

# Transcendence of values of logarithms of $E$ -functions

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## Abstract

Let  $f$  be an  $E$ -function (in Siegel's sense) not of the form  $e^{\beta z}$ ,  $\beta \in \overline{\mathbb{Q}}$ , and let  $\log$  denote any fixed determination of the complex logarithm. We first prove that there exists a finite set  $S(f)$  such that for all  $\xi \in \overline{\mathbb{Q}} \setminus S(f)$ ,  $\log(f(\xi))$  is a transcendental number. We then quantify this result when  $f$  is an  $E$ -function in the strict sense with rational coefficients, by proving an irrationality measure of  $\ln(f(\xi))$  when  $\xi \in \mathbb{Q} \setminus S(f)$  and  $f(\xi) > 0$ . This measure implies that  $\ln(f(\xi))$  is not an ultra-Liouville number, as defined by Marques and Moreira. The proof of our first result, which is in fact more general, uses in particular a recent theorem of Delaygue. The proof of the second result, which is independent of the first one, is a consequence of a new linear independence measure for values of linearly independent  $E$ -functions in the strict sense with rational coefficients, where emphasis is put on other parameters than on the height, contrary to the case in Shidlovskii's classical measure for instance.

## 1 Introduction

In this paper, we pursue our study of the properties of the values of  $E$ -functions at algebraic points, and more specifically of the logarithms of these values. We recall the definition of  $E$ -functions. As usual, we embed  $\overline{\mathbb{Q}}$  in  $\mathbb{C}$ . A power series  $F(z) = \sum_{n=0}^{\infty} a_n z^n / n! \in \overline{\mathbb{Q}}[[z]]$  is said to be a *strict*  $E$ -function if

(i)  $F(z)$  is solution of a non-zero linear differential equation with coefficients in  $\overline{\mathbb{Q}}(z)$ .

(ii) There exists  $C_1 > 0$  such that all Galois conjugates of  $a_n$  have modulus  $\leq C_1^{n+1}$ , for all  $n \geq 0$ .

(iii) There exists  $C_2 > 0$  and a sequence of positive integers  $d_n$ , with  $d_n \leq C_2^{n+1}$ , such that  $d_n a_m$  are algebraic integers for all  $m \leq n$ .

In fact,  $E$ -functions have been defined by Siegel [19] in 1929 in a more general way, ie the two bounds  $(\dots) \leq C_i^{n+1}$  are replaced by: for all  $\varepsilon > 0$ ,  $(\dots) \leq n!^\varepsilon$  for all  $n \geq N(\varepsilon)$ . It is believed that an  $E$ -function in Siegel's sense is automatically a strict  $E$ -function; see [2, p. 715] for a discussion. Unless otherwise specified,  $E$ -functions below will be understood in Siegel's sense. Note that if  $a_n \in \mathbb{Q}$ , (ii) and (iii) read  $|a_n| \leq C^{n+1}$  and  $d_n a_m \in \mathbb{Z}$ ; in (i), there exists such a differential equation with coefficients in  $\mathbb{Q}(z)$ , and the normalized one

of minimal order also has coefficients in  $\mathbb{Q}(z)$ . An  $E$ -function is either a polynomial or a transcendental function.

Let  $f$  and  $g$  be two  $E$ -functions. If  $f$  is transcendental and  $g$  is a polynomial, then  $f(\overline{\mathbb{Q}}) \cap g(\overline{\mathbb{Q}})$  is finite by [1]. If  $f$  and  $g$  are polynomials, two cases occur: if one of them is a constant,  $f(\overline{\mathbb{Q}}) \cap g(\overline{\mathbb{Q}})$  is finite, while if none is a constant,  $f(\overline{\mathbb{Q}}) \cap g(\overline{\mathbb{Q}})$  is infinite. Our first result completes the picture; it shows that a transcendental  $E$ -function is determined by the set of values it takes at algebraic numbers.

**Theorem 1.** *Let  $f, g$  be two transcendental  $E$ -functions such that  $f(z)$  is not of the form  $g(\beta z)$ ,  $\beta \in \overline{\mathbb{Q}}$ . Then  $\{(\xi, \eta) \in \overline{\mathbb{Q}}^2 : f(\xi) = g(\eta)\}$  is a finite set. Equivalently,  $f(\overline{\mathbb{Q}}) \cap g(\overline{\mathbb{Q}})$  is a finite set.*

The equivalence is a consequence of Proposition 2 proved in §2. The assumption in Theorem 1 is obviously also necessary to obtain finiteness when  $f$  and  $g$  are transcendental.

Applying Theorem 1 to  $f(z)$  transcendental and  $g(z) = e^z$ , we deduce that the set  $\{(\xi, \eta) \in \overline{\mathbb{Q}}^2 : f(\xi) = e^\eta\}$  is finite when  $f(z) \neq e^{\beta z}$  for all  $\beta \in \overline{\mathbb{Q}}$ . As  $\{(\xi, \eta) \in \overline{\mathbb{Q}}^2 : f(\xi) = e^\eta\}$  is also finite if  $f \in \overline{\mathbb{Q}}[z]$  by the Hermite-Lindemann Theorem, we obtain the following result.

**Corollary 1.** *Let  $f$  be an  $E$ -function not of the form  $e^{\beta z}$ ,  $\beta \in \overline{\mathbb{Q}}$ , and let  $\log$  denote any fixed determination of the complex logarithm. There exists a finite set  $S(f)$  such that for all  $\xi \in \overline{\mathbb{Q}} \setminus S(f)$ ,  $\log(f(\xi))$  is a transcendental number.*

As the proof shows, given  $\xi \in \overline{\mathbb{Q}}$ , there exists an algebraic determination of the logarithm of  $f(\xi)$  if, and only if, either  $f(\xi) = 1$  or the  $E$ -function  $f(z) - \exp(z/\varrho)$  vanishes at  $\xi$  for some  $\varrho$  in the finite set  $\mathfrak{S}(f)$  considered in §2.1. This provides an algorithm to determine  $S(f)$ .

This corollary applies to any  $E$ -function with a minimal differential equation of order  $\geq 2$ , for example to Bessel's function  $J_0(z) := \sum_{n=0}^{\infty} (-1)^n (z/2)^{2n} / n!^2$  whose minimal equation is  $zy''(z) + y'(z) + zy(z) = 0$ . But the property  $\{(\xi, \eta) \in \overline{\mathbb{Q}}^2 : J_0(\xi) = e^\eta\} = \{(0, 0)\}$  is not a new result as it is a consequence of the much more general [18, p. 219, Theorem 4]: for any  $\xi, \eta \in \overline{\mathbb{Q}}^*$ , the numbers  $J_0(\xi)$ ,  $J_0'(\xi)$  and  $e^\eta$  are algebraically independent over  $\overline{\mathbb{Q}}$ . Other examples of a similar flavor involving generalized hypergeometric series  ${}_pF_q[z^{q-p+1}]$  with rational parameters satisfying certain arithmetic conditions can be deduced from the very general algebraic independence result in [8, p. 300, Corollary 4.6]. However, these conditions do not exhaust all such series with rational parameters. To the best of our knowledge, Corollary 1 is new for  $f(z) := {}_pF_q[z^{q-p+1}]$  with  $1 \leq p \leq q$ ,  $1/2$  as a lower parameter and no upper parameter equal to  $1/2 \pmod{\mathbb{Z}}$ , because neither assumption A) nor assumption B) on page 280 of [8] is satisfied, for instance  ${}_1F_1[1/3; 1/2; z]$ .

Corollary 1 can be quantified in the rational and strict case. Our method to prove Theorem 2 is independent though of that of Theorem 1 (based on a recent result of Delaygue [12]), as it uses a new linear independence measure for values of  $E$ -functions (Proposition 2 stated in §3.1); see below for more details. When  $x$  is a positive real number, we denote by  $\ln(x)$  its Napierian logarithm.

