A PROOF OF KOLMOGOROV'S MAXIMAL INEQUALITY FOR RADEMACHER'S FUNCTIONS

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Abstract. It is a well-known and useful fact that Rademacher's functions (i.e, the binary bits of a real number x viewed as functions of x) provide a bridge between analysis and probability theory (sums of Bernoulli random variables). Thanks to a functional equation for a suitable moment generating function, we provide a simple analytical proof of Kolmogorov's famous maximal inequality, a now classical step in the proof of Khintchine's law of iterated logarithm. The proof is completely self-contained and does not use any deep properties of Rademacher's functions, particularly their independence – in this case, the proof can hardly be simpler than Kolmogorov's own proof.

1. Introduction

1.1. **Notations.** Given any real number $x \in [0,1]$, we denote by $\lfloor x \rfloor$ and $\{x\}$ its integer and fractional parts respectively. We define a function $T : [0,1] \to [0,1]$ by $T(x) = \{2x\}$. We will note $T^n x$ the n-th iterate of x under T, with $T^0 x = x$. Let $x = 0.x_1 x_2 x_3 \cdots$ denote the binary expansion of $x \in [0,1]$, where $x_j \in \{0,1\}$. Provided $x \notin \mathbb{D} = \{j/2^r : j \in \mathbb{N}, r \in \mathbb{Z}\}$, this expansion is unique. If $x \in \mathbb{D}$, we choose its finite expansion so that we have $x_j = \lfloor 2T^{j-1}x \rfloor$ for all $x \in [0,1]$ and $j \geq 1$.

Our main object of study is the "sum of digits" function B_n defined for $x \in [0,1]$ and $n \geq 1$ by $B_n(x) = \sum_{j=0}^{n-1} \lfloor 2T^j x \rfloor$. It is piecewise constant and for all $n \geq 1$, we have

$$B_{n+1}(x) = B_n\big(Tx\big) + \lfloor 2x \rfloor. \tag{1}$$

We will use the "centered" functions $S_n = 2B_n - n$. The functions $(1 - 2\lfloor 2T^n \rfloor)_{n\geq 1}$ coincide on $[0,1] \setminus \mathbb{D}$ with Rademacher's functions [18]. The later are important in analysis and probability, particularly because they mimic the game of heads or tails. For any predicate $P(x)$, we note $\mu(P)$ for $\mu(\{x \in [0,1] : P(x)\})$, where μ is Lebesgue measure on $\mathbb{R}$.

1.2. **The probabilistic and number theoretic contexts.** In this article, we are interested in the deviations from 0 of S_n and $\max_{1\leq k\leq n} S_k$ but before stating the main result of this article, it seems more appropriate to present the background. The almost sure (a.s.) asymptotic behavior of $B_n(x)$ with respect to μ has a long history going back to the seminal work of Borel [3] who proved in 1909 his law of large numbers:

$$B_n(x) = n/2 + o(n) \quad \text{a.s.} \tag{2}$$

See [1] for an account of Borel's proof of (2) and its place in the foundations of probability theory. In number theoretical terms, (2) says that almost all real numbers are simply normal in base 2. The more general statement that almost all real numbers are normal in base 2 was proved by Borel in [3]: a proof using products of Rademacher's functions was given by Mendès-France [16], see also Goodman [7]. It is difficult to give an exhaustive list of all the more or less elementary different proofs of (2); see for example Rademacher [19], Kac [10, 11] and, more recently, Nillsen [17].

In 1913, Hausdorff [9] improved (2) and showed that $B_n(x) = n/2 + \mathcal{O}(n^{1/2+\varepsilon})$ a.s. for all $\varepsilon > 0$. Hardy and Littlewood [8] made Hausdorff's result more precise in 1914: $B_n(x) = n/2 + \mathcal{O}(\sqrt{n\log(n)})$ a.s. and this was made completely effective by Steinhaus [23] in 1922:

$$\limsup_{n\to+\infty} \frac{|B_n(x) - n/2|}{\sqrt{\frac{1}{2}n\log(n)}} \leq 1 \quad \text{a.s.}.$$

The best possible estimate was obtained by Khintchine [12] in 1924:

$$\limsup_{n\to+\infty} \frac{B_n(x) - n/2}{\sqrt{\frac{1}{2}n\log\log(n)}} = 1, \quad \liminf_{n\to+\infty} \frac{B_n(x) - n/2}{\sqrt{\frac{1}{2}n\log\log(n)}} = -1 \;\; \text{a.s.}. \tag{3}$$

Lévy [15] addressed the more general question of the asymptotic expansion of the order of magnitude of $B_n - n/2$, a problem solved by Erdös [4] who found an asymptotic expansion on the scale of iterated logarithms $\log_k (= \log(\log_{k-1}))$.

