha\rangle) = 0$ then (8.5) holds. We may thus assume that $\rho(a,\langle t\alpha\rangle) = 1$, *i. e.* that $a > p\mathfrak{D}_p(\langle t\alpha\rangle) - \langle t\alpha\rangle$. In particular, we have $\langle t\alpha\rangle < 1$ and $a \geq 1$. For all $j \in \{0,\dots,K\}$, we have $a + jp \geq 1$ hence by Lemma 39,

$$\mathcal{Q}_{0,\cdot}(a+jp) \in g_0(a+jp)\mathcal{K}_b.$$

It follows that it is sufficient to show that

$$\frac{\rho(a,\langle t\alpha\rangle)}{\mathfrak{D}_p(\langle t\alpha\rangle)+j}\, g_0(a+jp) \in p\mathbb{Z}_p. \tag{8.6}$$

By Lemma 38 with $\langle t\alpha\rangle$ instead of x and j instead m, we obtain, for all $j \in \{0,\dots,K\}$ and all $\ell \in \big\{1,\dots,1+v_p\big(\mathfrak{D}_p(\langle t\alpha\rangle)+j\big)\big\}$, that $\big\{(a+jp)/p^\ell\big\} \geq \mathfrak{D}_p^\ell\big(\langle t\alpha\rangle\big) = \mathfrak{D}_{p_{c,\ell}}(\langle t\alpha\rangle)$ because $\langle t\alpha\rangle \in \mathbb{Z}_p$. We obtain $\big\{(a+jp)/p^\ell\big\} \geq \tau(0,\ell)$, thus

$$v_p\big(g_0(a+jp)\big) = \sum_{\ell=1}^{\infty} \mathbf{1}_{r,\ell}\left(\left\{\frac{a+jp}{p^\ell}\right\}\right) \geq v_p\big(\mathfrak{D}_p(\langle t\alpha\rangle)+j\big)+1,$$

and this completes the proof of (8.6) and also that of (8.5). □

8.2.2. *A combinatorial lemma.* We now use a combinatorial identity due to Dwork (see [**12**, Lemma 4.2, p. 308]) that enables us to write

$$\sum_{j=0}^{K} H_{\langle t^{(1)}\alpha\rangle}(j)\Big(\mathcal{Q}_{0,t}(a+jp)\mathcal{Q}_{1,t}(K-j) - \mathcal{Q}_{1,t}(j)\mathcal{Q}_{0,t}\big(a+(K-j)p\big)\Big)$$
$$= \sum_{s=0}^{r}\sum_{m=0}^{p^{r+1-s}-1} W_t(a,K,s,p,m),$$

where r is such that $K < p^r$,

$$W_t(a,K,s,p,m) := \big(H_{\langle t^{(1)}\alpha\rangle}(mp^s) - H_{\langle t^{(1)}\alpha\rangle}(\lfloor m/p\rfloor p^{s+1})\big)S_t(a,K,s,p,m)$$

and

$$S_t(a,K,s,p,m)=\sum_{j=mp^s}^{(m+1)p^s-1}\Big(\mathcal{Q}_{0,t}(a+jp)\mathcal{Q}_{1,t}(K-j)-\mathcal{Q}_{1,t}(j)\mathcal{Q}_{0,t}\big(a+(K-j)p\big)\Big),$$

where, for all $r\in\mathbb{N}$, we set $\mathcal{Q}_{r,t}(n)=0$ if $n<0$. Thus, to complete the proof, it is enough to show that, for all $s,m\in\mathbb{N}$, we have $\big(t\in\Omega_b\mapsto W_t(a,K,s,p,m)\big)\in p\mathcal{K}_b$. If $m=0$, this is obvious. We now assume that $m\geq 1$.

We write $m=k+qp$ with $k\in\{0,\dots,p-1\}$ and $q\in\mathbb{N}$, so that $mp^s=kp^s+qp^{s+1}$ and $\lfloor m/p\rfloor p^{s+1}=qp^{s+1}$. By [**12**, Lemma 4.1], we obtain

$$H_{\langle t^{(1)}\alpha\rangle}(mp^s)-H_{\langle t^{(1)}\alpha\rangle}(\lfloor m/p\rfloor p^{s+1})\equiv\frac{1}{p^{s+1}}\frac{\rho\big(k,\mathfrak{D}_p^s(\langle t^{(1)}\alpha\rangle)\big)}{\mathfrak{D}_p^{s+1}(\langle t^{(1)}\alpha\rangle)+q}\mod\frac{1}{p^s}\mathbb{Z}_p.$$

Let us show that, for all $s,m\in\mathbb{N}$, $m\geq 1$, we have

$$g_{s+1}(m)\frac{\rho\big(k,\mathfrak{D}_p^s(\langle t^{(1)}\alpha\rangle)\big)}{\mathfrak{D}_p^{s+1}(\langle t^{(1)}\alpha\rangle)+q}\in p\mathbb{Z}_p. \tag{8.7}$$

If $\rho\big(k,\mathfrak{D}_p^s(\langle t^{(1)}\alpha\rangle)\big)=0$, this is clear. Let us assume that $\rho\big(k,\mathfrak{D}_p^s(\langle t^{(1)}\alpha\rangle)\big)=1$. Since $\langle t^{(1)}\alpha\rangle\in\mathbb{Z}_p$, Eq. (8.3) yields $\mathfrak{D}_p^s(\langle t^{(1)}\alpha\rangle)=\langle t^{(s+1)}\alpha\rangle$ and $\mathfrak{D}_p^{s+1}(\langle t^{(1)}\alpha\rangle)=\mathfrak{D}_p(\langle t^{(s+1)}\alpha\rangle)$.

Using Lemma 38 with $\langle t^{(s+1)}\alpha\rangle$ for x, k for a and q for m, we get that, for all $\ell\in\big\{1,\dots,1+v_p\big(\mathfrak{D}_p(\langle t^{(s+1)}\alpha\rangle)+q\big)\big\}$, we have $\{m/p^\ell\}\geq\mathfrak{D}_p^\ell(\langle t^{(s+1)}\alpha\rangle)=\mathfrak{D}_{p_{c,\ell}}(\langle t^{(s+1)}\alpha\rangle)$ because $\langle t^{(s+1)}\alpha\rangle\in\mathbb{Z}_p$. We obtain $\{m/p^\ell\}\geq\tau(s+1,\ell)$ and

$$v_p\big(g_{s+1}(m)\big)=\sum_{\ell=1}^{\infty}\mathbf{1}_{s+1,\ell}\left(\left\{\frac{m}{p^\ell}\right\}\right)\geq v_p\big(\mathfrak{D}_p(\langle t^{(s+1)}\alpha\rangle)+q\big)+1,$$

which finishes the proof of (8.7).

By (8.7), for all $s,m\in\mathbb{N}$, $m\geq 1$, we have

$$\big(H_{\langle t^{(1)}\alpha\rangle}(mp^s)-H_{\langle t^{(1)}\alpha\rangle}(\lfloor m/p\rfloor p^{s+1})\big)p^{s+1}g_{s+1}(m)\in p\mathbb{Z}_p.$$

Hence, to complete the proof of Theorem 6, it is enough to show that, for all $s,m\in\mathbb{N}$, $m\geq 1$, we have

$$\big(t\in\Omega_b\mapsto S_t(a,K,s,p,m)\big)\in p^{s+1}g_{s+1}(m)\mathcal{K}_b. \tag{8.8}$$

We do this in the next section.

8.2.3. *Application of Theorem 30.* To prove (8.8), we will use Theorem 30 with the ring $\mathbb{Z}_p$ for $\mathcal{Z}$ and the $\mathbb{Z}_p$-algebra $\mathcal{A}$ defined as follows:

- $\mathcal{A}=\mathcal{A}_b$ if ($\boldsymbol{\beta}\in\mathbb{Z}^r$ or $p-1\nmid\lambda_p$) or if ($p=2$ and $\mathfrak{m}$ is odd);
- $\mathcal{A}=\mathcal{A}_b^*$ otherwise.

A map $f\in\mathcal{A}_b^*$ is regular if and only if, for all $t\in\Omega_b$, we have $f(t)\neq 0$. Moreover, we have $\mathcal{A}_b\subset\mathcal{A}_b^*$.

In particular, by Lemma 37 and Assertion (2) of Lemma 36, for all $r,m\in\mathbb{N}$, the map $\mathcal{Q}_{r,\cdot}(m)$ is a regular element of $\mathcal{A}_b$. In the sequel, for all $r,m\in\mathbb{N}$, we set $\mathbf{A}_r(m):=\mathcal{Q}_{r,\cdot}(m)$ and we define a function $\mathbf{g}_r$ as follows:

- If $\boldsymbol{\beta} \in \mathbb{Z}^r$ and $p \mid d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, then $\mathbf{g}_r(0) = 1$ and $\mathbf{g}_r(m) = \frac{1}{p}\Lambda_{b,r}(m)$ for $m \geq 1$;
- If $\boldsymbol{\beta} \notin \mathbb{Z}^r$ or $p \nmid d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, then $\mathbf{g}_r = g_r$.

We recall that, if $m \geq 1$ and if $p \mid d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, then for all $r \in \mathbb{N}$, we have $\Lambda_{b,r}(m) \in p\mathbb{Z}_p$. Hence, the maps $\mathbf{g}_r$ take their values in $\mathbb{Z}_p$.

We will show in the next sections that the sequences $\big(\mathbf{A}_r\big)_{r\geq 0}$ and $(\mathbf{g}_r)_{r\geq 0}$ satisfy Hypothesis (i), (ii) and (iii) of Theorem 30. Thus, for all $m, s \in \mathbb{N}$, $m \geq 1$, we will obtain that

$$S_{\cdot}(a, K, s, p, m) \in \begin{cases} p^s\Lambda_{b,s+1}(m)\mathcal{A}_b \text{ if } \boldsymbol{\beta} \in \mathbb{Z}^r \text{ and } p \mid d_{\boldsymbol{\alpha},\boldsymbol{\beta}};\\ p^{s+1}g_{s+1}(m)\mathcal{A}_b \text{ if } \boldsymbol{\beta} \notin \mathbb{Z}^r \text{ and } p-1 \nmid \lambda_p;\\ p^{s+1}g_{s+1}(m)\mathcal{A}_b \text{ if } \boldsymbol{\beta} \notin \mathbb{Z}^r,\ p=2 \text{ and } \mathfrak{m} \text{ is odd};\\ p^{s+1}g_{s+1}(m)\mathcal{A}_b^* \text{ otherwise.}\end{cases}$$

because, if $p \nmid d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, then $\mathcal{A}_b = \mathcal{A}_b^*$.

Proceeding in this way, we will obtain (8.8). Indeed, the only nonobvious case is the one for which $\boldsymbol{\beta} \in \mathbb{Z}^r$ and $p \mid d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. But in this case, by Lemma 39, we have

$$p^s\Lambda_{b,s+1}(m)\mathcal{A}_b \in p^{s+1}p^{-1-\left\lfloor\frac{\lambda_p}{p-1}\right\rfloor}g_{s+1}(m)\mathcal{A}_b = p^{s+1}g_{s+1}(m)\mathcal{K}_b.$$

In the next sections, we check the various hypotheses of Theorem 30.

8.2.4. *Verification of Conditions (i) and (ii) of Theorem 30.* For all $r \geq 0$, the map $\mathcal{Q}_{r,\cdot}(0)$ is constant on Ω_b with value 1, and thus it is invertible in $\mathcal{A}_b$.

By Lemmas 37 and 39, for all $m \in \mathbb{N}$, we have $\mathcal{Q}_{r,\cdot}(m) \in g_r(m)\mathcal{A}_b$ and $\mathcal{Q}_{r,\cdot}(m) \in \Lambda_{b,r}(m)\mathcal{A}_b$ so that in all these cases we have $\mathcal{Q}_{r,\cdot}(m) \in \mathbf{g}_r(m)\mathcal{A}_b$. This shows that Conditions (i) and (ii) of Theorem 30 hold.

8.2.5. *Verification of Condition (iii) of Theorem 30.* For all $r \in \mathbb{N}$, we set

$$\mathcal{N}_r := \bigcup_{t\geq 1}\bigg(\Big\{n \in \{0,\dots,p^t-1\} \,:\, \forall \ell \in \{1,\dots,t\},\, \{n/p^\ell\} \geq \tau(r,\ell)\Big\} \times \big\{t\big\}\bigg).$$

We apply Theorem 30 with the sequence $\mathcal{N} := (\mathcal{N}_r)_{r\geq 0}$. We observe that, for all $r, \ell \in \mathbb{N}$, we have $\tau(r,\ell) > 0$ and hence, if $(n,t) \in \mathcal{N}_r$, then $n \geq 1$. Moreover, in the sequel, we will often use the fact that, for all $h \in \mathbb{N}$, all $c \in \{1,\dots,p^\nu\}$ not divisible by p and all $t \in \Omega_b$, we have

$$\tau(r,\ell+h) = \tau(r+h,\ell), \quad \widetilde{\Delta}_{r,t}^{p,\ell+h} = \widetilde{\Delta}_{r+h,t}^{p,\ell} \quad \text{and} \quad \Delta_{r,t}^{c,\ell+h} = \Delta_{r+h,t}^{c,\ell}. \tag{8.9}$$

Indeed, let α be a term of the sequences $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$. Writing $\langle\alpha\rangle = k/N$ as an irreducible fraction, we obtain

$$\mathfrak{D}_{p_{c,\ell+h}}(\langle t^{(r)}\alpha\rangle) = \frac{\varpi_N(p_{c,\ell+h}^{-1}t^{(r)}k)}{N} = \frac{\varpi_N(p_{c,\ell}^{-1}t^{(r+h)}k)}{N} = \mathfrak{D}_{p_{c,\ell}}(\langle t^{(r+h)}\alpha\rangle),$$

so that $\tau(r,\ell+h) = \tau(r+h,\ell)$ and $\Delta_{r,t}^{c,\ell+h} = \Delta_{r+h,t}^{c,\ell}$. Furthermore, if $\alpha \in \mathbb{Z}_p$, then, by (8.3), we have

$$\mathfrak{D}_p^{\ell+h}(\langle t^{(r)}\alpha\rangle) = \mathfrak{D}_p^{\ell}\big(\mathfrak{D}_p^{h}(\langle t^{(r)}\alpha\rangle)\big) = \mathfrak{D}_p^{\ell}(\langle t^{(r+h)}\alpha\rangle),$$

which yields $\widetilde{\Delta}_{r,t}^{p,\ell+h} = \widetilde{\Delta}_{r+h,t}^{p,\ell}$.

8.2.6. *Verification of Condition (b) of Theorem 30.* Let $r, m \in \mathbb{N}$ and $(n,u) \in \mathcal{N}_r$. We want to show that $\mathbf{g}_r(n+mp^u) \in p^u \mathbf{g}_{r+u}(m)\mathbb{Z}_p$. We need to distinguish two cases.

- If $\boldsymbol{\beta} \in \mathbb{Z}^r$ and $p \mid d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, then

$$v_p\big(\Lambda_{b,r}(n+mp^u)\big) = \sum_{\ell=1}^{\infty} \widetilde{\Delta}_{r,t}^{p,\ell}\left(\left\{\frac{n+mp^u}{p^\ell}\right\}\right) - \lambda_p \frac{\mathfrak{s}_p(n+mp^u)}{p-1} + (n+mp^u)\left\{\frac{\lambda_p}{p-1}\right\}$$
$$> \sum_{\ell=1}^{u} \widetilde{\Delta}_{r,t}^{p,\ell}\left(\left\{\frac{n}{p^\ell}\right\}\right) + \sum_{\ell=u+1}^{\infty} \widetilde{\Delta}_{r,t}^{p,\ell}\left(\left\{\frac{n+mp^u}{p^\ell}\right\}\right) - \lambda_p \frac{\mathfrak{s}_p(m)}{p-1} + m\left\{\frac{\lambda_p}{p-1}\right\},$$

because $\lambda_p \leq -1$ and $n \geq 1$. Since $(n,u) \in \mathcal{N}_r$, for all $\ell \in \{1,\dots,u\}$, we have $\{n/p^\ell\} \geq \tau(r,\ell)$ and thus

$$v_p\big(\Lambda_{b,r}(n+mp^u)\big) > u + \sum_{\ell=u+1}^{\infty} \widetilde{\Delta}_{r,t}^{p,\ell}\left(\left\{\frac{n+mp^u}{p^\ell}\right\}\right) - \lambda_p \frac{\mathfrak{s}_p(m)}{p-1} + m\left\{\frac{\lambda_p}{p-1}\right\}.$$

We set $m = \sum_{k=0}^{\infty} m_k p^k$, where $m_k \in \{0,\dots,p-1\}$ is 0 for all but a finite number of k's. For all $\ell \geq u+1$,

$$\left\{\frac{n+mp^u}{p^\ell}\right\} = \frac{n + p^u\left(\sum_{k=0}^{\ell-u-1} m_k p^k\right)}{p^\ell} \geq \frac{p^u\left(\sum_{k=0}^{\ell-u-1} m_k p^k\right)}{p^\ell} = \left\{\frac{m}{p^{\ell-u}}\right\}.$$

Moreover, since $\langle\boldsymbol{\beta}\rangle = (1,\dots,1)$, the map $\widetilde{\Delta}_{r,t}^{p,\ell}$ is nondecreasing on $[0,1)$ and we obtain that

$$v_p\big(\Lambda_{b,r}(n+mp^u)\big) > u + \sum_{\ell=u+1}^{\infty} \widetilde{\Delta}_{r,t}^{p,\ell}\left(\left\{\frac{m}{p^{\ell-u}}\right\}\right) - \lambda_p \frac{\mathfrak{s}_p(m)}{p-1} + m\left\{\frac{\lambda_p}{p-1}\right\}.$$

But we have

$$\sum_{\ell=u+1}^{\infty} \widetilde{\Delta}_{r,t}^{p,\ell}\left(\left\{\frac{m}{p^{\ell-u}}\right\}\right) = \sum_{\ell=1}^{\infty} \widetilde{\Delta}_{r,t}^{p,\ell+u}\left(\left\{\frac{m}{p^{\ell}}\right\}\right) = \sum_{\ell=1}^{\infty} \widetilde{\Delta}_{r+u,t}^{p,\ell}\left(\left\{\frac{m}{p^{\ell}}\right\}\right),$$

which yields $v_p\big(\Lambda_{b,r}(n+mp^u)\big) > u + v_p\big(\Lambda_{b,r+u}(m)\big)$ and thus

$$v_p\big(\Lambda_{b,r}(n+mp^u)\big) \geq u + v_p\big(\Lambda_{b,r+u}(m)\big) + 1.$$

Since $n \geq 1$, we have $\mathbf{g}_r(n+mp^u) = \frac{1}{p}\Lambda_{b,r}(n+mp^u)$ and we obtain

$$v_p\big(\mathbf{g}_r(n+mp^u)\big) \geq u + v_p\big(\Lambda_{b,r+u}(m)\big) \geq u + v_p\big(\mathbf{g}_{r+u}(m)\big),$$

as expected.