**Theorem 2.** *Let  $f \in \mathbb{Q}[[z]]$  be a strict  $E$ -function, and  $\xi \in \mathbb{Q}^*$  be such that  $f(\xi) > 0$  and  $\ln(f(\xi)) \notin \mathbb{Q}$ . Then there exist  $c, d > 0$  such that, for all  $(a, b) \in \mathbb{Z} \times \mathbb{N}^*$ ,*

$$\left| \ln(f(\xi)) - \frac{a}{b} \right| \geq \frac{1}{\exp(cb^d)}. \quad (1.1)$$

In particular, this result implies that  $\ln(f(\xi))$  is not an ultra-Liouville number (as defined in [17]).

Theorem 2 applies to any  $\xi \in \mathbb{Q}^*$  such that  $J_0(\xi) > 0$  because  $\ln(J_0(\xi))$  is then a transcendental number by the above mentioned result; the irrationality measure for  $\ln(J_0(\xi))$  is new to our knowledge. Note that Theorem 2 can also be applied to any  $f \in \mathbb{Q}[z] \setminus \{0\}$ , but in this case the lower bound can be much improved because it is known that for any  $\alpha \in \mathbb{Q}_{>0}$ ,  $\alpha \neq 1$ , the number  $\ln(\alpha)$  is an irrational number and not a Liouville number; see [15, p. 150, Satz 5]. If  $f(\xi) < 0$ , the same result holds for  $-f(\xi)$  instead of  $f(\xi)$ .

The constants  $c, d$  in (1.1) depend on  $f$  and  $\xi$ ; they are effective but we did not try to compute them (this could be done in principle) because it is likely that the lower bound in (1.1) is not optimal and could be replaced by  $c/b^d$  for some other effective constants  $c, d > 0$ , proving that  $\ln(f(\xi))$  is not a Liouville number. It does not seem that this improvement could be obtained with our method, which relies on the following observation:

$\ln(f(\xi))$  is close to  $a/b$  if, and only if,  $f(\xi) - \exp(a/b)$  is small.

Considering the latter as a linear combination (with coefficients 1 and  $-1$ ) of values of  $E$ -functions, one may try to apply linear independence measures due to Shidlovskii [18]. A first problem is that such measures are optimized for linear forms with very large coefficients, which is not the case here. But another more important problem arises: the constants in such measures depend (usually in a non-explicit way) on the functions involved, and in our setting  $\exp(az/b)$  is amongst them. For this reason we need a linear independence measure in which the dependencies of the constants in terms of the functions are explicit. We prove such a result, namely Proposition 3, in §3.

We note that using instead Brownawell's more general algebraic independence measure <sup>(1)</sup> in [11] (and making explicit in it the dependence on the parameters we need, as with Shidlovskii's measure), we only found a lower bound of the form  $1/\exp(\exp(cb^d))$  on the right-hand side of (1.1), provided that  $f, f', \dots, f^{(\eta-1)}$  are algebraically independent (where  $\eta$  is the minimal order of a non-trivial differential equation satisfied by a strict  $E$ -function  $f \in \mathbb{Q}[[z]]$ ).

The structure of this paper is the following. In Section 2 we prove Theorem 1 using Delaygue's analogue of the Lindemann-Weierstrass theorem. Then we state and prove in Section 3 the linear independence measure that we shall use in Section 4 to prove Theorem 2.

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<sup>1</sup>Brownawell's measure is slightly ineffective because of the ineffectivity of Shidlovskii's constant  $n_0$ , that will also be used in the present paper. As we shall explain,  $n_0$  can now be bounded effectively, removing any form of ineffectivity in Shidlovskii's and Brownawell's measures.

## 2 Proof of Theorem 1

### 2.1 Delaygue's analogue of the Lindemann-Weierstrass theorem

Given an  $E$ -function  $f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$ , let  $\mathfrak{S}(f) \subset \overline{\mathbb{Q}}^*$  denote the set of finite singularities of the  $G$ -function  $\sum_{n=0}^{\infty} a_n z^n$ .

We shall derive Theorem 1 from the following special case of Delaygue's analogue of the linear version of the Lindemann-Weierstrass theorem (see [12, Corollary 2.2]).

**Proposition 1.** *Let  $f_1, f_2$  be  $E$ -functions and  $z_1, z_2$  be non-zero algebraic numbers such that  $f_1(z_1) = f_2(z_2)$  is transcendental. Then  $z_1/z_2$  can be written as  $\varrho_1/\varrho_2$  with  $\varrho_1 \in \mathfrak{S}(f_1)$  and  $\varrho_2 \in \mathfrak{S}(f_2)$ .*

The important point for us, in the conclusion of Proposition 1, is that  $z_1/z_2$  belongs to a finite set determined by  $f_1$  and  $f_2$ .

*Remark 1.* Proposition 1 is proved in [12] for  $E$ -functions in the strict sense. However, it also holds for  $E$ -functions in the large sense (and so does the general result [12, Theorem 2.1]) because only the following properties are used in the proof, and they hold for  $G$ -functions and  $E$ -functions in the large sense by the results proved in [4, 14]:

- (1) The point at infinity is regular or a regular singularity of any  $G$ -function, because it is solution of a fuchsian differential operator.
- (2) A  $G$ -function without finite singularity is a polynomial. Indeed such a function is entire, and has moderate growth at infinity by (1). By Liouville's theorem, it is a polynomial.
- (3) Any  $E$ -function is annihilated by an  $E$ -operator, without non-zero finite singularity.
- (4) Beukers' refined version of the Siegel-Shidlovskii theorem (*i.e.* [9, Theorem 1.3]) holds.

### 2.2 Application to Theorem 1

We first prove

**Proposition 2.** *Let  $f$  be a non-constant  $E$ -function and  $\chi \in \mathbb{C}$ . Then the set  $\{\alpha \in \overline{\mathbb{Q}} : f(\alpha) = \chi\}$  is finite.*

*Proof of Proposition 2.* If  $\chi \in \overline{\mathbb{Q}}$ , this is a consequence of the main result in [1]. Otherwise, let us fix  $\alpha_0 \in \overline{\mathbb{Q}}$  such that  $f(\alpha_0) = \chi$ ; if there is no such  $\alpha_0$ , the corresponding set is empty and therefore finite. For any  $\alpha \in \overline{\mathbb{Q}}$  such that  $f(\alpha) = \chi$ , Proposition 1 implies that  $\alpha/\alpha_0$  belongs to a finite set determined by  $f$ . This concludes the proof of Proposition 2.  $\square$

*Proof of Theorem 1.* First of all, let us consider the set of pairs  $(\xi, \eta) \in \overline{\mathbb{Q}}^2$  such that  $f(\xi) = g(\eta)$  is algebraic. Recall from [1] that a transcendental  $E$ -function takes algebraic values at only finitely many algebraic points. Therefore each of  $\xi, \eta$  belongs to a finite set determined by  $f$  and  $g$ : so does the pair  $(\xi, \eta)$ .