1.3. **Rényi's approach to large deviations inequalities.** Steinhaus and Khintchine's theorems rest on sharp bounds of the measure of the deviations from 0 of S_n (eq. (5)) and $\max_{1\leq k\leq n} S_k$ (eq. (6)) respectively. The proof of such bounds are more or less simple, according to what is assumed, in particular independence. Our main goal is to simplify an elementary proof of (6) given by Rényi.

Consider the moment generating function of $S_n(x)$ defined by $R_n(q) = \int_0^1 q^{S_n(x)}\mathrm{d}x$ for $q > 0$. As noticed by Rényi in [21], Hardy and Littlewood's theorem (and in fact also Steinhaus' theorem) is a consequence of the identity (where $q > 0$)

$$R_n(q) = \Big(\frac{q + 1/q}{2}\Big)^n, \tag{4}$$

because the later implies Bernstein's large deviations inequality [2]: for $\alpha \geq 0$, $n \geq 1$, we have

$$\mu(|S_n| \geq \alpha) \leq 2e^{-\alpha^2/(2n)}. \tag{5}$$

Steinhaus' theorem follows by the Borel-Cantelli lemma. In probabilistic terms, (4) is obvious because the sequence $(x_j)_{j\geq 1}$ is a i.i.d. Bernoulli process. It can also be proved analytically by mean of the Kac-Steinhaus' theory of independent functions [11], of which Rademacher's functions are a classical example. But Rényi found that (4) follows "by a simple combinatorial argument" (*sic*) without the use of any probability concepts. Since it is not clear to us what he meant, we provide a simple proof of (4) by an induction that uses no more than the recursive equation (1).

A different idea is needed to get the law of the iterated logarithm. Kolmogorov simplifed Khintchine's proof of (3) by introducing $S_n^* = \max_{1\leq k\leq n} S_k$, for which he (essentially) proved the following *maximal inequality*: for $\alpha \geq 0$ and $n \geq 1$, we have

$$\mu(|S_n^*| \geq \alpha) \leq 4e^{-\alpha^2/(2n)}. \tag{6}$$

That inequality then implies (3) with $= 1$ replaced by ≤ 1 and $= -1$ replaced by ≥ 1. Rényi showed that it was possible to deduce (6) using the explicit computation of the moment generating function of S_n^*. Setting $R_n^*(q) = \int_0^1 q^{S_n^*(x)} \mathrm{d}x$ for $n \geq 1$ and $q > 0$, he proved that

$$R_{2n}^*(q) = \frac{1}{2^{2n}}\Big(1 + \frac{1}{q}\Big)\left[\sum_{k=0}^{n-1} \binom{2n}{k} q^{2n-2k} + \binom{2n-1}{n-1}\right], \tag{7}$$

$$R_{2n+1}^*(q) = \frac{1}{2^{2n+1}}\left[\Big(1 + \frac{1}{q}\Big) \sum_{k=0}^{n-1} \binom{2n+1}{k} q^{2n+1-2k} \right.$$
$$\left. + \binom{2n+1}{n} q + \binom{2n}{n-1} + \binom{2n}{n}\frac{1}{q}\right]. \tag{8}$$

1.4. **The result.** Our main theorem is a simple analytical proof of the following identity, which simplifies part of Rényi's approach.

Theorem 1. *For all $n \geq 1$ and $q > 0$, we have*

$$R_n^*(q) + \frac{1}{q} R_n^*\Big(\frac{1}{q}\Big) = R_n(q) + \frac{1}{q} R_n\Big(\frac{1}{q}\Big). \tag{9}$$

Remarks. Since $R_n(q) = R_n(1/q)$ and $R_n(q)$ is known explicitely, we could simplify the right hand side of (9). But we find it aesthetic under this form.