• If $\boldsymbol{\beta} \notin \mathbb{Z}^r$ or $p \nmid d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, then we have to show that $g_r(n+mp^u) \in p^u g_{r+u}(m)\mathbb{Z}_p$. We have

$$\begin{aligned} v_p\big(g_r(n+mp^u)\big) &= \sum_{\ell=1}^{\infty} \mathbf{1}_{r,\ell}\left(\left\{\frac{n+mp^u}{p^\ell}\right\}\right) \\ &\geq \sum_{\ell=1}^{u} \mathbf{1}_{r,\ell}\left(\left\{\frac{n}{p^\ell}\right\}\right) + \sum_{\ell=u+1}^{\infty} \mathbf{1}_{r,\ell}\left(\left\{\frac{n+mp^u}{p^\ell}\right\}\right) \\ &\geq u + \sum_{\ell=u+1}^{\infty} \mathbf{1}_{r,\ell}\left(\left\{\frac{n+mp^u}{p^\ell}\right\}\right), \end{aligned} \tag{8.10}$$

because $(n,u) \in \mathcal{N}_r$. Hence, for all $\ell \in \{1,\dots,u\}$, we have $\{n/p^\ell\} \geq \tau(r,\ell)$. Furthermore, for all $h \in \mathbb{N}$, we have $\tau(r,\ell+h) = \tau(r+h,\ell)$ and consequently

$$\sum_{\ell=u+1}^{\infty} \mathbf{1}_{r,\ell}\left(\left\{\frac{n+mp^u}{p^\ell}\right\}\right) \geq \sum_{\ell=u+1}^{\infty} \mathbf{1}_{r,\ell}\left(\left\{\frac{m}{p^{\ell-u}}\right\}\right) = \sum_{\ell=1}^{\infty} \mathbf{1}_{r+u,\ell}\left(\left\{\frac{m}{p^\ell}\right\}\right) = v_p\big(g_{r+u}(m)\big).$$

Together with (8.10), we obtain $g_r(n+mp^u) \in p^u g_{r+u}(m)\mathbb{Z}_p$.

8.2.7. *Verification of Condition (a_2) of Theorem 30.* Let $r,s,m \in \mathbb{N}$, $u \in \Psi_{\mathcal{N}}(r,s)$ and $v \in \{0,\dots,p-1\}$ be such that $v+up \notin \Psi_{\mathcal{N}}(r-1,s+1)$. It is enough to show that

$$\mathbf{g}_r(v+up)\frac{\mathcal{Q}_{r+1,\cdot}(u+mp^s)}{\mathcal{Q}_{r+1,\cdot}(u)} \in p^{s+1}\mathbf{g}_{r+s+1}(m)\mathcal{A}_b. \tag{8.11}$$

We will first provide a few important properties concerning the set $\Psi_{\mathcal{N}}(r,s)$.

Lemma 40. *Let $r \in \mathbb{Z}$, $r \geq -1$ and $s \in \mathbb{N}$. Then $\Psi_{\mathcal{N}}(r,s)$ is the set of the $u \in \{0,\dots,p^s-1\}$ such that $\{u/p^s\} < \tau(r+1,s)$. Moreover, for all $u \in \Psi_{\mathcal{N}}(r,s)$ and all $\ell \geq s$, we have $\{u/p^\ell\} < \tau(r+1,\ell)$ and, for all $m \in \mathbb{N}$, we have*

$$\frac{\mathcal{Q}_{r+1,\cdot}(u+mp^s)}{\mathcal{Q}_{r+1,\cdot}(u)} \in 2^{\iota mp^s} p^{\left\{\frac{\lambda_p}{p-1}\right\}m(p^s-1)}\Lambda_{b,r+s+1}(m)\mathcal{A}_b.$$

By Lemma 40, to show (8.11) and thus to complete the verification of Condition (a_2), it is enough to show that $v_p\big(\mathbf{g}_r(v+up)\big) \geq s+1$.

We have $v+up \notin \Psi_{\mathcal{N}}(r-1,s+1)$, hence there exist $(n,t) \in \mathcal{N}_{r+s-t+1}$, $t \leq s+1$ and $j \in \{0,\dots,p^{s+1-t}-1\}$ such that $v+up = j + p^{s+1-t}n$. Since $u \in \Psi_{\mathcal{N}}(r,s)$, we necessarily have $s+1-t=0$, so that $(v+up, s+1) \in \mathcal{N}_r$, *i. e.*, for all $\ell \in \{1,\dots,s+1\}$, we have $\big\{(v+up)/p^\ell\big\} \geq \tau(r,\ell)$ and thus $g_r(v+up) \in p^{s+1}\mathbb{Z}_p$. Furthermore, if $\boldsymbol{\beta} \in \mathbb{Z}^r$ and $p \mid d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, then, since $v+up \geq 1$, we have $\mathbf{g}_r(v+up) = \frac{1}{p}\Lambda_{b,r}(v+up)\mathbb{Z}_p$ and by Lemma 39, we obtain

$$v_p\big(\mathbf{g}_r(v+up)\big) \geq v_p\big(g_r(v+up)\big) - 1 - \left\lfloor\frac{\lambda_p}{p-1}\right\rfloor \geq s+1,$$

because $\lambda_p \leq -1$. This completes the verification modulo Lemma 40.

Proof of Lemma 40. We first show that $\Psi_{\mathcal{N}}(r,s)$ is the set of the $u\in\{0,\dots,p^s-1\}$ such that $\{u/p^s\}<\tau(r+1,s)$. If $s=0$, then $\Psi_{\mathcal{N}}(r,0)=\{0\}$ and $\tau(r+1,0)>0$, thus this is obvious. We may then assume that $s\geq 1$. Let $u\in\{0,\dots,p^s-1\}$, $u=\sum_{k=0}^{s-1}u_kp^k$, with $u_k\in\{0,\dots,p-1\}$. It is sufficient to prove that the following assertions are equivalent.

(1) We have $\{u/p^s\}\geq\tau(r+1,s)$.
(2) There exist $(n,t)\in\mathcal{N}_{r+s-t+1}, t\leq s$ and $j\in\{0,\dots,p^{s-t}-1\}$ such that $u=j+p^{s-t}n$.

Proof of (2) $\Rightarrow$ (1): we have

$$\left\{\frac{u}{p^s}\right\}=\frac{u}{p^s}=\frac{j+p^{s-t}n}{p^s}\geq\frac{n}{p^t}=\left\{\frac{n}{p^t}\right\}.$$

Moreover, by definition of the sequence $\mathcal{N}$, we have $\{n/p^t\}\geq\tau(r+s-t+1,t)=\tau(r+1,s)$ and hence $\{u/p^s\}\geq\tau(r+1,s)$.

Proof of (1) $\Rightarrow$ (2): for all $s\geq 1$, we denote by $\mathcal{B}_s$ the assertion: "For all $u\in\{0,\dots,p^s-1\}$ and all $r\in\mathbb{Z}$, $r\geq -1$, such that $\{u/p^s\}\geq\tau(r+1,s)$, there exists $i\in\{0,\dots,s-1\}$ such that $\left(\sum_{k=i}^{s-1}u_kp^{k-i},s-i\right)\in\mathcal{N}_{r+i+1}$." It is enough to show by induction on s that, for all $s\geq 1$, $\mathcal{B}_s$ holds.

If $s=1$, then, for all $u\in\{0,\dots,p-1\}$ and all $r\in\mathbb{Z}$, $r\geq -1$, such that $\{u/p\}\geq\tau(r+1,1)$, we have $(u,1)\in\mathcal{N}_{r+1}$. Hence, $\mathcal{B}_1$ holds.

Let $s\geq 2$ be such that $\mathcal{B}_1,\dots,\mathcal{B}_{s-1}$ hold, let $u\in\{0,\dots,p^s-1\}$ and $r\in\mathbb{Z}$, $r\geq -1$, be such that $\{u/p^s\}\geq\tau(r+1,s)$. We further assume that $(u,s)\notin\mathcal{N}_{r+1}$. Hence, there exists $\ell\in\{1,\dots,s\}$ such that

$$a_\ell:=\frac{\sum_{k=0}^{\ell-1}u_kp^k}{p^\ell}=\left\{\frac{u}{p^\ell}\right\}<\tau(r+1,\ell).$$

We necessarily have $\ell\in\{1,\dots,s-1\}$. We write

$$\left\{\frac{u}{p^s}\right\}=\frac{u}{p^s}=\frac{p^\ell a_\ell+p^\ell\sum_{k=\ell}^{s-1}u_kp^{k-\ell}}{p^s}=\frac{a_\ell}{p^{s-\ell}}+\frac{\sum_{k=\ell}^{s-1}u_kp^{k-\ell}}{p^{s-\ell}}.$$

Since $\{u/p^s\}\geq\tau(r+1,s)$, we obtain that

$$\sum_{k=\ell}^{s-1}u_kp^{k-\ell}\geq p^{s-\ell}\tau(r+1,s)-a_\ell>p^{s-\ell}\tau(r+1,s)-\tau(r+1,\ell),$$

so that

$$\sum_{k=\ell}^{s-1}u_kp^{k-\ell}>p^{s-\ell}\tau(r+\ell+1,s-\ell)-\tau(r+\ell+1,0).$$

Let α be an element of the sequences $\widetilde{\boldsymbol{\alpha}}$ or $\widetilde{\boldsymbol{\beta}}$ such that $\tau(r+\ell+1, s-\ell) = \mathfrak{D}_p^{s-\ell}(\langle t^{(r+\ell+1)}\alpha\rangle)$. Then, we have $\tau(r+\ell+1, 0) \leq \langle t^{(r+\ell+1)}\alpha\rangle$ and thus

$$\sum_{k=\ell}^{s-1} u_k p^{k-\ell} > p^{s-\ell}\mathfrak{D}_p^{s-\ell}(\langle t^{(r+\ell+1)}\alpha\rangle) - \langle t^{(r+\ell+1)}\alpha\rangle. \tag{8.12}$$

Both sides of inequality (8.12) are integers, so that

$$\sum_{k=\ell}^{s-1} u_k p^{k-\ell} \geq p^{s-\ell}\mathfrak{D}_p^{s-\ell}(\langle t^{(r+\ell+1)}\alpha\rangle) - \langle t^{(r+\ell+1)}\alpha\rangle + 1 \geq p^{s-\ell}\mathfrak{D}_p^{s-\ell}(\langle t^{(r+\ell+1)}\alpha\rangle).$$

It follows that

$$\frac{\sum_{k=\ell}^{s-1} u_k p^{k-\ell}}{p^{s-\ell}} \geq \mathfrak{D}_p^{s-\ell}(\langle t^{(r+\ell+1)}\alpha\rangle) = \tau(r+\ell+1, s-\ell).$$

By $\mathcal{B}_{s-\ell}$, there exists $i \in \{0, \dots, s-\ell-1\}$ such that $\left(\sum_{k=\ell+i}^{s-1} u_k p^{k-\ell-i}, s-\ell-i\right) \in \mathcal{N}_{r+\ell+i+1}$. Hence there exists $j \in \{\ell, \dots, s-1\}$ such that $\left(\sum_{k=j}^{s-1} u_k p^{k-j}, s-j\right) \in \mathcal{N}_{r+j+1}$, which proves Assertion $\mathcal{B}_s$ and finishes the induction on s.

The equivalence of Assertions (1) and (2) is now proved and we have

$$\Psi_{\mathcal{N}}(r,s) = \left\{u \in \{0, \dots, p^s - 1\} \,:\, \{u/p^s\} < \tau(r+1, s)\right\}.$$

Let $u \in \Psi_{\mathcal{N}}(r,s)$. Let us prove that, for all $\ell \geq s$, we have $\{u/p^\ell\} < \tau(r+1, \ell)$.

To get a contradiction, let us assume that there exists $\ell \geq s$ such that $\{u/p^\ell\} \geq \tau(r+1, \ell)$. Let α be an element of the sequences $\widetilde{\boldsymbol{\alpha}}$ or $\widetilde{\boldsymbol{\beta}}$ such that $\tau(r+1, \ell) = \mathfrak{D}_p^{\ell}(\langle t^{(r+1)}\alpha\rangle)$. We obtain that

$$\begin{aligned}
\left\{\frac{u}{p^s}\right\} = p^{\ell-s}\left\{\frac{u}{p^\ell}\right\} &\geq p^{\ell-s}\mathfrak{D}_p^{\ell}(\langle t^{(r+1)}\alpha\rangle) \\
&\geq p^{\ell-s}\mathfrak{D}_p^{\ell}(\langle t^{(r+1)}\alpha\rangle) - \mathfrak{D}_p^{s}(\langle t^{(r+1)}\alpha\rangle) + \mathfrak{D}_p^{s}(\langle t^{(r+1)}\alpha\rangle) \\
&\geq \mathfrak{D}_p^{s}(\langle t^{(r+1)}\alpha\rangle) \\
&\geq \tau(r+1, s),
\end{aligned}$$

which is a contradiction. Hence, for all $\ell \geq s$, we have $\{u/p^\ell\} < \tau(r+1, \ell)$.

To complete the proof of Lemma 40, it remains to prove that, for all $u \in \Psi_{\mathcal{N}}(r,s)$ and all $m \in \mathbb{N}$, we have

$$\frac{\mathcal{Q}_{r+1,\cdot}(u + mp^s)}{\mathcal{Q}_{r+1,\cdot}(u)} \in 2^{\iota mp^s} p^{\left\{\frac{\lambda_p}{p-1}\right\} m(p^s-1)} \Lambda_{b,r+s+1}(m)\mathcal{A}_b. \tag{8.13}$$

By Lemma 37, we have

$$\frac{\mathcal{Q}_{r+1,\cdot}(u + mp^s)}{\mathcal{Q}_{r+1,\cdot}(u)} \in 2^{\iota mp^s} \frac{\Lambda_{b,r+1}(u + mp^s)}{\Lambda_{b,r+1}(u)} \mathcal{A}_b^{\times},$$

with

$$
\begin{aligned}
&v_p\left(\frac{\Lambda_{b,r+1}(u+mp^s)}{\Lambda_{b,r+1}(u)}\right)\\
&\quad=\sum_{\ell=1}^{\infty}\left(\widetilde{\Delta}_{r+1,t}^{p,\ell}\left(\left\{\frac{u+mp^s}{p^\ell}\right\}\right)-\widetilde{\Delta}_{r+1,t}^{p,\ell}\left(\left\{\frac{u}{p^\ell}\right\}\right)\right)-\lambda_p\frac{\mathfrak{s}_p(m)}{p-1}+mp^s\left\{\frac{\lambda_p}{p-1}\right\}\\
&\quad=\sum_{\ell=s+1}^{\infty}\widetilde{\Delta}_{r+1,t}^{p,\ell}\left(\left\{\frac{u+mp^s}{p^\ell}\right\}\right)-\lambda_p\frac{\mathfrak{s}_p(m)}{p-1}+mp^s\left\{\frac{\lambda_p}{p-1}\right\}, \qquad (8.14)
\end{aligned}
$$

because, for all $\ell\in\{1,\dots,s\}$, we have $\{u/p^\ell\}=\big\{(u+mp^s)/p^\ell\big\}$ and, for all $\ell\geq s+1$, we have $\{u/p^\ell\}<\tau(r+1,\ell)$, thus $\widetilde{\Delta}_{r+1,t}^{p,\ell}\big(\{u/p^\ell\}\big)=0$. Let us show that, for all $\ell\geq s+1$, we have

$$
\widetilde{\Delta}_{r+1,t}^{p,\ell}\left(\left\{\frac{u+mp^s}{p^\ell}\right\}\right)=\widetilde{\Delta}_{r+s+1,t}^{p,\ell-s}\left(\left\{\frac{m}{p^{\ell-s}}\right\}\right). \qquad (8.15)
$$

Let α be an element of the sequences $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ whose denominator is not divisible by p. To prove (8.15), it is enough to show that, for all $\ell\geq s+1$, we have

$$
\left\{\frac{u+mp^s}{p^\ell}\right\}\geq\mathfrak{D}_p^\ell(\langle t^{(r+1)}\alpha\rangle)\Longleftrightarrow\left\{\frac{m}{p^{\ell-s}}\right\}\geq\mathfrak{D}_p^{\ell-s}(\langle t^{(r+s+1)}\alpha\rangle). \qquad (8.16)
$$

We write $m=\sum_{k=0}^{\infty}m_kp^k$ with $m_k\in\{0,\dots,p-1\}$. Then, we have

$$
\left\{\frac{u+mp^s}{p^\ell}\right\}=\frac{u+\sum_{k=0}^{\ell-s-1}m_kp^{k+s}}{p^\ell}=\frac{u}{p^\ell}+\left\{\frac{m}{p^{\ell-s}}\right\}.
$$

We observe that $\mathfrak{D}_p^{\ell-s}(\langle t^{(r+s+1)}\alpha\rangle)=\mathfrak{D}_p^{\ell}(\langle t^{(r+1)}\alpha\rangle)$, so that

$$
\left\{\frac{m}{p^{\ell-s}}\right\}\geq\mathfrak{D}_p^{\ell-s}(\langle t^{(r+s+1)}\alpha\rangle)\Longrightarrow\left\{\frac{u+mp^s}{p^\ell}\right\}\geq\mathfrak{D}_p^{\ell}(\langle t^{(r+1)}\alpha\rangle).
$$