Now let us move to pairs  $(\xi, \eta) \in \overline{\mathbb{Q}}^2$  such that  $f(\xi) = g(\eta)$  is transcendental; this implies  $\xi, \eta \neq 0$ . Then Proposition 1 provides a finite set (determined by  $f$  and  $g$ ) that contains all quotients  $\eta/\xi$ . For each value  $\beta$  of the quotient, the  $E$ -function  $f(z) - g(\beta z)$  vanishes at  $\xi$ . Since this  $E$ -function is not identically zero by hypothesis,  $\xi$  belongs to a finite set determined by  $\beta$ . So does  $\eta$ , and this concludes the proof that  $\{(\xi, \eta) \in \overline{\mathbb{Q}}^2 : f(\xi) = g(\eta)\}$  is a finite set. This is equivalent to the fact that  $I := f(\overline{\mathbb{Q}}) \cap g(\overline{\mathbb{Q}})$  is a finite set. Indeed, if  $\{(\xi, \eta) \in \overline{\mathbb{Q}}^2 : f(\xi) = g(\eta)\}$  is finite, then obviously  $I$  is finite. Conversely, if  $I$  is finite, then for any  $\chi \in I$ , there are only finitely many  $(\xi, \eta) \in \overline{\mathbb{Q}}^2$  such that  $f(\xi) = \chi = g(\eta)$  by Proposition 2. This completes the proof of Theorem 1.  $\square$

### 3 An explicit version of Shidlovskii's linear independence measure

In this section, we prove a linear independence measure of values of  $E$ -functions, namely Proposition 3 stated in §3.1. The important point for our application to Theorem 2 is that when the last function is  $\exp(\beta z)$ , the constants are controlled in terms of  $\beta$ : this special case is studied in Corollary 2.

The structure of proof is similar to that of Shidlovskii's measure, so we recall it in §3.2. We proceed to the proof in §3.3.

#### 3.1 Statement of the measure

Assume  $Y := {}^t(f_1, \dots, f_m)$  is a vector of strict  $E$ -functions in  $\mathbb{Q}[[z]]$ , solution of a differential system  $Y' = AY$  with  $A \in M_m(\mathbb{Q}(z))$ ; assume moreover that  $f_1, \dots, f_m$  are  $\mathbb{Q}(z)$ -linearly independent. Let  $T \in \mathbb{Z}[z] \setminus \{0\}$  be a common denominator of minimal degree of the entries of  $A := (A_{i,j})_{1 \leq i, j \leq m}$ . Let  $\xi \in \mathbb{Q}$  be such that  $\xi T(\xi) \neq 0$ . We consider any integer  $H \geq 1$  and any vector  $(a_1, \dots, a_m) \in \mathbb{Z}^m \setminus \{0\}$  such that  $\max |a_k| \leq H$ .

As a special case  $\mathbb{K} = \mathbb{Q}$  of [18, p. 357, Theorem 1], Shidlovskii proved that for any  $\varepsilon > 0$ , there exists an ineffective constant  $c > 0$  such that, in the above situation, we have

$$\left| \sum_{j=1}^m a_j f_j(\xi) \right| > \frac{c}{H^{m-1+\varepsilon}}. \quad (3.1)$$

This constant  $c$  is now effective (because the integer  $n_0$  in Shidlovskii's multiplicity estimate can be bounded, see below). However it depends on  $f_1, \dots, f_m$  in a way which is not made explicit by Shidlovskii. This is a problem to prove Theorem 2, since in our setting  $f_m$  will be  $\exp(\beta z)$  and we need constants that we control explicitly in terms of  $\beta$ .

We shall prove a linear independence measure, namely Proposition 3, in which the dependencies of the constants on certain parameters important for us are made explicit, unlike the classical measures in the same context. With this aim in mind, we define as in [18, p. 93]:

$$p := \min_{1 \leq j \leq m} \text{ord}_{z=0} f_j(z) \quad \text{and} \quad q := \max(\deg T, \max_{i,j} \deg(TA_{i,j})). \quad (3.2)$$

Since the  $f_i = \sum_{k=0}^{\infty} \frac{\varphi_{k,i}}{k!} z^k$ ,  $i = 1, \dots, m$ , are  $E$ -functions in the strict sense, there exists a constant  $C > 0$  such  $|\varphi_{k,i}| \leq C^{k+1}$  for all  $k \geq 0, i \in \{1, \dots, m\}$ , and there exists a constant  $D > 0$  such the common denominator  $d_{k,i}$  of  $\varphi_{0,i}, \dots, \varphi_{k,i}$  satisfies  $d_{k,i} \leq D^{k+1}$  for all  $k \geq 0, i \in \{1, \dots, m\}$ .

We also denote by  $E$  the maximum modulus of the coefficients of the polynomial  $T(z)$  and of all the polynomials  $T(z)A_{i,j}(z)$ .

We denote by  $n_0 \geq 0$  the constant in Shidlovskii's zero estimate. Shidlovskii's proof of the existence of  $n_0$  is not effective (see the proof of [18, p. 93, Lemma 8] and the definition of  $n_0$  in [18, p. 99, Eq. (83)]). However, following the works of Bertrand, Beukers, Chirskii and Yebbou [5, 7], it is now known that the integer  $n_0$  can be bounded above using explicit quantities that depend on the matrix  $A$  of the differential system. Certain of these quantities are themselves bounded by means of the generalized local exponents at the singularities of  $A$  and the point at infinity (for their definition, see [6, Appendix] or [10, §2.3.4]). More precisely, from the discussion in [7, p. 252] we have that

$$n_0 \leq 2(q+1)m^2(\mathcal{E} + (q+1)m + 1), \quad (3.3)$$

where  $\mathcal{E}$  is the maximum of all the modulus of the generalized local exponents at the infinite point and at the finite singularities of  $A$ .

At last we denote by  $\kappa$  any real number such that

$$0 < \kappa \leq \max_{1 \leq j \leq m} |f_j(\xi)|$$

and for a given rational number  $r$ , we set  $H(r) := \max(|u|, |v|)$  where  $r = u/v$  with  $u, v$  coprime; we also write  $\text{den}(r) = |v|$ .

**Proposition 3.** *There exists an effective constant  $c$ , which depends on  $m, p, q, T(z), \kappa$  and polynomially on  $H(\xi), C, D$  and  $E$ , such that if  $H \geq \max(3, n_0^{n_0})$  then*

$$\left| \sum_{j=1}^m a_j f_j(\xi) \right| > \frac{1}{H^c}.$$

Using Eq. (3.3), the lower bound on  $H$  can be replaced by an explicit lower bound in terms of  $m, q, \mathcal{E}$ .

The important point in Proposition 3 is that  $c$  depends on  $f_1, \dots, f_m$  only through a given set of parameters, and also that the dependence on  $H(\xi), C, D, E$ , is polynomial.

The constant  $c$  is effective because the only potential source of ineffectivity of the proof, i.e.  $n_0$ , is now known to be effective. In principle it would be possible to make  $c$  completely explicit, but this would make the statement of our results much more complicated for no immediate application better than those we present here. Moreover such explicit formulas are not sharp in general.

In this proposition, and throughout this section, the polynomial dependence of  $c$  with respect to  $H(\xi), C, D, E$  means that there exists a polynomial  $P \in \mathbb{R}[W, X, Y, Z]$  with non-negative coefficients and degree depending only on  $m, p, q, T(z), \kappa$ , such that one may choose  $c = P(H(\xi), C, D, E)$ .

To prove Theorem 2 we shall be interested in the following situation:

$$f_m(z) = \exp(\beta z) \text{ and } f_1, \dots, f_{m-1} \text{ are independent from } \beta. \quad (3.4)$$

In precise terms, when we refer to (3.4) we shall assume that  $Z = {}^t(f_1, \dots, f_{m-1})$  is a vector of  $E$ -functions with rational coefficients, solution of a differential system  $Z' = BZ$  with  $B \in M_{m-1}(\mathbb{Q}(z))$ . Then  $Y = {}^t(f_1, \dots, f_m)$  is solution of  $Y' = AY$  where  $A \in M_m(\mathbb{Q}(z))$  is blockwise diagonal, with diagonal blocks  $B$  and  $\beta$ . The important point is that  $f_1, \dots, f_{m-1}$  and  $B$  are independent from  $\beta$ . In this setting we have the following special case of Proposition 3.