In a sense, this identity is not new because it is equivalent to (7) and (8) (note that since $S_n^*(x)$ takes values in $\{-1, 0, 1, \ldots, n\}$, the function $qR_n^*(q)$ is a polynomial in q of degree $n+1$). Furthermore, equating the coefficients of like powers on both sides of (9), we obtain

$$\mu(S_n^* = k) + \mu(S_n^* = -k-1) = \mu(S_n = k) + \mu(S_n = k+1)$$

for all $n \geq 1$ and all $k \in \mathbb{Z}$. For any integer N, summing over $k \geq N$, we get that

$$\mu(S_n^* \geq N) + \mu(S_n^* \leq -N-1) = 2\mu(S_n > N) + \mu(S_n = N).$$

Since $\mu(S_n^* \leq -N-1) = 0$ for $N \geq 1$, we recover in this case an identity mentioned by Lévy in [15, Lemme V].

What we hope to be new is our proof of (9). It is based on the function

$$\mathscr{R}_n(\sigma, q) = \int_0^1 q^{\max(\sigma, S_n^*(x))} \mathrm{d}x,$$

defined for $\sigma \in \mathbb{R}$ and $q > 0$ and which satisfies $\mathscr{R}_n(-1, q) = R_n^*(q)$ and the functional-recursive relation

$$\mathscr{R}_{n+1}(\sigma, q) = \frac{q}{2}\mathscr{R}_n\big(\max(0, \sigma - 1), q\big) + \frac{q^{-1}}{2}\mathscr{R}_n\big(\max(0, \sigma + 1), q\big),$$

which is the heart of our argument.

2. The proofs

As a warmup, we first present a simple proof of (4) and (5), we will then prove Theorem 1, from which Kolmogorov's inequality (6) (and then Khintchine's theorem) follows.

2.1. **Proof of Bernstein's large deviations inequality** (5)**.** For completeness, we first prove (4). We proceed by induction on $n \geq 1$. The result is true for $n = 1$ because

$$R_1(q) = \int_0^1 q^{2\lfloor 2x\rfloor -1}\mathrm{d}x = \int_0^{1/2} q^{-1}\mathrm{d}x + \int_{1/2}^1 q^{1}\mathrm{d}x = \frac{q+1/q}{2}.$$

Suppose now that (4) is true for n. Since $Tx = 2x$ on $[0,1/2]$ and $Tx = 2x-1$ on $[1/2,1]$, we have

$$\begin{aligned} R_{n+1}(q) &= \int_0^1 q^{S_n(Tx)+2\lfloor 2x\rfloor -1}\mathrm{d}x = \int_0^{1/2} q^{S_n(2x)-1}\mathrm{d}x + \int_{1/2}^1 q^{S_n(2x-1)+1}\mathrm{d}x \\ &= \frac{q^{-1}}{2}\int_0^1 q^{S_n(u)}\mathrm{d}u + \frac{q}{2}\int_0^1 q^{S_n(v)}\mathrm{d}v = \frac{q+1/q}{2}R_n(q) \end{aligned}$$

which proves (4) for $n+1$ by the induction hypothesis. The last line follows from the changes of variable $u = 2x$ and $v = 2x-1$.

A classical proof of Bernstein's large deviation inequality (5) is the following. For any real number u, we have $(e^u + e^{-u})/2 \leq e^{u^2/2}$ and hence (4) implies that $R_n(e^u) \leq e^{nu^2/2}$. Now, for any real number α, consider the set $E(n,\alpha) = \{x \in [0,1] : S_n(x) \geq \alpha\}$. For $u \geq 0$, we obviously have

$$R_n(e^u) \geq \int_{E(n,\alpha)} e^{uS_n(x)}\mathrm{d}x \geq e^{u\alpha}\mu(E(n,\alpha)).$$

Hence, the inequality $\mu(E(n,\alpha)) \leq \exp(nu^2/2 - \alpha u)$ holds for any $u \geq 0$ and any α. Provided that $\alpha \geq 0$, the minimum with respect to u is attained at $u = \alpha/(2n)$, yielding $\mu(S_n \geq \alpha) \leq \exp(-\alpha^2/2n)$ for all $\alpha \geq 0$. The inequality $\mu(S_n \leq \alpha) \leq \exp(-\alpha^2/2n)$ for all $\alpha \leq 0$ follows from the symmetry $S_n(-x) = -S_n(x)$ for $x \in [0,1] \setminus \mathbb{D}$ and $n \geq 1$, which itself follows from the identity $\lfloor 2T^j(1-x)\rfloor = 1 - \lfloor 2T^j(x)\rfloor$ for $x \in [0,1]\setminus\mathbb{D}$ and $j \geq 1$.