Moreover, we have

$$
\begin{aligned}
\left\{\frac{u+mp^s}{p^\ell}\right\}\geq\mathfrak{D}_p^{\ell}(\langle t^{(r+1)}\alpha\rangle)&\Longrightarrow\frac{u}{p^\ell}+\left\{\frac{m}{p^{\ell-s}}\right\}\geq\mathfrak{D}_p^{\ell}(\langle t^{(r+1)}\alpha\rangle)\\
&\Longrightarrow p^{\ell-s}\left\{\frac{m}{p^{\ell-s}}\right\}\geq p^{\ell-s}\mathfrak{D}_p^{\ell}(\langle t^{(r+1)}\alpha\rangle)-\frac{u}{p^s}\\
&\Longrightarrow p^{\ell-s}\left\{\frac{m}{p^{\ell-s}}\right\}> p^{\ell-s}\mathfrak{D}_p^{\ell}(\langle t^{(r+1)}\alpha\rangle)-\mathfrak{D}_p^{s}(\langle t^{(r+1)}\alpha\rangle)\\
&\Longrightarrow p^{\ell-s}\left\{\frac{m}{p^{\ell-s}}\right\}\geq p^{\ell-s}\mathfrak{D}_p^{\ell}(\langle t^{(r+1)}\alpha\rangle)-\mathfrak{D}_p^{s}(\langle t^{(r+1)}\alpha\rangle)+1\\
&\Longrightarrow\left\{\frac{m}{p^{\ell-s}}\right\}\geq\mathfrak{D}_p^{\ell-s}(\langle t^{(r+s+1)}\alpha\rangle).
\end{aligned}
$$

Equivalence (8.16) is thus proved and we have (8.15). Using (8.15) in (8.14), we obtain

$$\begin{aligned} v_p\left(\frac{\Lambda_{b,r+1}(u+mp^s)}{\Lambda_{b,r+1}(u)}\right) &= \sum_{\ell=s+1}^{\infty} \widetilde{\Delta}_{r+s+1,t}^{p,\ell-s}\left(\left\{\frac{m}{p^{\ell-s}}\right\}\right) - \lambda_p\frac{\mathfrak{s}_p(m)}{p-1} + mp^s\left\{\frac{\lambda_p}{p-1}\right\} \\ &= \sum_{\ell=1}^{\infty} \widetilde{\Delta}_{r+s+1,t}^{p,\ell}\left(\left\{\frac{m}{p^{\ell}}\right\}\right) - \lambda_p\frac{\mathfrak{s}_p(m)}{p-1} + mp^s\left\{\frac{\lambda_p}{p-1}\right\} \\ &= v_p\big(\Lambda_{b,r+s+1}(m)\big) + m(p^s-1)\left\{\frac{\lambda_p}{p-1}\right\}. \end{aligned}$$

This completes the proof of Lemma 40. □

8.2.8. *Verification of Conditions* (a) *and* (a_1) *of Theorem 30.* Let us fix $r\in\mathbb{N}$. For all $s\in\mathbb{N}$, all $v\in\{0,\dots,p-1\}$ and all $u\in\Psi_{\mathcal{N}}(r,s)$, we set $\theta_{r,s}(v+up):=\mathcal{Q}_{r,\cdot}(v+up)$ if $v+up\notin\Psi_{\mathcal{N}}(r-1,s+1)$, and $\theta_{r,s}(v+up):=\mathbf{g}_r(v+up)$ otherwise.

The aim of this section is to prove the following fact: for all $s,m\in\mathbb{N}$, all $v\in\{0,\dots,p-1\}$ and all $u\in\Psi_{\mathcal{N}}(r,s)$, we have

$$\theta_{r,s}(v+up)\left(\frac{\mathcal{Q}_{r,\cdot}(v+up+mp^{s+1})}{\mathcal{Q}_{r,\cdot}(v+up)} - \frac{\mathcal{Q}_{r+1,\cdot}(u+mp^s)}{\mathcal{Q}_{r+1,\cdot}(u)}\right) \in p^{s+1}\mathbf{g}_{r+s+1}(m)\mathcal{A}. \tag{8.17}$$

This will prove Conditions (a) and (a_1) of Theorem 30. Indeed, by Lemmas 37 and 39, for all $v\in\{0,\dots,p-1\}$ and all $u\in\Psi_{\mathcal{N}}(r,s)$, we have

$$\mathcal{Q}_{r,\cdot}(v+up)\in\Lambda_{b,r}(v+up)\mathcal{A}\subset\mathbf{g}_r(v+up)\mathcal{A}.$$

Hence, Congruence (8.17) implies Condition (a) of Theorem 30. Moreover, by definition of $\theta_{r,s}$, when $v+up\in\Psi_{\mathcal{N}}(r-1,s+1)$, Congruence (8.17) implies Condition (a_1) of Theorem 30.

If $m=0$, then we have (8.17). In the sequel, we write Ψ for $\Psi_{\mathcal{N}}$, we assume that $m\geq 1$ and we split the proof of (8.17) into four cases.

- Case 1: we assume that $v+up\notin\Psi(r-1,s+1)$.

We then have $\theta_{r,s}(v+up)=\mathcal{Q}_{r,\cdot}(v+up)\in\Lambda_{b,r}(v+up)\mathcal{A}_b$. Let us show that $\Lambda_{b,r}(v+up)\in p^{s+1}\mathbb{Z}_p$. We have

$$v_p\big(\Lambda_{b,r}(v+up)\big)=\frac{1}{\varphi(p^\nu)}\sum_{\ell=1}^{\infty}\sum_{\substack{c=1\\ \gcd(c,p)=1}}^{p^\nu}\Delta_{r,t}^{c,\ell}\left(\left\{\frac{v+up}{p^\ell}\right\}\right)+(v+up)\left\{\frac{\lambda_p}{p-1}\right\}.$$

Since $v+up\notin\Psi(r-1,s+1)$ and $u\in\Psi(r,s)$, we obtain that $(v+up,s+1)\in\mathcal{N}_r$ and, for all $\ell\in\{1,\dots,s+1\}$, we have $\{(v+up)/p^\ell\}\geq\tau(r,\ell)$. It follows that

$$\frac{1}{\varphi(p^\nu)}\sum_{\ell=1}^{s+1}\sum_{\substack{c=1\\ \gcd(c,p)=1}}^{p^\nu}\Delta_{r,t}^{c,\ell}\left(\left\{\frac{v+up}{p^\ell}\right\}\right)\geq s+1$$

and $v_p\big(\Lambda_{b,r}(v+up)\big)\geq s+1$ because the functions $\Delta_{r,t}^{c,\ell}$ are nonnegative on $[0,1)$.

Since $u\in\Psi(r,s)$, Lemma 40 yields

$$\mathcal{Q}_{r,\cdot}(v+up)\frac{\mathcal{Q}_{r+1,\cdot}(u+mp^s)}{\mathcal{Q}_{r+1,\cdot}(u)}\in p^{s+1}\Lambda_{b,r+s+1}(m)\mathcal{A}_b\subset p^{s+1}\mathbf{g}_{r+s+1}(m)\mathcal{A}_b.$$

Thus, to show (8.17), it is enough to show

$$\mathcal{Q}_{r,\cdot}(v+up+mp^{s+1})\in p^{s+1}\mathbf{g}_{r+s+1}(m)\mathcal{A}_b. \tag{8.18}$$

By Lemma 37, we have

$$v_p\big(\Lambda_{b,r}(v+up+mp^{s+1})\big)=\frac{1}{\varphi(p^\nu)}\sum_{\ell=1}^{\infty}\sum_{\substack{c=1\\ \gcd(c,p)=1}}^{p^\nu}\Delta_{r,t}^{c,\ell}\left(\left\{\frac{v+up+mp^{s+1}}{p^\ell}\right\}\right)$$
$$+(v+up+mp^{s+1})\left\{\frac{\lambda_p}{p-1}\right\},$$

hence

$$v_p\big(\Lambda_{b,r}(v+up+mp^{s+1})\big)\geq s+1+\frac{1}{\varphi(p^\nu)}\sum_{\ell=s+2}^{\infty}\sum_{\substack{c=1\\ \gcd(c,p)=1}}^{p^\nu}\Delta_{r,t}^{c,\ell}\left(\left\{\frac{v+up+mp^{s+1}}{p^\ell}\right\}\right)$$
$$+m\left\{\frac{\lambda_p}{p-1}\right\}.$$

If $\boldsymbol{\beta}\in\mathbb{Z}^r$, then the functions $\Delta_{r,t}^{c,\ell}$ are nondecreasing on $[0,1)$ and, by (8.9), for all $\ell\geq s+2$, we obtain

$$\begin{aligned}\sum_{\ell=s+2}^{\infty}\sum_{\substack{c=1\\ \gcd(c,p)=1}}^{p^\nu}\Delta_{r,t}^{c,\ell}\left(\left\{\frac{v+up+mp^{s+1}}{p^\ell}\right\}\right)&\geq\sum_{\ell=s+2}^{\infty}\sum_{\substack{c=1\\ \gcd(c,p)=1}}^{p^\nu}\Delta_{r,t}^{c,\ell}\left(\left\{\frac{mp^{s+1}}{p^\ell}\right\}\right)\\&\geq\sum_{\ell=1}^{\infty}\sum_{\substack{c=1\\ \gcd(c,p)=1}}^{p^\nu}\Delta_{r,t}^{c,\ell+s+1}\left(\left\{\frac{m}{p^\ell}\right\}\right)\\&\geq\sum_{\ell=1}^{\infty}\sum_{\substack{c=1\\ \gcd(c,p)=1}}^{p^\nu}\Delta_{r+s+1,t}^{c,\ell}\left(\left\{\frac{m}{p^\ell}\right\}\right).\end{aligned}$$

Consequently, if $\boldsymbol{\beta}\in\mathbb{Z}^r$, then

$$v_p\big(\Lambda_{b,r}(v+up+mp^{s+1})\big)\geq s+1+v_p\big(\Lambda_{b,r+s+1}(m)\big)\geq s+1+v_p\big(\mathbf{g}_{r+s+1}(m)\big),$$

as expected.

On the other hand, if $\boldsymbol{\beta} \notin \mathbb{Z}^r$, then we observe that, for all $\ell \in \mathbb{N}$, $\ell \geq 1$, we have

$$\begin{aligned}\left\{\frac{m}{p^\ell}\right\} \geq \tau(r+s+1,\ell) &\Longrightarrow \left\{\frac{mp^{s+1}}{p^{\ell+s+1}}\right\} \geq \tau(r,\ell+s+1)\\ &\Longrightarrow \left\{\frac{v+up+mp^{s+1}}{p^{\ell+s+1}}\right\} \geq \tau(r,\ell+s+1)\\ &\Longrightarrow \frac{1}{\varphi(p^\nu)}\sum_{\substack{c=1\\ \gcd(c,p)=1}}^{p^\nu} \Delta_{r,t}^{c,\ell+s+1}\left(\left\{\frac{v+up+mp^{s+1}}{p^{\ell+s+1}}\right\}\right) \geq 1,\end{aligned}$$

so that

$$\frac{1}{\varphi(p^\nu)}\sum_{\ell=s+2}^{\infty}\sum_{\substack{c=1\\ \gcd(c,p)=1}}^{p^\nu} \Delta_{r,t}^{c,\ell}\left(\left\{\frac{v+up+mp^{s+1}}{p^{\ell}}\right\}\right) \geq v_p\big(g_{r+s+1}(m)\big)$$

and thus $v_p\big(\Lambda_{b,r}(v+up+mp^{s+1})\big) \geq s+1+v_p\big(\mathbf{g}_{r+s+1}(m)\big)$, as expected. Hence (8.18) is proved, which finishes the proof of (8.17) when $v+up \notin \Psi(r-1,s+1)$.

- Case 2: we assume that $v+up \in \Psi(r-1,s+1)$ and that $p-1 \nmid \lambda_p$.

We have $\theta_{r,s}(v+up) = \mathbf{g}_r(v+up)$, $\mathcal{A}=\mathcal{A}_b$, and we have to show that

$$\mathbf{g}_r(v+up)\left(\frac{\mathcal{Q}_{r,\cdot}(v+up+mp^{s+1})}{\mathcal{Q}_{r,\cdot}(v+up)} - \frac{\mathcal{Q}_{r+1,\cdot}(u+mp^{s})}{\mathcal{Q}_{r+1,\cdot}(u)}\right) \in p^{s+1}\mathbf{g}_{r+s+1}(m)\mathcal{A}_b.$$

By Lemma 40,

$$\frac{\mathcal{Q}_{r+1,\cdot}(u+mp^{s})}{\mathcal{Q}_{r+1,\cdot}(u)} \in p^{\left\{\frac{\lambda_p}{p-1}\right\}m(p^s-1)}\Lambda_{b,r+s+1}(m)\mathcal{A}_b$$

and

$$\frac{\mathcal{Q}_{r,\cdot}(v+up+mp^{s+1})}{\mathcal{Q}_{r,\cdot}(v+up)} \in p^{\left\{\frac{\lambda_p}{p-1}\right\}m(p^{s+1}-1)}\Lambda_{b,r+s+1}(m)\mathcal{A}_b.$$

Since $p-1 \nmid \lambda_p$ and $m \geq 1$, we have

$$\left\{\frac{\lambda_p}{p-1}\right\}m(p^s-1) \geq m\frac{p^s-1}{p-1} \geq s \quad \text{and} \quad \left\{\frac{\lambda_p}{p-1}\right\}m(p^{s+1}-1) \geq s+1.$$

Thus, we obtain

$$\mathbf{g}_r(v+up)\frac{\mathcal{Q}_{r,\cdot}(v+up+mp^{s+1})}{\mathcal{Q}_{r,\cdot}(v+up)} \in p^{s+1}\Lambda_{b,r+s+1}(m)\mathcal{A}_b \subset p^{s+1}\mathbf{g}_{r+s+1}(m)\mathcal{A}_b,$$

because $\mathbf{g}_r(v+up) \in \mathbb{Z}_p$. It remains to show that

$$\mathbf{g}_r(v+up)\frac{\mathcal{Q}_{r+1,\cdot}(u+mp^{s})}{\mathcal{Q}_{r+1,\cdot}(u)} \in p^{s+1}\mathbf{g}_{r+s+1}(m)\mathcal{A}_b. \tag{8.19}$$

By Lemma 39,

$$v_p\big(\Lambda_{b,r+s+1}(m)\big) \geq v_p\big(g_{r+s+1}(m)\big) + m\left\{\frac{\lambda_p}{p-1}\right\}$$

and thus, since $p-1 \nmid \lambda_p$ and $m \geq 1$, we obtain that $\Lambda_{b,r+s+1}(m) \in p g_{r+s+1}(m)\mathbb{Z}_p$. Hence, we have $\Lambda_{b,r+s+1}(m) \in p\mathbf{g}_{r+s+1}(m)\mathbb{Z}_p$, as well as (8.19) because $\mathbf{g}_r(v+up) \in \mathbb{Z}_p$.

• Case 3: we assume that $v+up \in \Psi(r-1,s+1)$, $\boldsymbol{\beta} \notin \mathbb{Z}^r$, $p=2$, and that $\mathfrak{m}$ is odd.

We have $\theta_{r,s}(v+up) = \mathbf{g}_r(v+up) = g_r(v+up)$, $\mathcal{A} = \mathcal{A}_b$, and we have to show

$$g_r(v+up)\left(\frac{\mathcal{Q}_{r,\cdot}(v+up+mp^{s+1})}{\mathcal{Q}_{r,\cdot}(v+up)} - \frac{\mathcal{Q}_{r+1,\cdot}(u+mp^s)}{\mathcal{Q}_{r+1,\cdot}(u)}\right) \in p^{s+1}g_{r+s+1}(m)\mathcal{A}_b. \tag{8.20}$$

By Lemma 40, we have

$$\frac{\mathcal{Q}_{r+1,\cdot}(u+mp^s)}{\mathcal{Q}_{r+1,\cdot}(u)} \in 2^{mp^s}\Lambda_{b,r+s+1}(m)\mathcal{A}_b$$

and

$$\frac{\mathcal{Q}_{r,\cdot}(v+up+mp^{s+1})}{\mathcal{Q}_{r,\cdot}(v+up)} \in 2^{mp^{s+1}}\Lambda_{b,r+s+1}(m)\mathcal{A}_b.$$

Moreover, we have $m2^s \geq s+1$ and $m2^{s+1} \geq s+1$ because $m \geq 1$. Since $\Lambda_{b,r+s+1}(m) \in g_{r+s+1}(m)\mathbb{Z}_p$ and $g_r(v+up) \in \mathbb{Z}_p$, we get (8.20).

• Case 4: we assume that $v+up \in \Psi(r-1,s+1)$, $p-1$ divides λ_p and that, if $p=2$ and $\boldsymbol{\beta} \notin \mathbb{Z}^r$, then $\mathfrak{m}$ is even.

We set

$$X_{r,s}(v,u,m) := \frac{\mathcal{Q}_{r,\cdot}(v+up)}{\mathcal{Q}_{r+1,\cdot}(u)}\,\frac{\mathcal{Q}_{r+1,\cdot}(u+mp^s)}{\mathcal{Q}_{r,\cdot}(v+up+mp^{s+1})}.$$

Assertion (8.17) is satisfied if and only if, for all $s,m \in \mathbb{N}$, all $v \in \{0,\dots,p-1\}$ and all $u \in \Psi_{\mathcal{N}}(r,s)$, we have

$$\mathbf{g}_r(v+up)\big(X_{r,s}(v,u,m)-1\big)\frac{\mathcal{Q}_{r,\cdot}(v+up+mp^{s+1})}{\mathcal{Q}_{r,\cdot}(v+up)} \in p^{s+1}\mathbf{g}_{r+s+1}(m)\mathcal{A}. \tag{8.21}$$

The following lemma will give the conclusion.