**Corollary 2.** *In the situation (3.4), there exists an effective constant  $c$ , which depends only on  $f_1, \dots, f_{m-1}$  and polynomially on  $H(\xi)$  and  $H(\beta)$ , such that if  $H \geq \max(3, n_0^{n_0})$  then*

$$\left| \sum_{j=1}^m a_j f_j(\xi) \right| > \frac{1}{H^c}.$$

Moreover the assumption on  $H$  may be stated as a lower bound in terms of  $f_1, \dots, f_{m-1}$  only, independently of  $\beta$  and of  $\xi$ .

*Proof.* In the setting of (3.4), recall that  $Z = {}^t(f_1, \dots, f_{m-1})$  is a solution of  $Z' = BZ$  with  $B \in M_{m-1}(\mathbb{Q}(z))$ , and  $Y = {}^t(f_1, \dots, f_m)$  of  $Y' = AY$ . The matrix  $A \in M_m(\mathbb{Q}(z))$  is blockwise diagonal, with diagonal blocks  $B$  and  $\beta$ . Therefore  $T(z)$  is independent of  $\beta$ ; so are  $m, p, q$ , and also  $\mathcal{E}$  because the function  $\exp(\beta z)$  has null generalized local exponents everywhere. Hence, Eq. (3.3) shows that  $n_0$  can be bounded independently of  $\beta$ .

Moreover at least one of  $f_1, \dots, f_{m-1}$  does not vanish at  $\xi$ , because  $\xi$  is not a singularity of the differential system  $Z' = BZ$  (indeed it is not a pole of a coefficient of  $B$ , because all these coefficients are coefficients of  $A$ ); of course the functions  $f_1, \dots, f_{m-1}$  are not identically zero because they are linearly independent over  $\mathbb{Q}(z)$ . Therefore we may choose  $\kappa = \min(1, \max_{1 \leq j \leq m-1} |f_j(\xi)|)$ .

Since  $A_{m,m} = \beta$ , we may take  $C = \max(\tilde{C}, |\beta|)$ ,  $D = \text{den}(\beta)\tilde{D}$  and  $E \leq |\beta|\tilde{E}$ , where  $\tilde{C}, \tilde{D}$  and  $\tilde{E}$  are quantities analogous to  $C, D$  and  $E$  when we consider only  $f_1, \dots, f_{m-1}$ . Therefore  $C, D, E$  depend linearly on  $H(\beta)$ . Applying Proposition 3 concludes the proof of Corollary 2.  $\square$

*Remark 2.* In [18, p. 421, Theorem 2], Shidlovskii also proved for  $E$ -functions in the strict sense an effective <sup>(2)</sup> refinement of (3.1) in which he replaced the exponent  $m - 1 + \varepsilon$  by  $m - 1 + \gamma m^{7/2} (\ln \ln(H))^{-1/2}$  where  $\ln \ln(H) \geq \gamma^2 \max(m^2, \ln(n_0))$  and  $\gamma$  is a constant. However the dependence of  $\gamma$  with respect to  $f_1, \dots, f_m$  is unknown while to prove Theorem 2, it is necessary to know how the constants depend on  $\beta$  in the setting of (3.4).

*Remark 3.* Proposition 3 can also be used to provide an irrationality measure of a real irrational zero  $\zeta$  of any non-polynomial  $E$ -function  $F$  with rational coefficients. We assume further that  $\zeta$  is a simple zero and not a singularity of the minimal differential equation satisfied by  $F$ . Then, we have

$$\zeta - \frac{a}{b} = \frac{1}{F'(\eta)} \left( F(\zeta) - F\left(\frac{a}{b}\right) \right) = \frac{F(a/b)}{F'(\eta)}$$

for some real number  $\eta$  in the interval with endpoints  $\zeta$  and  $a/b \in \mathbb{Q}$ . Since  $\zeta$  is an irrational simple zero and not a singularity of the minimal differential equation satisfied by  $F$ , neither  $F(a/b)$  nor  $F'(\eta)$  is equal to 0 when  $a/b$  is close to  $\zeta$ , and  $a/b$  is not a singularity. Proposition 3 with  $\xi := a/b$  then implies that  $|F(a/b)| > \exp(-Q(b))$  for some effective polynomial  $Q$  with positive coefficients. Therefore for all  $(a, b) \in \mathbb{Z} \times \mathbb{N}^*$ ,

$$\left| \zeta - \frac{a}{b} \right| \geq \frac{1}{\exp(P(b))}$$

for some effective polynomial  $P$  with positive coefficients. This irrationality measure is not as strong as the one obtained by Galochkin who found a lower bound  $c_\varepsilon \exp(-b^\varepsilon)$  for all  $\varepsilon > 0$ , but  $c_\varepsilon > 0$  is ineffective; see [18, p. 436].

### 3.2 Shidlovskii's measure (3.1): sketch of proof

We first recall the statement of [18, p. 107, Lemma 14] when the number field  $\mathbb{K} = \mathbb{Q}$ . Assume  $Y := {}^t(f_1, \dots, f_m)$  is a vector of  $E$ -functions in Siegel's sense <sup>(3)</sup> in  $\mathbb{Q}[[z]]$  solution of a differential system  $Y' = AY$  with  $A \in M_m(\mathbb{Q}(z))$ . Let  $n \in \mathbb{N}$  and  $\varepsilon_1 \in (0, \frac{1}{2m-1})$ ; the reason of this technical assumption on  $\varepsilon_1$  will appear in §3.3.3. There exist  $P_1, \dots, P_m \in \mathbb{Z}[z]$  of degree at most  $n$  and not all zero such that:

$$b_{i,\nu} = \mathcal{O}(n^{(1+\varepsilon_1)n}), \quad i = 1, \dots, m, \quad \nu = 0, \dots, n,$$

where  $b_{i,\nu}$  is the coefficient of  $z^\nu$  in  $P_i(z)$  and the symbol  $\mathcal{O}$  is uniform in  $i$  and  $\nu$ , and such that the function

$$R := \sum_{i=1}^m P_i(z) f_i(z) = \sum_{\nu=\tau}^{\infty} \frac{a_\nu}{\nu!} z^\nu$$

<sup>2</sup>Again, strictly speaking, it was ineffective when Shidlovskii proved it because of the presence of  $n_0$ , but it is now effective.

<sup>3</sup>However, we shall make this proof explicit only for  $E$ -functions in the strict sense, otherwise it would be difficult to obtain a good control on the quantities we are interested in, as for instance the parameters  $C$  and  $D$  do not exist for  $E$ -functions in Siegel's sense.



satisfies  $\text{ord}_{z=0}R(z) \geq \tau$  with

$$\tau = m(n+1) - \lfloor \varepsilon_1 n \rfloor - 1,$$

and  $a_\nu = \nu^{\varepsilon_1 n} \mathcal{O}(n^n)$  for  $\nu \geq \tau$ .

In the same setting, define

$$R_k := \sum_{i=1}^m P_{k,i}(z) f_i(z)$$

with  $P_{k,i} \in \mathbb{Z}[z]$  by  $R_1 = R$  and  $R_{k+1} = T(z)R'_k(z)$ ; recall that  $T \in \mathbb{Z}[z] \setminus \{0\}$  is a common denominator of minimal degree of the entries of  $A$ .

From now on, we assume that  $f_1, \dots, f_m$  are  $\mathbb{Q}(z)$ -linearly independent, and that  $n \geq n_0$  where  $n_0$  was introduced in §3.1.