2.2. **Proof of Theorem 1.** Since the function $S_n^*(x)$ takes values in the set $\{-1, 0, \dots, n\}$, we have

$$\int_0^1 q^{\max(\sigma, S_n^*(x))}\mathrm{d}x = \sum_{k=-1}^{n} \mu(S_n^* = k)\, q^{\max(\sigma,k)}$$

for all $\sigma \in \mathbb{R}$ and $q > 0$: the sequence of coefficients of the power expansion on the right hand side is unique [1]). For simplicity, we set $a_{k,n} = 2^n\mu(S_n^* = k)$, so that

$$\mathscr{R}_n(\sigma, q) = \frac{1}{2^n}\sum_{k=-1}^{n} a_{k,n}\, q^{\max(\sigma,k)}.$$

[1] If two finite expansions of this type are equal, starting at k_1 and k_2 say, then for $\alpha = \min(k_1,k_2)$, these expansions are Laurent polynomials in q.

(From now on, we drop the variable q in $\mathscr{R}_n(\sigma,q)$.) We want to find a recursive way to compute the $a_{k,n}$'s. This will be achieved thanks to the following lemma.

Lemma 1. *For all $\alpha\in\mathbb{R}$, $q>0$ and $n\ge 1$, we have $\mathscr{R}_1(\sigma)=\frac{1}{2}\big(q^{\max(\sigma,-1)}+q^{\max(\sigma,1)}\big)$ and*

$$\mathscr{R}_{n+1}(\sigma)=\frac{q}{2}\mathscr{R}_n\big(\max(0,\sigma-1)\big)+\frac{q^{-1}}{2}\mathscr{R}_n\big(\max(0,\sigma+1)\big). \tag{10}$$

Proof. Like for $R_1(q)$, the assertion for $\mathscr{R}_1(\sigma)$ is straightforward.

To prove (10), we first make the trivial but crucial observation that

$$S^*_{n+1}(x)=2\lfloor 2x\rfloor-1+\max\big(0,S^*_n(Tx)\big) \tag{11}$$

because

$$\begin{aligned}\max_{1\le k\le n+1}S_k(x) &= \max\big(S_1(x),\max_{1\le k\le n}S_{k+1}(x)\big)\\ &= \max\big(S_1(x),\max_{1\le k\le n}\big(S_k(Tx)+S_1(x)\big)\big)\\ &= S_1(x)+\max\big(0,\max_{1\le k\le n}S_k(Tx)\big).\end{aligned}$$

The rest of the proof is now similar in principle to that of (4): we deduce from (11) that

$$\begin{aligned}\mathscr{R}_{n+1}(\sigma) &= \int_0^1 q^{\max\left(\sigma,2\lfloor 2x\rfloor-1+\max(0,S^*_n(Tx))\right)}\mathrm{d}x\\ &= q^{-1}\int_0^{1/2}q^{\max(\sigma+1,0,S^*_n(2x))}\mathrm{d}x+q\int_{1/2}^1 q^{\max(\sigma-1,0,S^*_n(2x-1))}\mathrm{d}x\\ &= \frac{q^{-1}}{2}\int_0^1 q^{\max(\sigma+1,0,S^*_n(u))}\mathrm{d}u+\frac{q}{2}\int_0^1 q^{\max(\sigma-1,0,S^*_n(v))}\mathrm{d}v \qquad (12)\\ &= \frac{q^{-1}}{2}\int_0^1 q^{\max(\sigma+1,0,S^*_n(u))}\mathrm{d}u+\frac{q}{2}\int_0^1 q^{\max(\sigma-1,0,S^*_n(v))}\mathrm{d}v\\ &= \frac{q^{-1}}{2}\mathscr{R}_n\big(\max(0,\sigma+1)\big)+\frac{q}{2}\mathscr{R}_n\big(\max(0,\sigma-1)\big).\end{aligned}$$

In (12), we made the changes of variable $u=2x$ and $v=2x-1$. ☐

We can now state the recursions satisfied by the $a_{k,n}$'s.