Lemma 41. *We assume that $p-1$ divides λ_p and that, if $p=2$ and $\boldsymbol{\beta} \notin \mathbb{Z}^r$, then $\mathfrak{m}$ is even. Then,*

(1) *For all $r,s \in \mathbb{N}$, all $v \in \{0,\dots,p-1\}$, all $u \in \Psi_{\mathcal{N}}(r,s)$ and all $m \in \mathbb{N}$, there exists $Y_{r,s}(v,u,m) \in \mathbb{Z}_p$ independent of $t \in \Omega_b$ such that*

$$X_{r,s}(v,u,m) \in \begin{cases} Y_{r,s}(v,u,m)(1+p^s\mathcal{A}_b) \text{ if } \boldsymbol{\beta} \in \mathbb{Z}^r \text{ and } p \mid d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\,; \\ Y_{r,s}(v,u,m)(1+p^{s+1}\mathcal{A}_b^*) \text{ otherwise;} \end{cases}.$$

(2) *If there exists $j \in \{1,\dots,s+1\}$ such that $\big\{(v+up)/p^j\big\} < \tau(r,j)$, then we have $Y_{r,s}(v,u,m) \in 1+p^{s-j+2}\mathbb{Z}_p$.*

Since $v+up \in \Psi(r-1, s+1)$, Lemma 40 implies that $\{(v+up)/p^{s+1}\} < \tau(r, s+1)$. Let j_0 be the smallest $j \in \{1, \dots, s+1\}$ such that $\{(v+up)/p^j\} < \tau(r, j)$. By Lemma 41 applied with j_0, we obtain that $Y_{r,s}(v,u,m) \in 1 + p^{s-j_0+2}\mathbb{Z}_p$ and that

$$X_{r,s}(v,u,m) \in \begin{cases} 1 + p^{s-j_0+1}\mathcal{A}_b \text{ if } \boldsymbol{\beta} \in \mathbb{Z}^r \text{ and } p \mid d_{\boldsymbol{\alpha},\boldsymbol{\beta}}; \\ 1 + p^{s-j_0+2}\mathcal{A}_b^* \text{ otherwise.} \end{cases}.$$

Hence, Lemma 40 yields

$$\big(X_{r,s}(v,u,m) - 1\big)\frac{\mathcal{Q}_{r,\cdot}(v+up+mp^{s+1})}{\mathcal{Q}_{r,\cdot}(v+up)} \in p^{s-j_0+2}\mathbf{g}_{r+s+1}(m) \times \begin{cases} \mathcal{A}_b \text{ if } \boldsymbol{\beta} \in \mathbb{Z}^r \text{ and } p \mid d_{\boldsymbol{\alpha},\boldsymbol{\beta}}; \\ \mathcal{A}_b^* \text{ otherwise.} \end{cases}.$$

Therefore to prove (8.21), it is enough to show that $\mathbf{g}_r(v+up) \in p^{j_0-1}\mathbb{Z}_p$. If $v+up=0$, then we have $j_0 = 1$ and the conclusion is clear. We may thus assume that $v+up \geq 1$. But for all $j \in \{1, \dots, j_0-1\}$, we have $\{(v+up)/p^j\} \geq \tau(r,j)$, hence $v_p\big(g_r(v+up)\big) \geq j_0 - 1$. Furthermore, if $\boldsymbol{\beta} \in \mathbb{Z}^r$ and if $p \mid d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, we have $\lambda_p \leq -1$ and, by Lemma 39, we have

$$\mathbf{g}_r(v+up) = \frac{\Lambda_{b,r}(v+up)}{p} \in g_r(v+up)\mathbb{Z}_p \subset p^{j_0-1}\mathbb{Z}_p,$$

as expected.

To complete the proof of (8.21) and that of Theorem 6, it remains to prove Lemma 41.

Proof of Lemma 41. We will show that Lemma 41 holds with

$$Y_{r,s}(v,u,m) := \frac{\prod_{\beta_i \in \mathbb{Z}_p}\left(1 + \frac{mp^s}{\langle t^{(r+1)}\beta_i\rangle + u}\right)^{\rho(v, \langle t^{(r)}\beta_i\rangle)}}{\prod_{\alpha_i \in \mathbb{Z}_p}\left(1 + \frac{mp^s}{\langle t^{(r+1)}\alpha_i\rangle + u}\right)^{\rho(v, \langle t^{(r)}\alpha_i\rangle)}}.$$

By Lemma 1 of [**13**], if α is an element of the sequences $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ whose denominator is not divisible by p, then for all $v \in \{0, \dots, p-1\}$, all $s, m \in \mathbb{N}$ and all $u \in \{0, \dots, p^s - 1\}$, we have

$$\frac{(\alpha)_{v+up+mp^{s+1}}\big(\mathfrak{D}_p(\alpha)\big)_u}{\big(\mathfrak{D}_p(\alpha)\big)_{u+mp^s}(\alpha)_{v+up}} \in \big((-p)^{p^s}\varepsilon_{p^s}\big)^m\left(1 + \frac{mp^s}{\mathfrak{D}_p(\alpha)+u}\right)^{\rho(v,\alpha)}(1+p^{s+1}\mathbb{Z}_p), \tag{8.22}$$

where $\varepsilon_k = -1$ if $k = 2$, and $\varepsilon_k = 1$ otherwise.

Similarly, using Dwork's method, we will show that if α is an element of the sequences $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ whose denominator is divisible by p, then for all $v \in \{0, \dots, p-1\}$, all $r, s, m \in \mathbb{N}$ and all $u \in \{0, \dots, p^s - 1\}$, we have

$$\left(t \in \Omega_b \mapsto d(\alpha)^{m\varphi(p^{s+1})}\frac{(\langle t^{(r)}\alpha\rangle)_{v+up+mp^{s+1}}(\langle t^{(r+1)}\alpha\rangle)_u}{(\langle t^{(r+1)}\alpha\rangle)_{u+mp^s}(\langle t^{(r)}\alpha\rangle)_{v+up}}\right) \in \varepsilon'_{p^s}(\alpha)^m(1+p^{s+1}\mathcal{A}_b^*), \tag{8.23}$$

where $\varepsilon'_k(\alpha) = \varepsilon_k$ if $v_p\big(d(\alpha)\big) = 1$ and $\varepsilon'_k(\alpha) = 1$ otherwise.

We first show that (8.22) and (8.23) imply the validity of Assertion (1) of Lemma 41. Indeed, by (8.22), we obtain

$$\frac{\Lambda_{b,r}(v+up+mp^{s+1})\Lambda_{b,r+1}(u)}{\Lambda_{b,r+1}(u+mp^{s})\Lambda_{b,r}(v+up)} \in \left(C\frac{\prod_{\beta_i\notin\mathbb{Z}_p} d(\beta_i)}{\prod_{\alpha_i\notin\mathbb{Z}_p} d(\alpha_i)}\right)^{m\varphi(p^{s+1})} \frac{\left((-p)^{p^s}\varepsilon_{p^s}\right)^{m\lambda_p}}{Y_{r,s}(v,u,m)}(1+p^{s+1}\mathbb{Z}_p). \quad (8.24)$$

We write

$$C\frac{\prod_{\beta_i\notin\mathbb{Z}_p} d(\beta_i)}{\prod_{\alpha_i\notin\mathbb{Z}_p} d(\alpha_i)} = \sigma p^{-\left\lfloor\frac{\lambda_p}{p-1}\right\rfloor} = \sigma p^{-\frac{\lambda_p}{p-1}},$$

with $\sigma\in\mathbb{Z}_p^\times$, so that

$$\left(C\frac{\prod_{\beta_i\notin\mathbb{Z}_p} d(\beta_i)}{\prod_{\alpha_i\notin\mathbb{Z}_p} d(\alpha_i)}\right)^{m\varphi(p^{s+1})} \in p^{-mp^s\lambda_p}(1+p^{s+1}\mathbb{Z}_p).$$

We thus have

$$\left(C\frac{\prod_{\beta_i\notin\mathbb{Z}_p} d(\beta_i)}{\prod_{\alpha_i\notin\mathbb{Z}_p} d(\alpha_i)}\right)^{m\varphi(p^{s+1})} \left((-p)^{p^s}\varepsilon_{p^s}\right)^{m\lambda_p} \in (-1)^{mp^s\lambda_p}\varepsilon_{p^s}^{m\lambda_p}(1+p^{s+1}\mathbb{Z}_p) \subset \varepsilon_{p^s}^{m\lambda_p}(1+p^{s+1}\mathbb{Z}_p), \quad (8.25)$$

because $-1\in\mathbb{Z}_p^\times$ and $\varphi(p^{s+1})=p^s(p-1)$ divides $mp^s\lambda_p$. Using (8.25) in (8.24), we obtain that

$$\frac{\Lambda_{b,r+1}(u)\Lambda_{b,r}(v+up+mp^{s+1})}{\Lambda_{b,r}(v+up)\Lambda_{b,r+1}(u+mp^{s})} \in \frac{\varepsilon_{p^s}^{m\lambda_p}}{Y_{r,s}(v,u,m)}(1+p^{s+1}\mathbb{Z}_p). \quad (8.26)$$

By (8.23), we also obtain

$$\frac{\mathfrak{R}_{r+1}(u+mp^s,\cdot)\mathfrak{R}_r(v+up,\cdot)}{\mathfrak{R}_r(v+up+mp^{s+1},\cdot)\mathfrak{R}_{r+1}(u,\cdot)} \in \left(\frac{\prod_{\beta_i\notin\mathbb{Z}_p}\varepsilon'_{p^s}(\beta_i)}{\prod_{\alpha_i\notin\mathbb{Z}_p}\varepsilon'_{p^s}(\alpha_i)}\right)^m (1+p^{s+1}\mathcal{A}_b^*).$$

If $p^s\neq 2$, then, for any element $\alpha\notin\mathbb{Z}_p$ of $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$, we have $\varepsilon'_{p^s}(\alpha)=\varepsilon_{p^s}=1$. If $p^s=2$ and if the number of elements α of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ that satisfy $v_2\big(d(\alpha)\big)\geq 2$ is even, then, since $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ have the same length, we have

$$\frac{\prod_{\beta_i\notin\mathbb{Z}_p}\varepsilon'_{p^s}(\beta_i)}{\prod_{\alpha_i\notin\mathbb{Z}_p}\varepsilon'_{p^s}(\alpha_i)} = (-1)^{\lambda_2} = \varepsilon_2^{\lambda_2}.$$

Moreover, we have $p\mathcal{A}_b^*\subset\mathcal{A}_b$ and $\varepsilon_{p^s},\varepsilon'_{p^s}(\alpha)\in 1+p^s\mathbb{Z}_p$. It follows that we obtain

$$\frac{\mathfrak{R}_{r+1}(u+mp^s,\cdot)\mathfrak{R}_r(v+up,\cdot)}{\mathfrak{R}_r(v+up+mp^{s+1},\cdot)\mathfrak{R}_{r+1}(u,\cdot)} \in \begin{cases} 1+p^s\mathcal{A}_b \text{ if } \boldsymbol{\beta}\in\mathbb{Z}^r \text{ and } p\mid d_{\boldsymbol{\alpha},\boldsymbol{\beta}};\\ \varepsilon_{p^s}^{m\lambda_p}(1+p^{s+1}\mathcal{A}_b^*) \text{ otherwise.}\end{cases} \quad (8.27)$$

By (8.26) and (8.27), we obtain

$$X_{r,s}(v,u,m)\in Y_{r,s}(v,u,m)\times\begin{cases}(1+p^s\mathcal{A}_b)\text{ if }\boldsymbol{\beta}\in\mathbb{Z}^r\text{ and }p\mid d_{\boldsymbol{\alpha},\boldsymbol{\beta}};\\(1+p^{s+1}\mathcal{A}_b^*)\text{ otherwise.}\end{cases}$$

To finish the proof of Assertion (1) of Lemma 41, we have to prove (8.23).

Let α be an element of $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ whose denominator is divisible by p. For all $s,m\in\mathbb{N}$ and all $u\in\{0,\dots,p^s-1\}$, we set

$$\mathfrak{q}_r(u,s,m):=t\in\Omega_b\mapsto d(\alpha)^{mp^s}\frac{(\langle t^{(r)}\alpha\rangle)_{u+mp^s}}{(\langle t^{(r)}\alpha\rangle)_u}=\prod_{k=0}^{mp^s-1}\big(d(\alpha)\langle t^{(r)}\alpha\rangle+d(\alpha)u+d(\alpha)k\big).$$

Hence, proving (8.23) amounts to proving that

$$\frac{\mathfrak{q}_r(v+up,s+1,m)}{\mathfrak{q}_{r+1}(u,s,m)}\in\varepsilon'_{p^s}(\alpha)^m(1+p^{s+1}\mathcal{A}_b^*).$$

As functions of t, we have

$$\begin{aligned}\mathfrak{q}_r(u,s,m)(t)&=\prod_{i=0}^{p^s-1}\prod_{j=0}^{m-1}\big(d(\alpha)\langle t^{(r)}\alpha\rangle+d(\alpha)u+d(\alpha)i+d(\alpha)jp^s\big)\\&\equiv\prod_{i=0}^{p^s-1}\big(d(\alpha)\langle t^{(r)}\alpha\rangle+d(\alpha)u+d(\alpha)i\big)^m\mod p^{s+1}\mathcal{A}_b\\&\equiv\prod_{i=0}^{p^s-1}\big(d(\alpha)\langle t^{(r)}\alpha\rangle+d(\alpha)i\big)^m\mod p^{s+1}\mathcal{A}_b.\end{aligned}$$

Since $d(\alpha)$ is divisible by p, we obtain that, for all $i\in\{0,\dots,p^s-1\}$, the map $t\in\Omega_b\mapsto d(\alpha)\langle t^{(r)}\alpha\rangle+d(\alpha)i$ is invertible in $\mathcal{A}_b$ and thus

$$\mathfrak{q}_r(u,s,m)\in\mathfrak{q}_r(0,s,1)^m(1+p^{s+1}\mathcal{A}_b).$$

Hence proving (8.23) amounts to proving that, for all $s\in\mathbb{N}$, we have

$$\frac{\mathfrak{q}_r(0,s+1,1)}{\mathfrak{q}_{r+1}(0,s,1)}\in\varepsilon'_{p^s}(\alpha)(1+p^{s+1}\mathcal{A}_b^*).\tag{8.28}$$

- Case 1: we assume that $s=0$.

As functions of t, we have

$$\frac{\mathfrak{q}_r(0,1,1)(t)}{\mathfrak{q}_{r+1}(0,0,1)(t)}\in\frac{\big(d(\alpha)\langle t^{(r)}\alpha\rangle\big)^p}{d(\alpha)\langle t^{(r+1)}\alpha\rangle}(1+p\mathcal{A}_b)$$

and

$$t^{(r)}\equiv\varpi_{p^\nu}\left(\frac{t}{D}\right)D+\varpi_D\left(\frac{b}{p^{\nu+r}}\right)p^\nu\mod p^\nu D.$$

Hence, with $\langle\alpha\rangle := \kappa/d(\alpha)$, we obtain the existence of $\eta(r,t)\in\mathbb{Z}$ such that

$$d(\alpha)\langle t^{(r)}\alpha\rangle = \varpi_{p^\nu}\left(\frac{t\kappa}{D}\right)D + \varpi_D\left(\frac{b\kappa}{p^{\nu+r}}\right)p^\nu + d(\alpha)\eta(r,t).$$

Moreover, by Assertions (2), (4) and (5) of Lemma 36, the maps $t\in\Omega_b\mapsto d(\alpha)\langle t^{(r)}\alpha\rangle$ and $f : t\in\Omega_b\mapsto \varpi_{p^\nu}(t\kappa/D)D$ are in $\mathcal{A}_b^\times$. Thus $t\in\Omega_b\mapsto d(\alpha)\eta(r,t)$ is in $\mathcal{A}_b$ and $t\in\Omega_b\mapsto d(\alpha)\eta(r,t)/p$ is in $\mathcal{A}_b^*$ because p divides $d(\alpha)$. It follows that

$$\big(t\in\Omega_b\mapsto d(\alpha)\langle t^{(r)}\alpha\rangle\big)\in f(1+p\mathcal{A}_b^*). \tag{8.29}$$

We obtain

$$\frac{\mathfrak{q}_r(0,1,1)}{\mathfrak{q}_{r+1}(0,0,1)}\in f^{p-1}(1+p\mathcal{A}_b^*)\subset\big(1+p(\mathfrak{E}_1\circ f)\big)(1+p\mathcal{A}_b^*)\subset 1+p\mathcal{A}_b^*,$$

as expected, where the final inclusion is obtained *via* Assertion (3) of Lemma 35.

- Case 2: we assume that $s\geq 1$.