Then by Lemma 10 of [18, p. 101], for any  $\xi \in \mathbb{C}$  such  $\xi T(\xi) \neq 0$ , the linear forms  $R_k(\xi)$ ,  $k = 1, \dots, m + t_1$ , include  $m$  linearly independent forms, where

$$t_1 := q \frac{(m-1)m}{2} + \lfloor \varepsilon_1 n \rfloor + p; \quad (3.5)$$

we recall from §3.1 (and [18, p. 93]) that

$$p := \min_{1 \leq i \leq m} \text{ord}_{z=0} f_i(z) \quad \text{and} \quad q := \max(\deg T, \max_{i,j} \deg(TA_{i,j}))$$

with  $A := (A_{i,j})_{1 \leq i,j \leq m}$ . Now Lemma 15 in [18, p. 110] says the following. Suppose that  $\varepsilon_1 = \varepsilon/(6(m+1))$  for some  $\varepsilon \in (0, 1)$ . Then for any  $\xi \in \mathbb{Q}$  such that  $\xi T(\xi) \neq 0$ , we have

$$R_k(\xi) = \mathcal{O}(n^{-(m-1-\varepsilon/2)n}), \quad k \leq m + t_1,$$

and

$$P_{k,i}(\xi) = \mathcal{O}(n^{(1+\varepsilon/2)n}), \quad k \leq m + t_1, \quad i = 1, \dots, m.$$

From these estimates, Shidlovskii deduces [18, p. 357, Theorem 1], ie (3.1) for  $E$ -functions in Siegel's sense.

With Shidlovskii's original method, the constant  $c$  in (3.1) was ineffective because of the ineffectivity of the integer  $n_0$ . As we have explained before, this is no longer the case. For our purpose, we need a different version of the measure (3.1), with a control of the dependencies of the constants on the parameters.

### 3.3 Proof of Proposition 3

In this section we shall prove Proposition 3.

In all what follows, as in §3.1, when we say that *a constant  $c$  depends polynomially on a parameter  $k$*  we mean that there exists a polynomial  $p$  with positive coefficients such that

$|c| \leq p(|k|)$ . The polynomial  $p$ , including its degree, may depend on all other parameters  $c$  depends on.

We shall follow now the sketch of proof given in the previous section; at each step we shall make all bounds explicit (to be precise, we shall make the dependencies in terms of the parameters explicit).

As in §3.1 we consider a vector  $Y = {}^t(f_1, \dots, f_m)$  of strict  $E$ -functions in  $\mathbb{Q}[[z]]$ , solution of a differential system  $Y' = AY$  with  $A \in M_m(\mathbb{Q}(z))$ . We assume that  $f_1, \dots, f_m$  are  $\mathbb{Q}(z)$ -linearly independent, and denote by  $T \in \mathbb{Z}[z] \setminus \{0\}$  a common denominator of minimal degree of the entries of  $A$ . We fix  $\xi \in \mathbb{Q}$  be such that  $\xi T(\xi) \neq 0$ . As in §3.2 we consider also  $\varepsilon_1 > 0$  such that  $\varepsilon_1 < \frac{1}{2m-1}$ .

### 3.3.1 Construction of the polynomials

Following the proof of [18, p. 107, Lemma 14] and using Siegel's lemma, we find that for any  $1 \leq i \leq m$  and any  $0 \leq \nu \leq n$ ,

$$|b_{i,\nu}| \leq n! 2(m(n+1)CD)^{2m/\varepsilon_1} (2CD)^{4m^2n/\varepsilon_1}$$

and for any  $\nu \geq \tau$ ,

$$|a_\nu| \leq n! m 2^{\nu+1} (m(n+1)CD)^{2m/\varepsilon_1} (2CD)^{4m^2n/\varepsilon_1} C^{\nu+1}.$$

Since

$$R(z) = \sum_{\nu=\tau}^{\infty} \frac{a_\nu}{\nu!} z^\nu,$$

we deduce that

$$|R(z)| \leq n! m (m(n+1)CD)^{2m/\varepsilon_1} (2CD)^{4m^2n/\varepsilon_1} 2C \sum_{\nu=\tau}^{\infty} \frac{(2C|z|)^\nu}{\nu!}.$$

As

$$\sum_{\nu=\tau}^{\infty} \frac{t^\nu}{\nu!} = \frac{1}{(\tau-1)!} \int_0^t (t-x)^{\tau-1} e^x dx \leq \frac{t^{\tau-1} e^t}{(\tau-1)!} \text{ for any } t \geq 0,$$

it follows that

$$|R(z)| \leq C_1 (2CD)^{6m^2n/\varepsilon_1} e^{2C|z|} \frac{n!}{(m(n+1) - \lfloor \varepsilon_1 n \rfloor)!} (2C|z|)^{m(n+1) - \lfloor \varepsilon_1 n \rfloor - 2}$$

where  $C_1 > 0$  depends on  $\varepsilon_1, m$ , polynomially on  $n$ , not on  $z$  and not on the Taylor coefficients of the  $f_j$ 's. This constant  $C_1$  also satisfies

$$|a_\nu| \leq n! C_1 (2CD)^{6m^2n/\varepsilon_1} (2C)^{\nu+1} \text{ for any } \nu \geq \tau \quad (3.6)$$

and

$$|b_{i,\nu}| \leq n! C_1 (2CD)^{6m^2n/\varepsilon_1} \text{ for any } 1 \leq i \leq m \text{ and any } 0 \leq \nu \leq n. \quad (3.7)$$

### 3.3.2 Upper bounds on the linear forms

By a similar analysis of the proof of [18, p. 110, Lemma 15], using Eq. (3.6) we see that

$$|R_k(z)| \leq C_2(2CD)^{6m^2n/\varepsilon_1} \frac{n!k!(2q)^k e^{2C|z|} (1+|z|)^{(k-1)q}}{(m(n+1) - \lfloor \varepsilon_1 n \rfloor - k - 2)!} (2C|z|)^{m(n+1) - \lfloor \varepsilon_1 n \rfloor - k - 2}$$

for all  $n \geq n_0$  and all  $k \in \{1, \dots, m + t_1\}$ , where  $C_2 > 0$  depends on  $\varepsilon_1, m, p, q$ , polynomially on  $n$ , linearly on the  $k$ -th power of the maximum modulus of the coefficients of the polynomial  $T(z)$ , and not on  $z$ .

Moreover, the degree of each  $P_{k,i}$  is less than  $n + (k - 1)q$  by [18, p. 115] and using Eq. (3.7) we have

$$|P_{k,i}(z)| \leq C_3(2CD)^{6m^2n/\varepsilon_1} (1+|z|)^{n+(k-1)q} (m+n)^k n!k!q^k, \quad k \leq m + t_1, \quad i = 1, \dots, m$$

where  $C_3$  depends on  $\varepsilon, m, p, q$ , polynomially on  $n$  and  $z$ , and linearly on  $E^k$ . As a polynomial in  $n$  and  $z$ , the degree of  $C_3$  depends only on  $\varepsilon_1, m, p, q$ .