If $s\geq 1$, then

$$\prod_{i=0}^{p^s-1}\big(d(\alpha)\langle t^{(r)}\alpha\rangle + d(\alpha)i\big) = \prod_{j=0}^{p^{s-1}-1}\prod_{a=0}^{p-1}\big(d(\alpha)\langle t^{(r)}\alpha\rangle + d(\alpha)j + d(\alpha)ap^{s-1}\big) \tag{8.30}$$

$$\equiv \prod_{j=0}^{p^{s-1}-1}\big(d(\alpha)\langle t^{(r)}\alpha\rangle + d(\alpha)j\big)^p \mod p^s\mathcal{A}_b. \tag{8.31}$$

Using (8.31) with $s+1$ for s, we obtain

$$\mathfrak{q}_r(0,s+1,1)\in\mathfrak{q}_r(0,s,1)^p(1+p^{s+1}\mathcal{A}_b)$$

and thus

$$\mathfrak{q}_r(0,s+1,1)\in\big(d(\alpha)\langle t^{(r)}\alpha\rangle\big)^{p^{s+1}}(1+p^{s+1}\mathcal{A}_b). \tag{8.32}$$

We set $P(x) := x^p - x\in\mathbb{Z}_p[x]$. For all $a\in\{0,\dots,p-1\}$, we have $a^p - a\equiv 0 \mod p\mathbb{Z}_p$. Since $P'(x) = px^{p-1}-1$, for all $a\in\{0,\dots,p-1\}$, we have $v_p\big(P'(a)\big)=0$ and, by Hensel's lemma (see [**30**]), there exists a root w_a of P in $\mathbb{Z}_p$ such that $w_a\equiv a \mod p\mathbb{Z}_p$. Consequently, for all $x\in\mathbb{Z}_p$ and all $s\in\mathbb{N}$, $s\geq 1$, we have

$$\prod_{a=0}^{p-1}\big(x+d(\alpha)ap^{s-1}\big)\equiv\prod_{i=0}^{p-1}\big(x-d(\alpha)w_ip^{s-1}\big) \mod p^{s+1}\mathbb{Z}_p$$

$$\equiv x^p - \big(d(\alpha)p^{s-1}\big)^{p-1}x \mod p^{s+1}\mathbb{Z}_p. \tag{8.33}$$

If $p\neq 2$, then $\big(d(\alpha)p^{s-1}\big)^{p-1}x\in p^{s+1}\mathbb{Z}_p$ thus, by (8.30), for all $s\in\mathbb{N}$, $s\geq 1$, we obtain

$$\mathfrak{q}_{r+1}(0,s,1)\in\prod_{j=0}^{p^{s-1}-1}\big(d(\alpha)\langle t^{(r+1)}\alpha\rangle + d(\alpha)j\big)^p(1+p^{s+1}\mathcal{A}_b),$$

hence $\mathfrak{q}_{r+1}(0,s,1) \in \mathfrak{q}_{r+1}(0,s-1,1)^p(1+p^{s+1}\mathcal{A}_b)$ and

$$\mathfrak{q}_{r+1}(0,s,1) \in \big(d(\alpha)\langle t^{(r+1)}\alpha\rangle\big)^{p^s}(1+p^{s+1}\mathcal{A}_b).$$

By (8.32) and (8.29), we obtain the existence of $f_1, f_2 \in \mathcal{A}_b^*$ such that

$$\begin{aligned}\frac{\mathfrak{q}_r(0,s+1,1)}{\mathfrak{q}_{r+1}(0,s,1)} &\in f^{\varphi(p^{s+1})}\frac{(1+pf_1)^{p^{s+1}}}{(1+pf_2)^{p^s}}(1+p^{s+1}\mathcal{A}_b)\\ &\subset \big(1+p^{s+1}(\mathfrak{E}_{s+1}\circ f)\big)(1+p^{s+1}\mathcal{A}_b^*) \subset 1+p^{s+1}\mathcal{A}_b^*,\end{aligned}$$

which proves (8.28) when $p \neq 2$ because in this case we have $\varepsilon'_{p^s}(\alpha) = 1$.

Let us now assume $p = 2$. Then by (8.30) and (8.33), for all $s \in \mathbb{N}$, $s \geq 1$, we obtain

$$\mathfrak{q}_{r+1}(0,s,1) \in \prod_{j=0}^{2^{s-1}-1}\big(d(\alpha)\langle t^{(r+1)}\alpha\rangle + d(\alpha)j\big)^2\left(1-\frac{d(\alpha)2^{s-1}}{d(\alpha)\langle t^{(r+1)}\alpha\rangle + d(\alpha)j}\right)(1+2^{s+1}\mathcal{A}_b).$$

Since 2 divides $d(\alpha)$, we have

$$\begin{aligned}\prod_{j=0}^{2^{s-1}-1}\left(1-\frac{d(\alpha)2^{s-1}}{d(\alpha)\langle t^{(r+1)}\alpha\rangle + d(\alpha)j}\right) &= \prod_{j=0}^{2^{s-1}-1}\left(1-\frac{d(\alpha)2^{s-1}}{1+2\mathfrak{E}_1\big(d(\alpha)\langle t^{(r+1)}\alpha\rangle + d(\alpha)j\big)}\right)\\ &\equiv \prod_{j=0}^{2^{s-1}-1}\big(1-d(\alpha)2^{s-1}\big) \mod 2^{s+1}\mathcal{A}_b^*\\ &\equiv 1-d(\alpha)2^{2s-2} \mod 2^{s+1}\mathcal{A}_b^*,\end{aligned}$$

with $1-d(\alpha)2^{2s-2} \equiv 1 \mod 2^{s+1}$ if $s \geq 2$ or $v_2\big(d(\alpha)\big) \geq 2$, and $1-d(\alpha)2^{2s-2} \equiv -1 \mod 4$ if $s = v_2\big(d(\alpha)\big) = 1$. It follows that

$$\mathfrak{q}_{r+1}(0,s,1) \in \varepsilon'_{2^s}(\alpha)\prod_{j=0}^{2^{s-1}-1}\big(d(\alpha)\langle t^{(r+1)}\alpha\rangle + d(\alpha)j\big)^2(1+2^{s+1}\mathcal{A}_b^*),$$

i. e. $\mathfrak{q}_{r+1}(0,s,1) \in \varepsilon'_{2^s}(\alpha)\mathfrak{q}_{r+1}(0,s-1,1)^2(1+2^{s+1}\mathcal{A}_b^*)$ and thus

$$\mathfrak{q}_{r+1}(0,s,1) \in \varepsilon'_{2^s}(\alpha)\big(d(\alpha)\langle t^{(r+1)}\alpha\rangle\big)^{2^s}(1+2^{s+1}\mathcal{A}_b^*).$$

By (8.32) and (8.29), we obtain the existence of $f_1, f_2 \in \mathcal{A}_b^*$ such that

$$\begin{aligned}\frac{\mathfrak{q}_r(0,s+1,1)}{\mathfrak{q}_{r+1}(0,s,1)} &\in \frac{1}{\varepsilon'_{2^s}(\alpha)}f^{\varphi(2^{s+1})}\frac{(1+2f_1)^{2^{s+1}}}{(1+2f_2)^{2^s}}(1+2^{s+1}\mathcal{A}_b^*)\\ &\subset \varepsilon'_{2^s}(\alpha)\big(1+2^{s+1}(\mathfrak{E}_{s+1}\circ f)\big)(1+2^{s+1}\mathcal{A}_b^*) \subset \varepsilon'_{2^s}(\alpha)(1+2^{s+1}\mathcal{A}_b^*),\end{aligned}$$

which proves (8.28) and completes the proof of (1) of Lemma 41.

Let us now prove Assertion (2) of Lemma 41. We have

$$Y_{r,s}(v,u,m)=\frac{\prod_{\beta_i\in\mathbb{Z}_p}\left(1+\frac{mp^s}{\langle t^{(r+1)}\beta_i\rangle+u}\right)^{\rho(v,\langle t^{(r)}\beta_i\rangle)}}{\prod_{\alpha_i\in\mathbb{Z}_p}\left(1+\frac{mp^s}{\langle t^{(r+1)}\alpha_i\rangle+u}\right)^{\rho(v,\langle t^{(r)}\alpha_i\rangle)}}.$$

Let $j\in\{1,\dots,s+1\}$ be such that $\big\{(v+up)/p^j\big\}<\tau(r,j)$. We set $u=\sum_{k=0}^{\infty}u_kp^k$. For all elements $\alpha\in\mathbb{Z}_p$ of the sequences $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$, we have

$$\begin{aligned}\left\{\frac{v+up}{p^j}\right\}<\tau(r,j)&\Longrightarrow v+p\sum_{k=0}^{j-2}u_kp^k<p^j\mathfrak{D}_p^j(\langle t^{(r)}\alpha\rangle)\\&\Longrightarrow v+p\sum_{k=0}^{j-2}u_kp^k\leq p^j\mathfrak{D}_p^j(\langle t^{(r)}\alpha\rangle)-\langle t^{(r)}\alpha\rangle\\&\Longrightarrow v+p\sum_{k=0}^{j-2}u_kp^k\leq\sum_{k=0}^{j-1}p^k\big(p\mathfrak{D}_p^{k+1}(\langle t^{(r)}\alpha\rangle)-\mathfrak{D}_p^k(\langle t^{(r)}\alpha\rangle)\big)\\&\Longrightarrow\left(\rho(v,\langle t^{(r)}\alpha\rangle)=0\quad\text{or}\quad\sum_{k=0}^{j-2}u_kp^k<p^{j-1}\mathfrak{D}_p^j(\langle t^{(r)}\alpha\rangle)-\mathfrak{D}_p(\langle t^{(r)}\alpha\rangle)\right)\\&\Longrightarrow\left(\rho(v,\langle t^{(r)}\alpha\rangle)=0\quad\text{or}\quad\sum_{k=0}^{j-2}u_kp^k<p^{j-1}\mathfrak{D}_p^{j-1}(\langle t^{(r+1)}\alpha\rangle)-\langle t^{(r+1)}\alpha\rangle\right)\\&\Longrightarrow\big(\rho(v,\langle t^{(r)}\alpha\rangle)=0\quad\text{or}\quad v_p(u+\langle t^{(r+1)}\alpha\rangle)\leq j-2\big)\\&\Longrightarrow\left(1+\frac{mp^s}{\langle t^{(r+1)}\alpha\rangle+u}\right)^{\rho(v,\langle t^{(r)}\alpha\rangle)}\in 1+p^{s-j+2}\mathbb{Z}_p,\end{aligned}$$

as expected. This completes the proof of Lemma 41 and that of Theorem 6. □

9. Proof of Theorem 9

We shall prove the following more precise statement.

PROPOSITION 42. *Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be tuples of parameters in $\mathbb{Q}\setminus\mathbb{Z}_{\leq 0}$ such that $\langle\boldsymbol{\alpha}\rangle$ and $\langle\boldsymbol{\beta}\rangle$ are disjoint. Let $a\in\{1,\dots,d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ be coprime to $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ such that, for all $x\in\mathbb{R}$, we have $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a,x)\geq 0$. Then, all the Taylor coefficients at the origin of $q_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(z)$ are positive but its constant term, which is 0.*

To prove Proposition 42, we follow the method used by Delaygue in [**11**, Section 10.3], itself inspired by the work of Krattenthaler and Rivoal in [**21**]. We state three lemmas which enable us to prove Proposition 42.

LEMMA 43 (Lemma 2.1 in [**21**]). *Let $a(z)=\sum_{n=0}^{\infty}a_nz^n\in\mathbb{R}[[z]]$, $a_0=1$, be such that all Taylor coefficients at the origin of $\mathfrak{a}(z)=1-1/a(z)$ are nonnegative. Let*

$b(z)=\sum_{n=0}^{\infty} a_n h_n z^n$ *where* $(h_n)_{n\geq 0}$ *is a nondecreasing sequence of nonnegative real numbers. Then, all Taylor coefficients at the origin of* $b(z)/a(z)$ *are nonnegative.*

Furthermore, if all Taylor coefficients of $a(z)$ *and* $\mathfrak{a}(z)$ *are positive (excepted the constant term of* $\mathfrak{a}(z)$*) and if* $(h_n)_{n\geq 0}$ *is an increasing sequence, then all Taylor coefficients at the origin of* $b(z)/a(z)$ *are positive, except its constant term if* $h_0 = 0$.

The following lemma is a refined version of Kaluza's Theorem [**15**, Satz 3]. Initially, Satz 3 did not cover the case $a_{n+1}a_{n-1} > a_n^2$.

LEMMA 44 (Lemma 2.2 in [**21**]). *Let* $a(z)=\sum_{n=0}^{\infty} a_n z^n \in \mathbb{R}[[z]]$, $a_0 = 1$, *be such that* $a_1 > 0$ *and* $a_{n+1}a_{n-1} \geq a_n^2$ *for all positive integers* n. *Then, all Taylor coefficients of* $\mathfrak{a}(z) = 1 - 1/a(z)$ *are nonnegative.*

Furthermore, if we have $a_{n+1}a_{n-1} > a_n^2$ *for all positive integers* n*, then all Taylor coefficients of* $\mathfrak{a}(z)$ *are positive (except its constant term).*

For all $n \in \mathbb{N}$, we set

$$\mathcal{Q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(n) := \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n}.$$

By Lemmas 43 and 44, to prove Proposition 42, it suffices to prove the following result.

LEMMA 45. *Let* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r)$ *and* $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_s)$ *be tuples of parameters in* $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ *such that* $\langle \boldsymbol{\alpha} \rangle$ *and* $\langle \boldsymbol{\beta} \rangle$ *are disjoint. Let* $a \in \{1, \ldots, d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ *be coprime to* $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ *such that, for all* $x \in \mathbb{R}$, *we have* $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a, x) \geq 0$. *Then, for all positive integers* n, *we have*

$$\mathcal{Q}_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(n+1)\mathcal{Q}_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(n-1) > \mathcal{Q}_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(n)^2.$$

Furthermore, $\left(\sum_{i=1}^{r} H_{\langle a\alpha_i\rangle}(n) - \sum_{j=1}^{s} H_{\langle a\beta_j\rangle}(n)\right)_{n\geq 0}$ *is an increasing sequence.*

To prove Lemma 45, we first prove the following lemma that we also use in the proof of Theorem 8.

LEMMA 46. *Let* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r)$ *and* $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_s)$ *be tuples of parameters in* $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ *such that* $\langle \boldsymbol{\alpha} \rangle$ *and* $\langle \boldsymbol{\beta} \rangle$ *are disjoint. Let* $a \in \{1, \ldots, d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ *be coprime to* $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. *Let* $\gamma_1, \ldots, \gamma_t$ *be rational numbers such that* $\langle a\gamma_1 \rangle < \cdots < \langle a\gamma_t \rangle$ *and such that* $\big\{\langle a\gamma_1 \rangle, \ldots, \langle a\gamma_t \rangle\big\}$ *is the set of the numbers* $\langle a\gamma \rangle$ *when* γ *describes all the elements of* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$. *For all* $i \in \{1, \ldots, t\}$, *we define* $m_i := \#\big\{1 \leq j \leq r \,:\, \langle a\alpha_j \rangle = \langle a\gamma_i \rangle\big\} - \#\big\{1 \leq j \leq s \,:\, \langle a\beta_j \rangle = \langle a\gamma_i \rangle\big\}$.

Assume that, for all $x \in \mathbb{R}$, *we have* $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a, x) \geq 0$. *Then, for all* $i \in \{1, \ldots, t\}$ *and all* $b \in \mathbb{R}$, $b \geq 0$, *we have*

$$\sum_{k=1}^{i} \frac{m_k}{\langle a\gamma_k \rangle + b} > 0 \quad \text{and} \quad \prod_{k=1}^{i} \left(1 + \frac{1}{\langle a\gamma_k \rangle + b}\right)^{m_k} > 1.$$

PROOF OF LEMMA 46. First, observe that by Proposition 16, for all $j \in \{1, \ldots, t\}$, we have

$$\sum_{i=1}^{j} m_i = \xi_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(1, \langle a\gamma_j \rangle) \geq 0.$$

Furthermore, since $\langle a\boldsymbol{\alpha}\rangle$ and $\langle a\boldsymbol{\beta}\rangle$ are disjoint, for all $i \in \{1,\ldots,t\}$, we have $m_i \neq 0$. In particular, we obtain that $m_1 \geq 1$. It follows that we have

$$\frac{m_1}{\langle a\gamma_1\rangle + b} > 0 \quad \text{and} \quad \left(1 + \frac{1}{\langle a\gamma_1\rangle + b}\right)^{m_1} > 1.$$

Now assume that $t \geq 2$. We shall prove by induction on i that, for all $i \in \{2,\ldots,t\}$, we have

$$\sum_{k=1}^{i} \frac{m_k}{\langle a\gamma_k\rangle + b} > \frac{\sum_{k=1}^{i} m_k}{\langle a\gamma_i\rangle + b} \quad \text{and} \quad \prod_{k=1}^{i}\left(1 + \frac{1}{\langle a\gamma_k\rangle + b}\right)^{m_k} > \left(1 + \frac{1}{\langle a\gamma_i\rangle + b}\right)^{\sum_{k=1}^{i} m_k}. \tag{9.1}$$

We have $\langle a\gamma_1\rangle < \langle a\gamma_2\rangle$ and $m_1 > 0$, thus we get

$$\frac{m_1}{\langle a\gamma_1\rangle + b} + \frac{m_2}{\langle a\gamma_2\rangle + b} > \frac{m_1 + m_2}{\langle a\gamma_2\rangle + b}$$

and

$$\left(1 + \frac{1}{\langle a\gamma_1\rangle + b}\right)^{m_1}\left(1 + \frac{1}{\langle a\gamma_2\rangle + b}\right)^{m_2} > \left(1 + \frac{1}{\langle a\gamma_2\rangle + b}\right)^{m_1+m_2},$$

so that (9.1) holds for $i = 2$. We now assume that $t \geq 3$, and let $i \in \{2,\ldots,t-1\}$ be such that (9.1) holds. We obtain that

$$\sum_{k=1}^{i+1} \frac{m_k}{\langle a\gamma_k\rangle + b} > \frac{\sum_{k=1}^{i} m_k}{\langle a\gamma_i\rangle + b} + \frac{m_{i+1}}{\langle a\gamma_{i+1}\rangle + b} \tag{9.2}$$

and

$$\prod_{k=1}^{i+1}\left(1 + \frac{1}{\langle a\gamma_k\rangle + b}\right)^{m_k} > \left(1 + \frac{1}{\langle a\gamma_i\rangle + b}\right)^{\sum_{k=1}^{i} m_k}\left(1 + \frac{1}{\langle a\gamma_{i+1}\rangle + b}\right)^{m_{i+1}}. \tag{9.3}$$

Since $\langle a\gamma_i\rangle < \langle a\gamma_{i+1}\rangle$ and $\sum_{k=1}^{i} m_k \geq 0$, we obtain that

$$\frac{\sum_{k=1}^{i} m_k}{\langle a\gamma_i\rangle + b} \geq \frac{\sum_{k=1}^{i} m_k}{\langle a\gamma_{i+1}\rangle + b} \quad \text{and} \quad \left(1 + \frac{1}{\langle a\gamma_i\rangle + b}\right)^{\sum_{k=1}^{i} m_k} \geq \left(1 + \frac{1}{\langle a\gamma_{i+1}\rangle + b}\right)^{\sum_{k=1}^{i} m_k},$$

which, together with (9.2) and (9.3), finishes the induction on i. By (9.1) together with $\sum_{k=1}^{t} m_k \geq 0$, this completes the proof of Lemma 46. □

We can now prove Lemma 45 and hence complete the proof of Proposition 42 and Theorem 9.

Proof of Lemma 45. Throughout this proof, we use the notations defined in Lemma 46. For all nonnegative integers n, we have

$$\frac{\mathcal{Q}_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(n+1)}{\mathcal{Q}_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(1)\mathcal{Q}_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(n)}=\frac{1}{\mathcal{Q}_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(1)}\cdot\frac{\prod_{i=1}^{r}(\langle a\alpha_i\rangle+n)}{\prod_{j=1}^{s}(\langle a\beta_j\rangle+n)}$$
$$=\frac{\prod_{i=1}^{r}(1+n/\langle a\alpha_i\rangle)}{\prod_{j=1}^{s}(1+n/\langle a\beta_j\rangle)}$$
$$=\prod_{k=1}^{t}\left(1+\frac{n}{\langle a\gamma_k\rangle}\right)^{m_k}.$$

We deduce that, for all positive integers n, we obtain

$$\frac{\mathcal{Q}_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(n+1)\mathcal{Q}_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(n-1)}{\mathcal{Q}_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(n)^2}=\prod_{k=1}^{t}\left(\frac{1+n/\langle a\gamma_k\rangle}{1+(n-1)/\langle a\gamma_k\rangle}\right)^{m_k}$$
$$=\prod_{k=1}^{t}\left(1+\frac{1}{\langle a\gamma_k\rangle+n-1}\right)^{m_k}>1,$$

where the last inequality is obtained by Lemma 46 with $n-1$ instead of b.

Furthermore, for all $n\in\mathbb{N}$, we have

$$\sum_{i=1}^{r}H_{\langle a\alpha_i\rangle}(n+1)-\sum_{j=1}^{s}H_{\langle a\beta_j\rangle}(n+1)-\left(\sum_{i=1}^{r}H_{\langle a\alpha_i\rangle}(n)-\sum_{j=1}^{s}H_{\langle a\beta_j\rangle}(n)\right)$$
$$=\sum_{i=1}^{r}\frac{1}{\langle a\alpha_i\rangle+n}-\sum_{j=1}^{s}\frac{1}{\langle a\beta_j\rangle+n}$$
$$=\sum_{k=1}^{t}\frac{m_k}{\langle a\gamma_k\rangle+n}>0,$$

where the last inequality is obtained by Lemma 46 with n instead of b. It follows that $\left(\sum_{i=1}^{r}H_{\alpha_i}(n)-\sum_{j=1}^{s}H_{\beta_j}(n)\right)_{n\geq 0}$ is an increasing sequence and Lemma 45 is proved. □

10. Proof of Theorem 12

Throughout this section, we fix two tuples $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ of parameters in $\mathbb{Q}\setminus\mathbb{Z}_{\leq 0}$ of the same length such that $\langle\boldsymbol{\alpha}\rangle$ and $\langle\boldsymbol{\beta}\rangle$ are disjoint. Furthermore, we assume that $H_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ holds, that is, for all $a\in\{1,\dots,d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ coprime to $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and all $x\in\mathbb{R}$ satisfying $\min_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a)\preceq x\prec a$, we have $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a,x)\geq 1$. We will also use the notations defined at the beginning of Section 8.2.

10.1. A p-adic reformulation of Theorem 12. To prove Theorem 12, we have to prove that

$$\exp\left(\frac{S_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)}{\mathfrak{n}_{\boldsymbol{\alpha},\boldsymbol{\beta}}}\right)\in\mathbb{Z}[[z]].\qquad(10.1)$$

A classical method to prove the integrality of the Taylor coefficients of exponential of a power series is to reduce the problem to a p-adic one for all primes p and to use Dieudonné-Dwork's lemma as follows. Assertion (10.1) holds if and only if, for all primes p, we have

$$\exp\left(\frac{S_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)}{\mathfrak{n}_{\boldsymbol{\alpha},\boldsymbol{\beta}}}\right)\in\mathbb{Z}_p[[z]]. \tag{10.2}$$

Let us recall that we have

$$S_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)=\sum_{\substack{a=1\\ \gcd(a,d)=1}}^{d}\frac{G_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(z)}{F_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(z)}\in z\mathbb{Q}[[z]],$$

with $d=d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. By Proposition 2 applied to (10.2), we obtain that (10.1) holds if and only if, for all primes p, we have

$$S_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z^p)-pS_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)\in p\mathfrak{n}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\mathbb{Z}_p[[z]]. \tag{10.3}$$

The map $t\mapsto t^{(1)}$ is a permutation of the elements of $\{1,\dots,d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ coprime to $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. Hence, we have

$$S_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C'z^p)-pS_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C'z)=\sum_{\substack{t=1\\ \gcd(t,d)=1}}^{d}\left(\frac{G_{\langle t^{(1)}\boldsymbol{\alpha}\rangle,\langle t^{(1)}\boldsymbol{\beta}\rangle}}{F_{\langle t^{(1)}\boldsymbol{\alpha}\rangle,\langle t^{(1)}\boldsymbol{\beta}\rangle}}(C'z^p)-p\frac{G_{\langle t\boldsymbol{\alpha}\rangle,\langle t\boldsymbol{\beta}\rangle}}{F_{\langle t\boldsymbol{\alpha}\rangle,\langle t\boldsymbol{\beta}\rangle}}(C'z)\right),$$

with $d=d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and $C'=C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. By Theorem 6, we obtain

$$\begin{aligned}S_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z^p)-pS_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)&=p\sum_{\substack{b=1\\ \gcd(b,D)=1}}^{D}\sum_{t\in\Omega_b}\sum_{k=0}^{\infty}R_{k,b}(t)z^k\\ &=p\sum_{\substack{b=1\\ \gcd(b,D)=1}}^{D}\sum_{k=0}^{\infty}\left(\sum_{t\in\Omega_b}R_{k,b}(t)\right)z^k,\end{aligned}$$

with $R_{k,b}\in\mathcal{A}_b^*$ and, moreover if p divides $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, then we have

$$R_{k,b}\in\begin{cases}p^{-1-\lfloor\lambda_p/(p-1)\rfloor}\mathcal{A}_b \text{ if } \boldsymbol{\beta}\in\mathbb{Z}^r;\\ \mathcal{A}_b \text{ if } \boldsymbol{\beta}\notin\mathbb{Z}^r \text{ and } p-1\nmid\lambda_p;\\ \mathcal{A}_b \text{ if } \boldsymbol{\beta}\notin\mathbb{Z}^r,\ \mathfrak{m}_{\boldsymbol{\alpha},\boldsymbol{\beta}} \text{ is odd and } p=2.\end{cases}$$

By point (7) of Lemma 36, we have

$$\sum_{t\in\Omega_b}R_{k,b}(t)\in\mathfrak{n}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\mathbb{Z}_p. \tag{10.4}$$

Indeed, if p does not divide $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, then p does not divide $\mathfrak{n}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and $R_{k,b}(t)\in\mathbb{Z}_p$. Let us now assume that p divides $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ so that $\nu\geq 1$.

If $\boldsymbol{\beta}\in\mathbb{Z}^r$, then we have $v_p(\mathfrak{n}_{\boldsymbol{\alpha},\boldsymbol{\beta}})=\nu-2-\lfloor\lambda_p/(p-1)\rfloor$. If $\boldsymbol{\beta}\notin\mathbb{Z}^r$ and if $p-1\nmid\lambda_p$, then we have $p\neq 2$ and $v_p(\mathfrak{n}_{\boldsymbol{\alpha},\boldsymbol{\beta}})=\nu-1$. Let us now assume that $\boldsymbol{\beta}\notin\mathbb{Z}^r$ and that $p-1\mid\lambda_p$.

If $p \neq 2$ then $v_p(\mathfrak{n}_{\boldsymbol{\alpha},\boldsymbol{\beta}}) = 0$ or $\nu - 2$. On the other hand, if $p = 2$, then either $\mathfrak{m}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is even and $v_2(\mathfrak{n}_{\boldsymbol{\alpha},\boldsymbol{\beta}}) = 0$ or $\nu - 2$, or $\mathfrak{m}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is odd and $v_2(\mathfrak{n}_{\boldsymbol{\alpha},\boldsymbol{\beta}}) = \nu - 1$.

It follows that, in all cases, we have (10.3) and Theorem 12 is proved.

11. Proof of Theorem 8

Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be tuples of parameters in $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ such that $\langle \boldsymbol{\alpha} \rangle$ and $\langle \boldsymbol{\beta} \rangle$ are disjoint (this is equivalent to the irreducibility of $\mathcal{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$) and such that $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is N-integral. Theorem 12 implies Assertion $(iii) \Rightarrow (i)$ of Theorem 8. Indeed, if (iii) holds, then we have

$$z^{-1}\tilde{q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z) = \left(z^{-1}q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)\right)^{\varphi(d_{\boldsymbol{\alpha},\boldsymbol{\beta}})}$$

and, according to Theorem 12, $z^{-1}\tilde{q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$ is N-integral. Hence, it suffices to prove the following result.

Proposition 47. *Let $f(z) \in 1 + z\mathbb{Q}[[z]]$ be an N-integral power series and let a be a positive integer. Then $f(z)^{1/a}$ is an N-integral power series.*

Proof. We write $f(z) = 1 + zg(z)$ with $g(z) \in \mathbb{Q}[[z]]$. Thus, we obtain that

$$f(z)^{1/a} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{(-1/a)_n}{n!} z^n g(z)^n.$$

Since $f(z)$ is N-integral, there exists $C \in \mathbb{N}$ such that $g(Cz) \in \mathbb{Z}[[z]]$. Furthermore, by Theorem 3 applied with $\boldsymbol{\alpha} = (-1/a)$ and $\boldsymbol{\beta} = (1)$, we obtain that there exists $K \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$, we have

$$K^n \frac{(-1/a)_n}{n!} \in \mathbb{Z}.$$

It follows that $f(CKz)^{1/a} \in \mathbb{Z}[[z]]$, *i. e.* $f(z)^{1/a}$ is N-integral. $\square$

Furthermore, by definition, we have $(ii) \Rightarrow (i)$ of Theorem 8. Thus, we only have to prove that $(i) \Rightarrow (iii)$, $(i) \Rightarrow (ii)$ and that, if (i) holds, then we have either $\boldsymbol{\alpha} = (1/2)$ and $\boldsymbol{\beta} = (1)$ or there are at least two elements equal to 1 in $\langle \boldsymbol{\beta} \rangle$. Throughout this section, we assume that (i) holds, *i. e.* that $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is N-integral.

11.1. Proof of Assertion (iii) of Theorem 8. The aim of this section is to prove that $r = s$, that $H_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ holds and that, for all $a \in \{1, \dots, d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ coprime to $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, we have $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z) = q_{\langle a\boldsymbol{\alpha} \rangle, \langle a\boldsymbol{\beta} \rangle}(z)$. Since $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ are N-integral, there exists $C \in \mathbb{Q} \setminus \{0\}$ such that

$$F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(Cz) \in \mathbb{Z}[[z]] \quad \text{and} \quad q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(Cz) = Cz \exp\left(\frac{G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(Cz)}{F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(Cz)}\right) \in \mathbb{Z}[[z]].$$

Thus, for almost all primes p, we have

$$F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z) \in \mathbb{Z}_p[[z]] \quad \text{and} \quad \exp\left(\frac{G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)}{F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)}\right) \in \mathbb{Z}_p[[z]]. \tag{11.1}$$

We shall use Dieudonné-Dwork's lemma in order to get rid of the exponential map in (11.1).

Let p be a prime such that (11.1) holds. By Proposition 2 applied to (11.1), we obtain that

$$\frac{G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z^p)}{F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z^p)}-p\frac{G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)}{F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)}\in pz\mathbb{Z}_p[[z]].$$

Since $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)\in\mathbb{Z}_p[[z]]$, we get

$$G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z^p)F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)-pG_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z^p)\in pz\mathbb{Z}_p[[z]]. \tag{11.2}$$

In the sequel of the proof of Theorem 8, we use several times that (11.2) holds for almost all primes p.

11.1.1. *Proof of $r=s$.* We give a proof by contradiction assuming that $r\neq s$. Since $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is N-integral, Christol's criterion ensures that, for all $a\in\{1,\dots,d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ coprime to $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and all $x\in\mathbb{R}$, we have $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a,x)\geq 0$. In particular, since $r-s$ is the limit of $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(1,n)$ when $n\in\mathbb{Z}$ tends to $-\infty$, we obtain that $r-s\geq 1$. For all $n\in\mathbb{N}$, we write A_n for the assertion

$$\sum_{i=1}^r H_{\alpha_i}(n)-\sum_{j=1}^s H_{\beta_j}(n)=0.$$

First, we prove by induction on n that A_n is true for all $n\in\mathbb{N}$.

Assertion A_0 holds. Let n be a positive integer such that, for all integers k, $0\leq k<n$, A_k holds. The coefficient $\Phi_p(np)$ of z^{np} in (11.2) belongs to $p\mathbb{Z}_p$ and is equal to

$$\sum_{j=0}^n \mathcal{Q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(jp)\mathcal{Q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(n-j)\left(\sum_{i=1}^r\big(H_{\alpha_i}(n-j)-pH_{\alpha_i}(jp)\big)-\sum_{i=1}^s\big(H_{\beta_i}(n-j)-pH_{\beta_i}(jp)\big)\right).$$

By induction, we obtain that

$$\Phi_p(np)=\mathcal{Q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(n)\left(\sum_{i=1}^r H_{\alpha_i}(n)-\sum_{i=1}^s H_{\beta_i}(n)\right)\\ -p\sum_{j=1}^n\mathcal{Q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(jp)\mathcal{Q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(n-j)\left(\sum_{i=1}^r H_{\alpha_i}(jp)-\sum_{i=1}^s H_{\beta_i}(jp)\right).$$

Furthermore, according to Lemma 26, there exists a constant $M_{\boldsymbol{\alpha},\boldsymbol{\beta}}>0$ such that, for all $x\in[0,1/M_{\boldsymbol{\alpha},\boldsymbol{\beta}}[$, all primes p not dividing $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and all $\ell\in\mathbb{N}$, $\ell\geq 1$, we have $\Delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{p,\ell}(x)=0$. Hence, for almost all primes p and all $j\in\{1,\dots,n\}$, we have

$$v_p\big(\mathcal{Q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(jp)\big)=\sum_{\ell=1}^\infty\Delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{p,\ell}\left(\frac{jp}{p^\ell}\right)=\Delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{p,1}(j)+\sum_{\ell=1}^\infty\Delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{p,\ell+1}\left(\frac{j}{p^\ell}\right)=\Delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{p,1}(j)=j(r-s). \tag{11.3}$$

According to Lemma 23, for almost all primes p and all the elements α in $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$, we have $\mathfrak{D}_p(\alpha)=\mathfrak{D}_p(\langle\alpha\rangle)$, so that $\mathfrak{D}_p(\alpha)=\langle\omega\alpha\rangle$ where $\omega\in\{1,\dots,d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ satisfies $\omega p\equiv 1$

mod $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. Thus we get

$$pH_\alpha(jp) = p\sum_{k=0}^{p-1}\sum_{i=0}^{j-1}\frac{1}{\alpha+k+ip}$$
$$= H_{\mathfrak{D}_p(\alpha)}(j) + p\sum_{\substack{k=0\\ k\neq p\mathfrak{D}_p(\alpha)-\alpha}}^{p-1}\sum_{i=0}^{j-1}\frac{1}{\alpha+k+ip} \in H_{\langle\omega\alpha\rangle}(j) + p\mathbb{Z}_p,$$

which leads to

$$p\left(\sum_{i=1}^{r} H_{\alpha_i}(jp) - \sum_{i=1}^{s} H_{\beta_i}(jp)\right) \equiv \sum_{i=1}^{r} H_{\langle\omega\alpha_i\rangle}(j) - \sum_{i=1}^{s} H_{\langle\omega\beta_i\rangle}(j) \mod p\mathbb{Z}_p. \tag{11.4}$$

Furthermore, for almost all primes p, we have

$$\left\{\sum_{i=1}^{r} H_{\langle\omega\alpha_i\rangle}(j) - \sum_{i=1}^{s} H_{\langle\omega\beta_i\rangle}(j) \,:\, 1\leq j\leq n,\, 1\leq\omega\leq d_{\boldsymbol{\alpha},\boldsymbol{\beta}},\, \gcd(\omega, d_{\boldsymbol{\alpha},\boldsymbol{\beta}})=1\right\} \subset \mathbb{Z}_p,$$

which, together with (11.3) and (11.4), gives us that

$$-p\mathcal{Q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(jp)\mathcal{Q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(n-j)\left(\sum_{i=1}^{r} H_{\alpha_i}(jp) - \sum_{i=1}^{s} H_{\beta_i}(jp)\right) \in p^{r-s}\mathbb{Z}_p,$$

for almost all primes p and all $j\in\{1,\dots,n\}$. In addition, for almost all primes p, we have

$$\mathcal{Q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(n)\left(\sum_{i=1}^{r} H_{\alpha_i}(n) - \sum_{j=1}^{s} H_{\beta_j}(n)\right) \in \mathbb{Z}_p^\times\cup\{0\} \quad\text{and}\quad \mathcal{Q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(n)\neq 0.$$

Since $\Phi_p(np)\in p\mathbb{Z}_p$ and $r-s\geq 1$, we obtain that A_n holds, which finishes the induction on n.