In the above upper bounds for  $R_k$  and  $P_{k,i}$ , we now use the fact that  $k \leq m + t_1 \leq \lfloor \varepsilon_1 n \rfloor + C_4$ , where  $C_4$  depends only on  $m, p, q$  (by Eq. (3.5)), but not on  $n$  or  $z$ . As in [18] we take  $z = \xi$  and multiply by a common denominator. After some simplifications, the situation can now be summarized as follows (this makes explicit [18, p. 114, Lemma 16]): for every  $\xi \in \mathbb{Q}$  such that  $\xi T(\xi) \neq 0$  and for all  $n \geq n_0$ , there exist  $m$  linearly independent linear forms

$$L_j := \sum_{i=1}^m a_{j,i} f_i(\xi), \quad a_{j,i} \in \mathbb{Z}, \quad j = 1, \dots, m,$$

(that depend on  $n$ ) such that for any  $1 \leq j \leq m$ ,

$$|L_j| \leq C_5(2CD)^{6m^2n/\varepsilon_1} e^{2C|\xi|} (2q \max(\varepsilon_1, C_4))^{\varepsilon_1 n} \times (1+|\xi|)^{q\varepsilon_1 n} \text{den}(\xi)^{(1+q\varepsilon_1)n} \frac{n^{n+\varepsilon_1 n} (2C|\xi|)^{mn-2\varepsilon_1 n}}{(mn - 2\lfloor \varepsilon_1 n \rfloor)!}, \quad (3.8)$$

where the factor  $C_5$  depends on  $\varepsilon_1, m, p, q$ , polynomially on  $H(\xi)$  and  $n$ , and linearly on the  $(m + t_1)$ -th power of the maximum modulus of the coefficients of the polynomial  $T(z)$ . Since this exponent is  $\leq \varepsilon_1 n + C_4$  we have  $C_5 \leq \tilde{c}_5^n$  where  $\tilde{c}_5$  depends on  $\varepsilon_1, m, p, q, T(z)$ , polynomially on  $H(\xi)$  but not on  $n$ . Moreover, for any  $1 \leq i, j \leq m$ ,

$$|a_{j,i}| \leq C_6(2CD)^{6m^2n/\varepsilon_1} (2q \max(\varepsilon_1, C_4))^{\varepsilon_1 n} ((1+|\xi|)\text{den}(\xi))^{(1+q\varepsilon_1)n} n^{n+2\varepsilon_1 n}, \quad (3.9)$$

where  $C_6$  depends on  $\varepsilon_1, m, p, q$ , polynomially on  $H(\xi)$  and  $n$ , and linearly in  $E^{m+t_1}$ . As a polynomial in  $H(\xi)$  and  $n$ , the total degree of  $C_6$  depends only on  $\varepsilon_1, m, p, q$ .

### 3.3.3 Conclusion

We are now ready to analyze the proof of [18, p. 357, Theorem 1] in the case  $\mathbb{K} = \mathbb{Q}$  in order to make Shidlovskii's measure explicit. Shidlovskii proves that for any  $\varepsilon \in (0, 1)$ , any integer  $H \geq 1$  and any vector  $(a_1, \dots, a_m) \in \mathbb{Z}^m \setminus \{0\}$  such that  $\max |a_i| \leq H$ , we have

$$\left| \sum_{i=1}^m a_i f_i(\xi) \right| \geq b_4 n^{-(m-1+\varepsilon)n} (1 - b_5 H n^{-(1-\varepsilon)n}),$$

where  $b_4, b_5 > 0$  are not computed but are known to be independent of  $H$ . He then chooses the smallest  $n$  such that  $n \geq n_0$  and  $n^{(1-\varepsilon)n} > 2b_5 H$  (such an  $n$  obviously exists) to deduce the expected linear independence measure:

$$\left| \sum_{i=1}^m a_i f_i(\xi) \right| \geq \frac{b_7}{H^{m-1+2m\varepsilon}}$$

for a constant  $b_7 > 0$  which is again not computed but is known to be independent of  $H$ .

In the proof of [18, p. 357, Theorem 1] in the case  $\mathbb{K} = \mathbb{Q}$ , we have  $i = h = 1$ . Shidlovskii defines the determinant  $\Delta$  whose entries are the coefficients (in  $\mathbb{Z}$ ) of the  $m$  linear forms  $L_1, \dots, L_{m-1}, L_0 := \sum_{i=1}^m a_i f_i(\xi)$  (which up to reordering can be assumed to be linearly independent without loss of generality when  $n \geq n_0$ ), and the determinant  $\Delta_{j,\ell}$  which is the cofactor of the entry in the  $j$ -th row and  $\ell$ -column of  $\Delta$ . (Each line corresponds to a linear form, with  $L_1$  at the top and  $L_0$  at the bottom of the determinant.) In [18, p. 358, Eq. (41)], each occurrence of  $i(=1)$  can be deleted and we have

$$|\Delta_{m,\ell}| \cdot |L_0| \geq |f_\ell(\xi)| \cdot |\Delta| - (m-1) \max_{1 \leq j \leq m-1} |\Delta_{j,\ell}| \cdot \max_{1 \leq j \leq m-1} |L_j|. \quad (3.10)$$

This inequality holds for any  $\ell \in \{1, \dots, m\}$  such that  $f_\ell(\xi) \neq 0$ ; such an  $\ell$  exists because  $\xi T(\xi) \neq 0$ . We have  $|\Delta| \geq 1$  when  $n \geq n_0$ . Then using (3.8) and (3.9), for any  $n \geq n_0$ , we can replace the three bounds in [18, p. 359, Eq. (42)] by

$$\begin{aligned} \max_{1 \leq j \leq m-1} |L_j| &\leq C_5 (2CD)^{6m^2n/\varepsilon_1} e^{2C|\xi|} (2q \max(\varepsilon_1, C_4))^{\varepsilon_1 n} \\ &\quad \times (1 + |\xi|)^{q\varepsilon_1 n} \text{den}(\xi)^{(1+q\varepsilon_1)n} \frac{n^{n+\varepsilon_1 n} (2C|\xi|)^{mn-2\varepsilon_1 n}}{(mn - 2\lfloor \varepsilon_1 n \rfloor)!}, \end{aligned}$$

$$|\Delta_{m,\ell}| \leq C_7 ((2CD)^{6m^2n/\varepsilon_1} (2q \max(\varepsilon_1, C_4))^{\varepsilon_1 n} ((1 + |\xi|) \text{den}(\xi))^{(1+q\varepsilon_1)n} n^{n+2\varepsilon_1 n})^{m-1}$$

and

$$\max_{1 \leq j \leq m-1} |\Delta_{j,\ell}| \leq C_8 H ((2CD)^{6m^2n/\varepsilon_1} (2q \max(\varepsilon_1, C_4))^{\varepsilon_1 n} ((1 + |\xi|) \text{den}(\xi))^{(1+q\varepsilon_1)n} n^{n+2\varepsilon_1 n})^{m-2},$$

where  $C_7, C_8 > 0$  have the same dependencies as  $C_6^m$ , where  $C_6$  is considered in (3.9), and both can be bounded accordingly. We recall from §3.3.2 that:

- $C_4$  depends only on  $m, p, q$ , but not on  $n, \xi$ .
- $C_5 \leq \tilde{c}_5^n$  where  $\tilde{c}_5$  depends on  $\varepsilon_1, m, p, q, T(z)$ , polynomially on  $H(\xi)$  but not on  $n$ .
- $C_6 \leq \tilde{c}_6^n$  where  $\tilde{c}_6$  depends on  $\varepsilon_1, m, p, q$ , polynomially on  $H(\xi)$  and on  $E$ , but is independent of  $n$ . All polynomials involved here have degrees bounded in terms of  $\varepsilon_1, m, p, q$ .

Now the three bounds obtained above yield, for any  $n \geq n_0$ ,

$$\max_{1 \leq j \leq m-1} |L_j| \leq C_9 e^{2C|\xi|} n^{C_{10}} C_{11}^n n^{-n(m-1-3\varepsilon_1)}$$

$$|\Delta_{m,\ell}| \leq C_9 n^{C_{10}} C_{11}^n n^{n(m-1)(1+2\varepsilon_1)},$$

and

$$\max_{1 \leq j \leq m-1} |\Delta_{j,\ell}| \leq C_9 H n^{C_{10}} C_{11}^n n^{n(m-2)(1+2\varepsilon_1)},$$

where :

- $C_9 \geq 1$  depends on  $\varepsilon_1, m, p, q$ , polynomially on  $H(\xi), C$  and  $D$ , but not on  $n$ .
- $C_{10} \geq 0$  depends on  $\varepsilon_1, m, p, q$ , polynomially on  $H(\xi)$ , not on  $n$  and not on the Taylor coefficients of the  $f_j$ 's.
- $C_{11} \geq 1$  depends on  $\varepsilon_1, m, p, q, T(z)$ , polynomially on  $H(\xi), C, D$  and  $E$ , but not on  $n$ .
- The total degree of  $C_9$  as a polynomial in  $H(\xi), C$  and  $D$ , and the one of  $C_{11}$  in  $H(\xi), C, D$  and  $E$ , are bounded in terms of  $\varepsilon_1, m, p, q$ .