It follows that, for all $n\in\mathbb{N}$, we obtain that

$$\sum_{i=1}^{r}\frac{1}{\alpha_i+n} - \sum_{i=1}^{s}\frac{1}{\beta_i+n} = \sum_{i=1}^{r}\big(H_{\alpha_i}(n+1)-H_{\alpha_i}(n)\big) - \sum_{i=1}^{s}\big(H_{\beta_i}(n+1)-H_{\beta_i}(n)\big) = 0,$$

contradicting that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are disjoint since

$$\sum_{i=1}^{r}\frac{1}{\alpha_i+X} - \sum_{i=1}^{s}\frac{1}{\beta_i+X} \in \mathbb{Q}(X)$$

must be a nontrivial rational fraction in this case. Thus we have $r=s$ as expected. $\square$

11.1.2. *Proof of $H_{\boldsymbol{\alpha},\boldsymbol{\beta}}$.* Let us recall that, since $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is N-integral, for all $a \in \{1,\dots,d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ coprime to $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and all $x \in \mathbb{R}$, we have $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a,x) \geq 0$. We give a proof of $H_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ by contradiction, assuming that there exist $a \in \{1,\dots,d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ coprime to $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and $x_0 \in \mathbb{R}$ such that $\min_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a) \preceq x_0 \prec a$ and $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a,x_0) = 0$. Let α and β be such that

$$a\beta = \max\big(\{a\gamma \,:\, a\gamma \preceq x_0,\, \gamma \text{ is in } \boldsymbol{\alpha} \text{ or } \boldsymbol{\beta}\}, \preceq\big)$$

and

$$a\alpha = \min\big(\{a\gamma \,:\, x_0 \prec a\gamma,\, \gamma \text{ equals } 1 \text{ or is in } \boldsymbol{\alpha} \text{ or } \boldsymbol{\beta}\}, \preceq\big).$$

It follows that, for all $x \in \mathbb{R}$ satisfying $a\beta \preceq x \prec a\alpha$, we have $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a,x) = 0$. Observe that, since $\langle\boldsymbol{\alpha}\rangle$ and $\langle\boldsymbol{\beta}\rangle$ are disjoint, $\langle a\boldsymbol{\alpha}\rangle$ and $\langle a\boldsymbol{\beta}\rangle$ are also disjoint, thus β is a component of $\boldsymbol{\beta}$ and α equals 1 or is an element of $\boldsymbol{\alpha}$ because $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a,\cdot)$ is nonnegative on $\mathbb{R}$.

Let us write $\mathfrak{P}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a)$ for the set of all primes p such that $ap \equiv 1 \mod d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. For all large enough $p \in \mathfrak{P}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a)$, Lemma 23 gives us that $\mathfrak{D}_p(\alpha) = \mathfrak{D}_p(\langle\alpha\rangle) = \langle a\alpha\rangle$ and $\mathfrak{D}_p(\beta) = \langle a\beta\rangle$. On the one hand, if $\langle a\beta\rangle < \langle a\alpha\rangle$, then, for almost all $p \in \mathfrak{P}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a)$, we obtain that

$$\mathfrak{D}_p(\alpha) + \frac{\lfloor 1-\alpha\rfloor}{p} - \mathfrak{D}_p(\beta) - \frac{\lfloor 1-\beta\rfloor}{p} \geq \frac{1}{d_{\boldsymbol{\alpha},\boldsymbol{\beta}}} + \frac{\lfloor 1-\alpha\rfloor}{p} - \frac{\lfloor 1-\beta\rfloor}{p} \geq \frac{1}{p}.$$

On the other hand, if $\langle a\beta\rangle = \langle a\alpha\rangle$ and $\beta > \alpha$, then we have $\langle\beta\rangle = \langle\alpha\rangle$ so $\beta \geq 1+\alpha$ and

$$\mathfrak{D}_p(\alpha) + \frac{\lfloor 1-\alpha\rfloor}{p} - \mathfrak{D}_p(\beta) - \frac{\lfloor 1-\beta\rfloor}{p} = \frac{\lfloor 1-\alpha\rfloor}{p} - \frac{\lfloor 1-\beta\rfloor}{p} \geq \frac{1}{p}.$$

In both cases, we obtain that, for almost all $p \in \mathfrak{P}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a)$, there exists $v_p \in \{0,\dots,p-1\}$ such that

$$\mathfrak{D}_p(\beta) + \frac{\lfloor 1-\beta\rfloor}{p} \leq \frac{v_p}{p} < \mathfrak{D}_p(\alpha) + \frac{\lfloor 1-\alpha\rfloor}{p},$$

which, together with Lemma 28, gives us that $\Delta^{p,1}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(v_p/p) = 0$ for all large enough $p \in \mathfrak{P}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a)$. Furthermore, by Lemma 26, for almost all $p \in \mathfrak{P}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a)$ and all $\ell \in \mathbb{N}$, $\ell \geq 1$, $\Delta^{p,\ell}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ vanishes on $[0,1/p]$ so that

$$v_p\big(\mathcal{Q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(v_p)\big) = \sum_{\ell=1}^{\infty} \Delta^{p,\ell}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\left(\frac{v_p}{p^\ell}\right) = \Delta^{p,1}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\left(\frac{v_p}{p}\right) = 0,$$

i. e. $\mathcal{Q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(v_p) \in \mathbb{Z}_p^\times$. Now looking at the coefficient of z^{v_p} in (11.2), one obtains that

$$-p\mathcal{Q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(v_p)\sum_{i=1}^{r}\big(H_{\alpha_i}(v_p) - H_{\beta_i}(v_p)\big) \in p\mathbb{Z}_p.$$

To get a contradiction, we shall prove that, for all large enough $p \in \mathfrak{P}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a)$, we have

$$p\left(\sum_{i=1}^{r} H_{\alpha_i}(v_p) - \sum_{i=1}^{r} H_{\beta_i}(v_p)\right) \in \mathbb{Z}_p^\times. \tag{11.5}$$

Indeed, for all elements γ of $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ and all large enough $p \in \mathfrak{P}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a)$, we have

$$pH_\gamma(v_p) = p\sum_{k=0}^{v_p-1}\frac{1}{\gamma+k} \equiv \frac{\rho(v_p,\gamma)}{\mathfrak{D}_p(\gamma)} \mod p\mathbb{Z}_p$$
$$\equiv \frac{\rho(v_p,\gamma)}{\langle a\gamma\rangle} \mod p\mathbb{Z}_p.$$

Furthermore, we have

$$\rho(v_p,\gamma)=1 \iff v_p \geq p\mathfrak{D}_p(\gamma)-\gamma+1 \iff v_p \geq p\mathfrak{D}_p(\gamma)+\lfloor 1-\gamma\rfloor$$
$$\iff \frac{v_p}{p} \geq \mathfrak{D}_p(\gamma)+\frac{\lfloor 1-\gamma\rfloor}{p},$$

because $p\mathfrak{D}_p(\gamma)-\gamma \in \mathbb{Z}$ which leads to $v_p \geq p\mathfrak{D}_p(\gamma)+\lfloor 1-\gamma\rfloor \Rightarrow v_p \geq p\mathfrak{D}_p+\lfloor 1-\gamma\rfloor+\{1-\gamma\}$. Thus, by Lemma 28, for all large enough $p\in\mathfrak{P}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a)$, we have $\rho(v_p,\gamma)=1$ if $a\gamma \preceq a\beta$ and $\rho(v_p,\gamma)=0$ otherwise.

Now, let $\gamma_1,\cdots,\gamma_t$ be rational numbers such that $\langle a\gamma_1\rangle < \cdots < \langle a\gamma_t\rangle$ and such that $\{\langle a\gamma_1\rangle,\dots,\langle a\gamma_t\rangle\}$ is the set of the numbers $\langle a\gamma\rangle$ when γ describes all the elements of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ satisfying $a\gamma\preceq a\beta$. For all $i\in\{1,\dots,t\}$, we define

$$m_i := \#\big\{1\leq j\leq r\,:\,\langle a\alpha_j\rangle=\langle a\gamma_i\rangle\big\}-\#\big\{1\leq j\leq r\,:\,\langle a\beta_j\rangle=\langle a\gamma_i\rangle\big\}.$$

Then, we obtain that

$$p\left(\sum_{i=1}^r H_{\alpha_i}(v_p)-\sum_{j=1}^r H_{\beta_j}(v_p)\right) \equiv \sum_{i=1}^r\frac{\rho(v_p,\alpha_i)}{\langle a\alpha_i\rangle}-\sum_{j=1}^r\frac{\rho(v_p,\beta_j)}{\langle a\beta_j\rangle} \mod p\mathbb{Z}_p$$
$$\equiv \sum_{i=1}^t\frac{m_i}{\langle a\gamma_i\rangle} \mod p\mathbb{Z}_p.$$

For almost all primes p, we have $\sum_{i=1}^t(m_i/\langle a\gamma_i\rangle)\in\mathbb{Z}_p^\times\cup\{0\}$. Thus, to prove (11.5), it suffices to prove that

$$\sum_{i=1}^t\frac{m_i}{\langle a\gamma_i\rangle}\neq 0,$$

which follows by Lemma 46 applied with $b=0$. This finishes the proof of $H_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. □

11.1.3. *Last step in the proof of Assertion (iii) of Theorem 8.* To finish the proof of Assertion (iii) of Theorem 8, it remains to prove that, for all $a\in\{1,\dots,d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ coprime to $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, we have $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)=q_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(z)$. For that purpose, we shall use Dwork's results presented in [**12**] on the integrality of Taylor coefficients at the origin of power series similar to $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. We remind the reader that, by Sections 11.1.1 and 11.1.2, we have $r=s$ and $H_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ holds.

More precisely, we prove the following lemma which shows that, under these assumptions, we can apply Dwork's result [**12**, Theorem 4.1] for almost all primes.

LEMMA 48. *Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be two tuples of parameters in $\mathbb{Q}\setminus\mathbb{Z}_{\leq 0}$ with the same numbers of elements. If $\langle\boldsymbol{\alpha}\rangle$ and $\langle\boldsymbol{\beta}\rangle$ are disjoint (this is equivalent to the irreducibility of $\mathcal{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$) and if $H_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ holds, then for almost all primes p not dividing $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, we have*

$$\frac{G_{\mathfrak{D}_p(\boldsymbol{\alpha}),\mathfrak{D}_p(\boldsymbol{\beta})}(z^p)}{F_{\mathfrak{D}_p(\boldsymbol{\alpha}),\mathfrak{D}_p(\boldsymbol{\beta})}(z^p)}-p\frac{G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)}{F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)}\in p\mathbb{Z}_p[[z]].$$

REMARK 49. *Lemma 48 in combination with Lemma 18 gives us that $\tilde{q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)\in\mathbb{Z}_p[[z]]$ for almost all primes p.*

PROOF. If p is a prime not dividing $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, then the elements of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ lie in $\mathbb{Z}_p$ and

$$\frac{G_{\mathfrak{D}_p(\boldsymbol{\alpha}),\mathfrak{D}_p(\boldsymbol{\beta})}(z^p)}{F_{\mathfrak{D}_p(\boldsymbol{\alpha}),\mathfrak{D}_p(\boldsymbol{\beta})}(z^p)}-p\frac{G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)}{F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)}\in \mathbb{Q}_p[[z]].$$

Furthermore, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ have the same number of elements so that Lemma 48 follows from the conclusion of Dwork's theorem [**12**, Theorem 4.1]. In the sequel of this proof, we check that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ satisfy the hypotheses of [**12**, Theorem 4.1] for almost all primes p. We use the notations defined in Section 4.2.1. For a given fixed prime p not dividing $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, the hypotheses of [**12**, Theorem 4.1] read

- (v) for all $i\in\{1,\dots,r'\}$ and all $k\in\mathbb{N}$, we have $\mathfrak{D}_p^k(\beta_i)\in\mathbb{Z}_p^\times$;
- (vi) for all $a\in[0,p)$ and all $k\in\mathbb{N}$, we have either $N^k_{p,\boldsymbol{\alpha}}(a)=N^k_{p,\boldsymbol{\beta}}(a+)=0$ or $N^k_{p,\boldsymbol{\alpha}}(a)-N^k_{p,\boldsymbol{\beta}}(a+)\geq 1$.

If p is a large enough prime, then, by Lemma 23, for all $i\in\{1,\dots,r'\}$, we have $\mathfrak{D}_p(\beta_i)=\mathfrak{D}_p(\langle\beta_i\rangle)$ so that

$$\mathfrak{D}_p(\beta_i)\in\left\{\frac{1}{d_{\boldsymbol{\alpha},\boldsymbol{\beta}}},\dots,\frac{d_{\boldsymbol{\alpha},\boldsymbol{\beta}}-1}{d_{\boldsymbol{\alpha},\boldsymbol{\beta}}},1\right\}\subset\mathbb{Z}_p^\times. \tag{11.6}$$

Thus, for all large enough primes p, $\boldsymbol{\beta}$ satisfies Assertion (v).

Let α and β be elements of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. First, we prove that, for all large enough primes p, we have

$$p\mathfrak{D}_p(\alpha)-\alpha\leq p\mathfrak{D}_p(\beta)-\beta\Longleftrightarrow \omega\alpha\preceq\omega\beta, \tag{11.7}$$

where $\omega\in\{1,\dots,d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ satisfies $\omega p\equiv 1 \mod d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. Assume that p is large enough so that, by Lemma 23, we get $\mathfrak{D}_p(\alpha)=\langle\omega\alpha\rangle$ and $\mathfrak{D}_p(\beta)=\langle\omega\beta\rangle$. In particular, we obtain that

$$\mathfrak{D}_p(\alpha)=\mathfrak{D}_p(\beta)\quad\text{or}\quad\big|\mathfrak{D}_p(\alpha)-\mathfrak{D}_p(\beta)\big|\geq\frac{1}{d_{\boldsymbol{\alpha},\boldsymbol{\beta}}}.$$

Thus, for all large enough primes p, we have

$$\begin{aligned}p\mathfrak{D}_p(\alpha)-\alpha\leq p\mathfrak{D}_p(\beta)-\beta&\Longleftrightarrow\mathfrak{D}_p(\alpha)-\mathfrak{D}_p(\beta)\leq\frac{\alpha-\beta}{p}\\&\Longleftrightarrow\Big(\mathfrak{D}_p(\alpha)<\mathfrak{D}_p(\beta)\quad\text{or}\quad\big(\mathfrak{D}_p(\alpha)=\mathfrak{D}_p(\beta)\quad\text{and}\quad\alpha\geq\beta\big)\Big)\\&\Longleftrightarrow\omega\alpha\preceq\omega\beta,\end{aligned}$$

as expected. Now, we observe that, if $N_{p,\boldsymbol{\beta}}^k(a+)=0$, then Assertion (vi) is trivial, so we may assume that $N_{p,\boldsymbol{\beta}}^k(a+)\geq 1$. We set $\boldsymbol{\beta}':=(\beta_1,\dots,\beta_{r'})$. Let us write $\theta_p^k(x)$ for $p\mathfrak{D}_p^{k+1}(x)-\mathfrak{D}_p^k(x)$, and let γ be the component of $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}'$ such that $\theta_p^k(\gamma)$ is the largest element of

$$\Big\{\theta_p^k(\alpha_i)\,:\,1\leq i\leq r,\,\theta_p^k(\alpha_i)<a\Big\}\bigcup\Big\{\theta_p^k(\beta_j)\,:\,1\leq j\leq r',\,\theta_p^k(\beta_j)\leq a\Big\}.$$

Since $\langle\boldsymbol{\alpha}\rangle$ and $\langle\boldsymbol{\beta}\rangle$ are disjoint, $\mathfrak{D}_p^k(\boldsymbol{\alpha})$ and $\mathfrak{D}_p^k(\boldsymbol{\beta}')$ are also disjoint and, according to (11.7), $\theta_p^k(\boldsymbol{\alpha})$ and $\theta_p^k(\boldsymbol{\beta}')$ are disjoint. It follows that $N_{p,\boldsymbol{\alpha}}^k(a)-N_{p,\boldsymbol{\beta}}^k(a+)$ is equal to

$$\begin{aligned}&\#\big\{1\leq i\leq r\,:\,\theta_p^k(\alpha_i)\leq\theta_p^k(\gamma)\big\}-\#\big\{1\leq i\leq r'\,:\,\theta_p^k(\beta_j)\leq\theta_p^k(\gamma)\big\}\\&\qquad=\#\big\{1\leq i\leq r\,:\,\omega\mathfrak{D}_p^k(\alpha_i)\preceq\omega\mathfrak{D}_p^k(\gamma)\big\}-\#\big\{1\leq i\leq r'\,:\,\omega\mathfrak{D}_p^k(\beta_j)\preceq\omega\mathfrak{D}_p^k(\gamma)\big\}.\end{aligned}$$

If $k=0$, then we obtain that $\omega\mathfrak{D}_p^k(\alpha)\preceq\omega\mathfrak{D}_p^k(\gamma)\Leftrightarrow\omega\alpha\preceq\omega\gamma$ with $\min_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\omega)\preceq\omega\gamma\prec\omega$ since $\gamma\neq 1$. Indeed, if γ is an element of $\boldsymbol{\beta}'$ then $\gamma\neq 1$, else γ is an element of $\boldsymbol{\alpha}$ and $\theta_p^k(\gamma)<a$ so that $\gamma\neq 1$. Thus we have $N_{p,\boldsymbol{\alpha}}^0(a)-N_{p,\boldsymbol{\beta}}^0(a+)=\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\omega,\omega\gamma)$ and, by $H_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, we get $N_{p,\boldsymbol{\alpha}}^0(a)-N_{p,\boldsymbol{\beta}}^0(a+)\geq 1$ as expected.

If $k\geq 1$, then, for all elements α of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}'$, we have $\mathfrak{D}_p^k(\alpha)=\langle\omega^k\alpha\rangle$ and $\big\langle\omega\mathfrak{D}_p^k(\alpha)\big\rangle=\big\langle\omega\langle\omega^k\alpha\rangle\big\rangle=\langle\omega^{k+1}\alpha\rangle$. We deduce that we have $\omega\mathfrak{D}_p^k(\alpha)\preceq\omega\mathfrak{D}_p^k(\gamma)\Leftrightarrow\langle\omega^{k+1}\alpha\rangle\leq\langle\omega^{k+1}\gamma\rangle$ because

$$\langle\omega^{k+1}\alpha\rangle=\langle\omega^{k+1}\gamma\rangle\Longleftrightarrow\langle\alpha\rangle=\langle\gamma\rangle\Longleftrightarrow\langle\omega^k\alpha\rangle=\langle\omega^k\gamma\rangle.$$

If $\langle\gamma\rangle<1$, then $\langle\omega^{k+1}\gamma\rangle<1$ and we obtain that

$$N_{p,\boldsymbol{\alpha}}^k(a)-N_{p,\boldsymbol{\beta}}^k(a+)=\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\omega^{k+1},\langle\omega^{k+1}\gamma\rangle+)\geq 1.$$

On the other hand, if $\langle\gamma\rangle=1$, then we get $N_{p,\boldsymbol{\alpha}}^k(a)-N_{p,\boldsymbol{\beta}}^k(a+)=r-r'$. Note that $r'<r$ since there is at least one element of $\boldsymbol{\beta}$ equal to 1. Indeed, according to $H_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, if $x\in\mathbb{R}$ satisfies $\min_{\boldsymbol{\alpha},\boldsymbol{\beta}}(1)\preceq x\prec 1$, then we have $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(1,x)\geq 1$. Since $\langle\boldsymbol{\alpha}\rangle$ and $\langle\boldsymbol{\beta}\rangle$ are disjoint, we have $\langle\min_{\boldsymbol{\alpha},\boldsymbol{\beta}}(1)\rangle<1$ so that $\min_{\boldsymbol{\alpha},\boldsymbol{\beta}}(1)\preceq 2\prec 1$ and

$$1\leq\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(1,2)=\#\{1\leq i\leq r\,:\,\alpha_i\neq 1\}-\#\{1\leq j\leq r\,:\,\beta_j\neq 1\}.$$

We deduce that there is at least one $j\in\{1,\dots,r\}$ such that $\beta_j=1$ and we obtain that

$$N_{p,\boldsymbol{\alpha}}^k(a)-N_{p,\boldsymbol{\beta}}^k(a+)=r-r'\geq 1,$$

as expected. Thus Assertion (vi) holds and Lemma 48 is proved. □

Now we fix $a\in\{1,\dots,d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ coprime to $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. For all large enough primes $p\in\mathfrak{P}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a)$ and all the elements α of $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$, we have $\mathfrak{D}_p(\alpha)=\langle a\alpha\rangle$. By Lemma 48, we obtain that, for almost all primes $p\in\mathfrak{P}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a)$, we have

$$\frac{G_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(z^p)}{F_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(z^p)}-p\frac{G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)}{F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)}\in p\mathbb{Z}_p[[z]].$$

Furthermore, since $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$ is N-integral, for almost all primes p, we have

$$\frac{G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z^p)}{F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z^p)}-p\frac{G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)}{F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)}\in p\mathbb{Z}_p[[z]].$$

Thus, for almost all primes $p\in\mathfrak{P}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a)$, we obtain that

$$\frac{G_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(z^p)}{F_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(z^p)}-\frac{G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z^p)}{F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z^p)}\in p\mathbb{Z}_p[[z]],$$

which leads to

$$\frac{G_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(z)}{F_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(z)}-\frac{G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)}{F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)}\in p\mathbb{Z}_p[[z]].$$

By Dirichlet's theorem, there are infinitely many primes in $\mathfrak{P}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a)$ so that we have

$$\frac{G_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(z)}{F_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(z)}=\frac{G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)}{F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)},$$

which implies that $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)=q_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(z)$ as expected. This finishes the proof of Assertion (iii) of Theorem 8.

11.2. Proof of Assertion (ii) of Theorem 8. We have to prove that $(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)^{-1}q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)$ is in $\mathbb{Z}[[z]]$. By Section 11.1, Assertion (iii) of Theorem 8 holds, *i. e.* we have $r=s$, $H_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ holds and, for all $a\in\{1,\dots,d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ coprime to $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, we have $q_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(z)=q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$ so that

$$\frac{G_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(z)}{F_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(z)}=\frac{G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)}{F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)}. \tag{11.8}$$

By Theorem 6 in combination with (11.8), we obtain that

$$\frac{G_{\boldsymbol{\alpha},\boldsymbol{\beta}}}{F_{\boldsymbol{\alpha},\boldsymbol{\beta}}}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z^p)-p\frac{G_{\boldsymbol{\alpha},\boldsymbol{\beta}}}{F_{\boldsymbol{\alpha},\boldsymbol{\beta}}}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)\in p\mathbb{Z}_p[[z]],$$

so that, according to Proposition 2, we have $(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)^{-1}q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)\in\mathbb{Z}_p[[z]]$. Since p is an arbitrary prime; we get $(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)^{-1}q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)\in\mathbb{Z}[[z]]$, as expected.

11.3. Last step in the proof of Theorem 8. To complete the proof of Theorem 8, we have to prove that we have either $\boldsymbol{\alpha}=(1/2)$ and $\boldsymbol{\beta}=(1)$, or $r\geq 2$ and there are at least two 1's in $\boldsymbol{\beta}$. We shall distinguish two cases.

- Case 1: We assume that $r=1$.

As already proved at the end of the proof of Lemma 48, there is at least one element of $\boldsymbol{\beta}$ equal to 1. Thus we obtain that $\boldsymbol{\beta}=(1)$. We write $\boldsymbol{\alpha}=(\alpha)$. Since Assertion (iii) of Theorem 8 holds, for all $a\in\{1,\dots,d(\alpha)\}$ coprime to $d(\alpha)$, we have $G_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(z)/F_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(z)=G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)/F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$, *i. e.*

$$F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)G_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(z)=F_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(z)G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z). \tag{11.9}$$

Now looking at the coefficient of z in the power series involved in (11.9), one obtains that

$$\langle a\alpha\rangle\left(\frac{1}{\langle a\alpha\rangle}-1\right)=\alpha\left(\frac{1}{\alpha}-1\right).$$

We deduce that, for all $a\in\{1,\dots,d(\alpha)\}$ coprime to $d(\alpha)$, we have $\langle a\alpha\rangle=\alpha$. Thus we get that

$$\left\{\frac{\kappa}{d(\alpha)}\,:\,1\leq\kappa\leq d(\alpha),\,\gcd\big(\kappa,d(\alpha)\big)=1\right\}=\big\{\langle a\alpha\rangle\,:\,1\leq a\leq d(\alpha),\,\gcd\big(a,d(\alpha)\big)=1\big\}$$
$$=\{\alpha\},$$

which implies that $\alpha=1/2$ as expected.

- Case 2: We assume that $r\geq 2$.

We already know that there is at least one element of $\boldsymbol{\beta}$ equal to 1. Since $\langle\boldsymbol{\alpha}\rangle$ and $\langle\boldsymbol{\beta}\rangle$ are disjoint, for all the elements α of $\boldsymbol{\alpha}$, we have $\langle\alpha\rangle<1$. Furthermore, for all $a\in\{1,\dots,d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ coprime to $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, we have

$$\begin{aligned}\xi_{\langle\boldsymbol{\alpha}\rangle,\langle\boldsymbol{\beta}\rangle}(a,1-)&=\#\big\{1\leq i\leq r\,:\,\langle\alpha_i\rangle\neq 1\big\}-\#\big\{1\leq i\leq r\,:\,\langle\beta_i\rangle\neq 1\big\}\\&=r-\#\big\{1\leq i\leq r\,:\,\langle\beta_i\rangle\neq 1\big\}.\end{aligned}$$

It follows that we have to prove that $\xi_{\langle\boldsymbol{\alpha}\rangle,\langle\boldsymbol{\beta}\rangle}(a,1-)\geq 2$.

Let γ be an element of $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ with the largest exact denominator. Then, there exists $a\in\{1,\dots,d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ coprime to $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ such that $\langle a\gamma\rangle=1/d(\gamma)$. By $H_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ in combination with Lemma 18, we obtain that $H_{\langle\boldsymbol{\alpha}\rangle,\langle\boldsymbol{\beta}\rangle}$ holds. In addition, we have $\big\langle a\langle\gamma\rangle\big\rangle=\langle a\gamma\rangle=1/d(\gamma)$ so that $\xi_{\langle\boldsymbol{\alpha}\rangle,\langle\boldsymbol{\beta}\rangle}\big(a,1/d(\gamma)+\big)\geq 1$. Since $\langle a\boldsymbol{\alpha}\rangle$ and $\langle a\boldsymbol{\beta}\rangle$ are disjoint and have elements larger than or equal to $1/d(\gamma)$, we obtain that γ is a component of $\boldsymbol{\alpha}$.

Furthermore, there exists $a\in\{1,\dots,d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ coprime to $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ such that

$$\langle a\gamma\rangle=\frac{d(\gamma)-1}{d(\gamma)}=:\kappa.$$

Thus κ is the largest element distinct from 1 in $\langle a\boldsymbol{\alpha}\rangle$ and $\langle a\boldsymbol{\beta}\rangle$, and we obtain that $\xi_{\langle\boldsymbol{\alpha}\rangle,\langle\boldsymbol{\beta}\rangle}(a,\kappa+)=\xi_{\langle\boldsymbol{\alpha}\rangle,\langle\boldsymbol{\beta}\rangle}(a,1-)$. If $\langle\min_{\langle\boldsymbol{\alpha}\rangle,\langle\boldsymbol{\beta}\rangle}(a)\rangle=\kappa$, then all the elements of $\langle\boldsymbol{\beta}\rangle$ are equal to 1 and the result is proved. Otherwise, we have $\langle\min_{\langle\boldsymbol{\alpha}\rangle,\langle\boldsymbol{\beta}\rangle}(a)\rangle<\kappa$ so that $\xi_{\langle\boldsymbol{\alpha}\rangle,\langle\boldsymbol{\beta}\rangle}(a,\kappa-)\geq 1$. Since γ is an element of $\boldsymbol{\alpha}$, we obtain that $\xi_{\langle\boldsymbol{\alpha}\rangle,\langle\boldsymbol{\beta}\rangle}(a,\kappa+)\geq 2$ as expected. This finishes the proof of Theorem 8.

12. Proof of Theorem 10

According to Dieudonné-Dwork's lemma, we have to prove that, for all primes p,

$$\frac{G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z^p)}{F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z^p)}-p\frac{G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)}{F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)}\in p\mathfrak{n}'_{\boldsymbol{\alpha},\boldsymbol{\beta}}\mathbb{Z}_p[[z]].$$

Note that Theorem 8 ensures that the hypotheses of Theorem 6 are satisfied. According to Theorem 6, for any prime p, there exists $a \in \{1, ..., d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ (with the notations of Theorem 6, one may take $a = t^{(1)}$ with $t = 1$) such that $\gcd(a, d_{\boldsymbol{\alpha},\boldsymbol{\beta}}) = 1$ and

$$\frac{G_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z^p)}{F_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z^p)} - p\frac{G_{\langle \boldsymbol{\alpha}\rangle,\langle \boldsymbol{\beta}\rangle}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)}{F_{\langle \boldsymbol{\alpha}\rangle,\langle \boldsymbol{\beta}\rangle}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z)} \in p\mathfrak{n}'_{\boldsymbol{\alpha},\boldsymbol{\beta}}\mathbb{Z}_p[[z]].$$

The conclusion follows from assertion (iii) of Theorem 8, which ensures that

$$\frac{G_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z^p)}{F_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z^p)} = \frac{G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z^p)}{F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(C'_{\boldsymbol{\alpha},\boldsymbol{\beta}}z^p)}$$

and

$$\frac{G_{\langle \boldsymbol{\alpha}\rangle,\langle \boldsymbol{\beta}\rangle}(z)}{F_{\langle \boldsymbol{\alpha}\rangle,\langle \boldsymbol{\beta}\rangle}(z)} = \frac{G_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)}{F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)}.$$

13. Proof of Corollary 14

According to Theorem 8, for all $a \in \{1, ..., d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ such that $\gcd(a, d_{\boldsymbol{\alpha},\boldsymbol{\beta}}) = 1$, we have $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z) = q_{\langle a\boldsymbol{\alpha}\rangle,\langle a\boldsymbol{\beta}\rangle}(z)$. Therefore, we have $z^{-1}\widetilde{q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z) = (z^{-1}q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z))^{\varphi(d_{\boldsymbol{\alpha},\boldsymbol{\beta}})}$. Now, the result follows from Theorem 12.

Bibliography

[1] G. Almkvist, C. van Enckevort, D. van Straten, W. Zudilin, *Tables of Calabi-Yau equations*, arXiv:math/0507430v2 [math.AG] (2010).

[2] V. V. Batyrev, D. van Straten, *Generalized hypergeometric functions and rational curves on Calabi-Yau complete intersections in toric varieties*, Comm. Math. Phys. **168** (1995), no. 3, 493-533.

[3] F. Beukers, *Algebraic A-hypergeometric functions*, Invent. Math. **180** (2010), no. 3, 589-610.

[4] F. Beukers, G. Heckman, *Monodromy for the hypergeometric function* ${}_nF_{n-1}$. Invent. Math. **95** (1989), no. 2, 325-354.

[5] M. Bogner, S. Reiter, *On symplectically rigid local systems of rank four and Calabi-Yau operators*, J. Symbolic Comput. **48** (2013), 64-100.

[6] P. Candelas, X. C. de la Ossa, P. S. Green, L. Parkes, *A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory*, Nuclear Phys. B **359** (1991), no. 1, 21-74.

[7] G. Christol, *Fonctions hypergéométriques bornées*, Groupe de travail d'analyse ultramétrique, tome **14** (1986-1987), exp. 8, 1-16.

[8] E. Delaygue, *Critère pour l'intégralité des coefficients de Taylor des applications miroir*, J. Reine Angew. Math. **662** (2012), 205-252.

[9] E. Delaygue, *Intégralité des coefficients de Taylor de racines d'applications miroir*, J. Théor. Nombres Bordeaux **24** (2012), no. 3, 623-638.

[10] E. Delaygue, *A criterion for the integrality of the Taylor coefficients of mirror maps in several variables*, Adv. Math. **234** (2013), 414-452.

[11] E. Delaygue, *Propriétés arithmétiques des applications miroir*, PhD thesis (2011), available online at http://math.univ-lyon1.fr/ delaygue/articles/PhD_Thesis_Delaygue.pdf.

[12] B. Dwork, *On p-adic differential equations. IV. Generalized hypergeometric functions as p-adic analytic functions in one variable.* Ann. Sci. École Norm. Sup. (4), **6** (1973), 295-315.

[13] B. Dwork, *p-adic cycles.* Inst. Hautes Études Sci. Publ. Math., **37** (1969), 27-115.

[14] N. Heninger, E. M. Rains, N. J. A. Sloan, *On the integrality of nth roots of generating functions*, J. Combin. Theory Ser. A **113** (2006), 1732-1745.

[15] T. Kaluza, *Über die Koeffizienten reziproker Potenzreihen*, Math. Z. **28** (1928), 161-170.

[16] N. Katz, *Algebraic solutions of differential equations.* Invent. Math. **18** (1972), 1-118.

[17] N. Koblitz, *p-adic numbers, p-adic analysis, and zeta-functions.* Second edition. Graduate Texts in Mathematics, **58**. Springer-Verlag, New York, 1984. xii+150 pp. ISBN: 0-387-96017-1.

[18] C. Krattenthaler, T. Rivoal, *On the integrality of the Taylor coefficients of mirror maps*, Duke Math. J. **151** (2010), 175-218.

[19] C. Krattenthaler, T. Rivoal, *On the integrality of the Taylor coefficients of mirror maps, II*, Commun. Number Theory Phys. **3** (2009), no. 3, 555-591.

[20] C. Krattenthaler, T. Rivoal, *Multivariate p-adic formal congruences and integrality of Taylor coefficients of mirror maps*, Séminaire et Congrès **23** (2011), 301-329. Actes de la conférence *Théories galoisiennes et arithmétiques des équations différentielles* (CIRM, septembre 2009).

[21] C. Krattenthaler, T. Rivoal, *Analytic properties of mirror maps*, J. Austr. Math. Soc. **92** (2012), 195-235.

[22] E. LANDAU, *Sur les conditions de divisibilité d'un produit de factorielles par un autre*, collected works, **I**, page 116. Thales-Verlag (1985).

[23] S. LANG, *Cyclotomic fields, I, II*, Combined 2nd edition, vol. **121**, Graduate Texts in Math., Springer-Verlag, New York, 1990.

[24] B. H. LIAN, S. T. YAU, *The nth root of the mirror map*, in: Calabi-Yau varieties and mirror symmetry, Proceedings of the Workshop on Arithmetic, Geometry and Physics around Calabi-Yau Varieties and Mirror Symmetry, Toronto, ON, 2011, N. Yui and J. D. Lewis (eds.), Fields Inst. Commun., **38**, Amer. Math. Soc., Providence, RI, 2003, pp. 195-199.

[25] B. H. LIAN, K. LIU, S. T. YAU, *Mirror principle. I.*, Asian J. Math. **1** (1997), no. 4, 729-763.

[26] B. H. LIAN, S. T. YAU, *Integrality of certain exponential series*, Algebra and geometry (Taipei, 1995), 215-227, Lect. Algebra Geom., **2**, Int. Press, Cambridge, MA, 1998.

[27] A. MELLIT, M. VLASENKO, *Dwork's congruences for the constant terms of powers of a Laurent polynomial*, preprint (2013), available at arXiv:1306.5811v1 [math.NT].

[28] Y. MORITA, *A p-adic analogue of the Γ-function*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **22** (1975), no. 2, 255-266.

[29] K. OTA, *On special values of generalized p-adic hypergeometric functions*, Acta Arith. **67** (1994), no. 2, 141-163.

[30] A. M. ROBERT, *A course in p-adic analysis*, Graduate Texts in Mathematics, **198**. Springer-Verlag, New York, 2000. xvi+437 pp. ISBN: 0-387-98669-3

[31] J. ROQUES, *Arithmetic properties of mirror maps associated with Gauss's hypergeometric equation*, Monatshefte für Mathematik, Volume **171**, Issue 2, pp 241–253.

[32] J. ROQUES, *On generalized hypergeometric equations and mirror maps*, Proceedings of the American Mathematical Society, **142** (2014), 3153–3167.

[33] J. STIENSTRA, *GKZ hypergeometric structures*, Arithmetic and geometry around hypergeometric functions, 313-371, Progr. Math., **260**, Birkhauser, Basel, 2007.

[34] W. ZUDILIN, *On the integrality of power expansions related to hypergeometric series*, Math. Notes **71** (2002), no. 5-6, 604-616.