Now we choose  $\ell$  such that

$$|f_\ell(\xi)| = \max_{1 \leq j \leq m} |f_j(\xi)| \geq \kappa$$

and we deduce from (3.10) that, for  $n \geq n_0$ ,

$$|L_0| \geq \frac{\kappa}{C_9 n^{C_{10}} C_{11}^n n^{n(m-1)(1+2\varepsilon_1)}} (1 - un^v w^n n^{-\delta n} H) \quad (3.11)$$

where

$$u := \max(1, (m-1)C_9^2 e^{2C|\xi|}/\kappa), \quad v := 2C_{10}, \quad w := C_{11}^2,$$

and  $\delta := 1 - (2m-1)\varepsilon_1 \in (0, 1)$  by the assumption made on  $\varepsilon_1$  at the beginning of §3.3. This implies

$$|L_0| \geq \frac{\kappa}{2C_9 n^{C_{10}} C_{11}^n n^{n(m-1)(1+2\varepsilon_1)}}$$

provided

$$n \geq n_0 \quad \text{and} \quad 2uH \leq w^{-n} n^{\delta n - v}. \quad (3.12)$$

Since  $u \geq 1$  and we assume  $H \geq \max(3, n_0^{n_0})$  in Proposition 3, we have  $2uH \geq n_0^{n_0}$ . Accordingly for any  $n < n_0$  we have  $2uH > n^n \geq w^{-n}n^{\delta n - v}$ , so that the minimal value of  $n$  (denoted by  $N$  from now on) that satisfies  $2uH \leq w^{-n}n^{\delta n - v}$  is automatically  $\geq n_0$ : it satisfies the assumptions (3.12).

We want to find an upper bound for  $N$  in terms of  $2uH, w, v, \delta$ . An equivalent definition of  $N$  is that it is the largest integer such that

$$(N - 1)^{N-1} < (2uH)^{1/\delta} (w^{1/\delta})^{N-1} (N - 1)^{v/\delta}. \quad (3.13)$$

For any  $X \geq 1$  we have the following lower bounds:

$$\begin{aligned} \frac{X^X}{(2uH)^{1/\delta} w^{X/\delta} X^{v/\delta}} &\geq \frac{X^X}{(2uH)^{1/\delta} X^{X/4} X^{v/\delta}} \quad \text{assuming that } X \geq w^{4/\delta}, \\ &= \frac{X^{3X/4}}{(2uH)^{1/\delta} X^{v/\delta}} \\ &\geq \frac{X^{X/4}}{(2uH)^{1/\delta}} \quad \text{assuming that } X \geq \frac{2v}{\delta} \\ &= \left( \frac{X^X}{(2uH)^{4/\delta}} \right)^{1/4} \\ &\geq 1 \quad \text{assuming that } X \geq \frac{8 \ln(2uH)}{\delta \ln \ln((2uH)^{4/\delta})}, \end{aligned}$$

where in the last line we use the elementary fact that for any  $y > e$ , if  $x \geq \frac{2 \ln(y)}{\ln \ln(y)}$ , then  $x^x \geq y$ . Note that  $H \geq 3$  and  $u \geq 1$  so that  $(2uH)^{4/\delta} \geq 6^{4/\delta} > e$  because  $\delta \in (0, 1)$  and thus we can use this fact with  $y := (2uH)^{4/\delta}$ . All assumptions in the lower bound above are satisfied for instance for any

$$X \geq 1 + w^{4/\delta} + \frac{2v}{\delta} + \frac{8 \ln(2uH)}{\delta \ln \ln((2uH)^{4/\delta})}.$$

Since for  $X = N - 1$  the lower bound  $\frac{X^X}{(2uH)^{1/\delta} w^{X/\delta} X^{v/\delta}} \geq 1$  does not hold, we deduce that

$$N \leq 1 + w^{4/\delta} + \frac{2v}{\delta} + \frac{8 \ln(2uH)}{\delta \ln \ln((2uH)^{4/\delta})} =: \Phi. \quad (3.14)$$

Hence, since (3.12) holds with  $n := N$ , we have

$$\begin{aligned} |L_0| &\geq \frac{\kappa}{2C_9 N^{C_{10}} C_{11}^N N^{N(m-1)(1+2\varepsilon_1)}} \\ &= \frac{\kappa}{2C_9 e^{C_{10} \ln(N) + \ln(C_{11})N + (m-1)(1+2\varepsilon_1)N \ln(N)}} \\ &\geq \frac{1}{H^c} \end{aligned}$$

because

$$N \ln(N) \leq \Phi \ln(\Phi) \leq C_{12} \ln(H) \text{ and } H \geq 3,$$

where  $c$  and  $C_{12}$  depend on  $\varepsilon_1, m, p, q, \kappa, T(z)$ , and polynomially on  $H(\xi), C, D$  and  $E$ . The total degrees of these polynomials are bounded in terms of  $\varepsilon_1, m, p, q$ . To conclude the proof of Proposition 3, we choose  $\varepsilon_1 = \frac{1}{2m}$ .

## 4 Proof of Theorem 2

### 4.1 First reductions

Let  $f \in \mathbb{Q}[[z]]$  be a strict  $E$ -function, and  $\xi \in \mathbb{Q}^*$  be such that  $f(\xi) > 0$ . Considering  $f(\xi z)$  instead of  $f$ , we may assume that  $\xi = 1$ . Recall that  $f(1) \neq 1$ , because  $\ln(f(1)) \notin \mathbb{Q}$  is an assumption of Theorem 2. Accordingly, if  $f(1)$  is algebraic then  $\ln(f(1))$  is not a Liouville number by [16, p. 386, Theorem 3], and the conclusion of Theorem 2 follows at once. Therefore we may assume that  $f(1)$  is transcendental and apply the following consequence of [13, Proposition 2], which is a variant of Beukers' desingularization lemma [9, Theorem 1.5].

**Proposition 4.** *Let  $g_1, \dots, g_k$  be  $E$ -functions with rational coefficients, such that  $1, g_1, \dots, g_k$  are linearly independent over  $\mathbb{Q}(z)$  and  $g_1(1)$  is transcendental. Assume also that the vector  ${}^t(1, g_1, \dots, g_k)$  is solution of a differential system  $Y' = AY$  with  $A \in M_{k+1}(\mathbb{Q}(z))$ .*

*Then there exist  $E$ -functions  $f_1, \dots, f_k$  with rational coefficients, such that  $1, f_1, \dots, f_k$  are linearly independent over  $\mathbb{Q}(z)$ ,  $f_1(1) = g_1(1)$ , and the vector  ${}^t(1, f_1, \dots, f_k)$  is solution of a differential system  $Y' = BY$  with  $B \in M_{k+1}(\mathbb{Q}[z, \frac{1}{z}])$ .*

*Moreover, when the  $g_j$ 's are strict  $E$ -functions, then the  $f_j$ 's are strict  $E$ -functions as well.*

To apply Proposition 4, we consider the inhomogeneous differential equation of minimal order satisfied by  $f$ . We denote its order by  $\mu - 1$ , with  $\mu \geq 2$  because  $f$  is transcendental (since  $f(1)$  is). The functions  $1, f, f', \dots, f^{(\mu-2)}$  are linearly independent over  $\mathbb{Q}(z)$ : Proposition 4 provides  $E$ -functions  $f_1, \dots, f_{\mu-1}$  with rational coefficients such that  $f_1(1) = f(1)$  is the number we are interested in the logarithm of. Letting  $f_\mu = 1$ , the functions  $f_1, \dots, f_\mu$  are linearly independent over  $\mathbb{Q}(z)$  and make up a vector solution of a differential system without non-zero singularity; in particular 1 is not a singularity.

Now recall that the functions  $\exp(\beta z)$ ,  $\beta \in \overline{\mathbb{Q}}$ , are linearly independent over  $\mathbb{Q}(z)$ . Therefore at most  $\mu$  of them belong to the  $\mathbb{Q}(z)$ -vector space spanned by  $f_1, \dots, f_\mu$ . Since  $\ln(f(1))$  is irrational, we may exclude these finitely many values of  $\beta = a/b$  in proving the lower bound (1.1) (up to changing the values of  $c$  and  $d$ ). Therefore we may restrict to rationals  $\beta$  such that  $f_1, \dots, f_\mu$ , and  $\exp(\beta z)$  are linearly independent over  $\mathbb{Q}(z)$ .

## 4.2 Application of the effective linear independence measure

As explained in the previous section, we are trying to bound

$$\left| \ln(f_1(1)) - \frac{a}{b} \right|$$

from below, with  $a \in \mathbb{Z}$  and  $b \in \mathbb{N}^*$ . We may assume that  $a$  and  $b$  are coprime. By the mean value theorem (see Eq. (4.2) below), it is essentially equivalent to bounding below

$$|f_1(1) - \exp(a/b)|,$$

which is a  $\mathbb{Z}$ -linear combination of the values at 1 of the  $E$ -functions  $f_1(z)$  and  $\exp(\beta z)$ , with  $\beta := a/b$ . We point out that the coefficients of this linear combination are only 0, 1 and  $-1$  whereas in general, we are always interested in linear combinations with large coefficients.

As explained at the end of §4.1, we may assume that  $f_1, \dots, f_\mu$  and  $f_{\mu+1}(z) = \exp(\beta z)$  are strict  $E$ -functions linearly independent over  $\mathbb{Q}(z)$ , solution of a first-order differential system without non-zero singularity. Moreover  $f_1, \dots, f_\mu$  are solution of a first-order differential system without non-zero singularity, independent from  $\beta$ : we are in the setting of (3.4).

Corollary 2 yields a constant  $c$ , which depends on  $f_1, \dots, f_{m-1}$  and polynomially on  $H(\beta)$ , such that with  $H = \max(3, n_0^{n_0})$  we have

$$|f_1(1) - \exp(a/b)| > \frac{1}{H^c}.$$

An important feature of this corollary is that this value of  $H$  depends only on  $f_1, \dots, f_{m-1}$ , not on  $\beta$ .

If  $|f_1(1) - \exp(a/b)| \geq f_1(1)/2$ , the lower bound of Theorem 2 holds trivially. Therefore we may assume that

$$|f_1(1) - \exp(a/b)| < f_1(1)/2. \quad (4.1)$$

Accordingly  $|\beta|$  is bounded in terms of  $f_1(1)$  only. Since  $H(\beta) \leq b \cdot \max(1, |\beta|)$ , there exists a polynomial  $Q \in \mathbb{R}[X]$  with positive coefficients such that  $c \leq Q(b)$ ; this polynomial  $Q$  depends only on  $f_1, \dots, f_{m-1}$ . Therefore

$$|f_1(1) - \exp(a/b)| > \frac{1}{HQ(b)}$$

and this concludes the proof of Theorem 2. Indeed by the mean value theorem, for all  $(a, b) \in \mathbb{Z}^* \times \mathbb{N}^*$ ,  $a, b$  coprime, there exists  $\omega > 0$  in the interval with endpoints  $\exp(a/b)$  and  $f_1(1)$  such that

$$\left| \ln(f_1(1)) - \frac{a}{b} \right| = \frac{1}{\omega} |f_1(1) - \exp(a/b)|, \quad (4.2)$$

and finally in this equality, the coefficient  $\omega$  can be bounded above in terms of  $f_1(1)$  only due to Eq. (4.1).

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## References

- [1] B. Adamczewski, T. Rivoal, Exceptional values of  $E$ -functions at algebraic points, *Bull. Lond. Math. Soc.* **50.4** (2018), 697–708.
- [2] Y. André, Séries Gevrey de type arithmétique I. Théorèmes de pureté et de dualité, *Ann. of Math.* **151.2** (2000), 705–740.
- [3] Y. André, Séries Gevrey de type arithmétique II. Transcendance sans transcendance, *Ann. of Math.* **151.2** (2000), 741–756.
- [4] Y. André, Algèbres de solutions d'équations différentielles et variétés quasi-homogènes : une nouvelle correspondance de Galois différentielle, *Ann. Sci. École Norm. Sup.* **47.2** (2014), 449–467.
- [5] D. Bertrand, F. Beukers, Équations différentielles linéaires et majorations de multiplicités, *Ann. Sci. École Norm. Sup.* **18.1** (1985), 181–192.
- [6] D. Bertrand, Exposants des systèmes différentiels, vecteurs cycliques et majorations de multiplicités, Équations différentielles dans le champ complexe, Vol. I (Strasbourg, 1985), *Publ. Inst. Rech. Math. Av.*, Univ. Louis Pasteur, Strasbourg, 1988, pp. 61–85.
- [7] D. Bertrand, V. Chirskii, J. Yebbou, Effective estimates for global relations on Euler-type series, *Annales de la faculté des sciences de Toulouse* **13.2** (2004), 241–260.
- [8] F. Beukers, W. Dale Brownawell, G. Heckman, Siegel normality, *Ann. Math.* **127.2** (1988), 279–308.
- [9] F. Beukers, A refined version of the Siegel-Shidlovskii theorem, *Ann. of Math.* **163.1** (2006), 369–379.
- [10] A. Bostan, T. Rivoal, B. Salvy, Minimization of differential equations and algebraic values of  $E$ -functions, *Math. Comp.* **93** (2024), 1427–1472.
- [11] W. D. Brownawell, Effectivity in independence measures for values of  $E$ -functions, *J. Aust. Math. Soc. Ser. A* **39** (1985), 227–240.
- [12] E. Delaygue, A Lindemann-Weierstrass theorem for  $E$ -functions, preprint arXiv 2210.12046, 8 pages, submitted.
- [13] S. Fischler, T. Rivoal, Rational approximations to values of  $E$ -functions, preprint arXiv 2312.12043, 32 pages, submitted.
- [14] G. Lepetit, Le théorème d'André-Chudnovsky-Katz au sens large, *North-West. Eur. J. Math.* **7** (2021), 83–149.
- [15] K. Mahler, Zur Approximation der Exponentialfunktion und des Logarithmus. Teil I, *J. reine angew. Math.* **166** (1931), 118–136.

- [16] K. Mahler, On the approximation of logarithms of algebraic numbers, *Philos. Trans. Roy. Soc. London Ser. A* **245** (1953), 371–398.
- [17] D. Marques, C. Moreira, On a variant of a question proposed by K. Mahler concerning Liouville numbers, *Bull. Aust. Math. Soc.* **91** (2015), 29–33.
- [18] A. B. Shidlovskii, *Transcendental numbers*, de Gruyter Studies in Math. **12**, de Gruyter, Berlin, 1989.
- [19] C. Siegel, Über einige Anwendungen diophantischer Approximationen, vol. 1 S. *Abhandlungen Akad.*, Berlin, 1929.

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