Lemma 2. *The coefficients $a_{k,n}$ can be computed from the following recursions: for any $n\ge 1$,*

$$\begin{cases}a_{-1,n+1}=a_{-1,n}+a_{0,n}\\ a_{0,n+1}=a_{1,n}\\ a_{1,n+1}=a_{-1,n}+a_{0,n}+a_{2,n}\\ a_{k,n+1}=a_{k-1,n}+a_{k+1,n}\quad \text{for } k\ge 2\end{cases} \tag{13}$$

where by convention $a_{k,n}=0$ for $k\ge n+1$ and the initial values are $a_{-1,1}=a_{1,1}=1$, $a_{k,1}=0$ for other k's.

Proof. We apply (10) with $\sigma = -1$: after simplification of the powers of 2, we obtain

$$\begin{aligned}\sum_{k=-1}^{n+1} a_{k,n+1}\, q^k &= \sum_{k=-1}^{n} a_{k,n}\, q^{\max(0,k)+1} + \sum_{k=-1}^{n} a_{k,n}\, q^{\max(1,k)-1} \\ &= a_{-1,n}q + \sum_{k=0}^{n} a_{k,n}\, q^{k+1} + a_{-1,n}q^{-1} + a_{0,n}q^{-1} + \sum_{k=1}^{n} a_{k,n}\, q^{k-1} \\ &= a_{-1,n}q^{-1} + a_{0,n}q^{-1} + a_{-1,n}q + \sum_{k=1}^{n+1} a_{k-1,n}\, q^k + \sum_{k=0}^{n-1} a_{k+1,n}\, q^k.\end{aligned}$$

The recursions (13) readily follow by polynomial identification. □

Remark. The recursions (13) were obtained by Rényi by a different method. He then verified identities (7) and (8) from (13) by induction. I am still not sure that I would have been able to guess the binomial expressions for the $a_{k,n}$'s just by looking at (13). Instead, I tried to solved them by "generating-functionology" and finally found (9), whose proof is given below.

We are now ready to conclude. We want to show that, for any $n \geq 1$, we have

$$R_n^*(q) + \frac{1}{q} R_n^*\Big(\frac{1}{q}\Big) = \Big(1 + \frac{1}{q}\Big)\Big(\frac{q + 1/q}{2}\Big)^n.$$

We will proceed by induction on n. Since $R_1^*(q) = \mathscr{R}_1(-1, q) = (q + q^{-1})/2$, this is true for $n = 1$. Let us suppose that this is true for some n and let $A_k = a_{k,n+1}$ and $a_k = a_{k,n}$. Using recursions (13), we have by straightforward formal manipulations:

$$\begin{aligned}2^{n+1}\big(R_{n+1}^*(q) + q^{-1}R_{n+1}^*(q^{-1})\big) &= \sum_{k=-1}^{\infty} A_k q^k + \sum_{k=0}^{\infty} A_{k-1} q^{-k} \\ &= \frac{A_{-1}}{q} + A_0 + A_1 q + \sum_{k=2}^{\infty} A_k q^k + A_{-1} + \frac{A_0}{q} + \frac{A_1}{q^2} + \sum_{k=3}^{\infty} A_{k-1} q^{-k} \\ &= \sum_{k=2}^{\infty} \big(a_{k-1} + a_{k+1}\big) q^k + \sum_{k=3}^{\infty} \big(a_{k-2} + a_k\big) q^{-k} \\ &\qquad + A_{-1}(1 + q^{-1}) + A_0(1 + q^{-1}) + A_1(q + q^{-2}) \\ &= 2^n (q + q^{-1})\big(R_n^*(q) + q^{-1}R_n^*(q^{-1})\big) + \rho_n(q),\end{aligned}$$

where

$$\begin{aligned}\rho_n(q) = -\sum_{k=0}^{1} a_{k-1}q^k - \sum_{k=-2}^{1} a_{k+1}q^k - \sum_{k=1}^{2} a_{k-2}q^{-k} - \sum_{k=-1}^{2} a_k q^{-k} \\ + A_{-1}(1 + q^{-1}) + A_0(1 + q^{-1}) + A_1(q + q^{-2}).\end{aligned}$$

In this expression for $\rho_n(q)$, thanks to (13), we can replace A_{-1} by $a_{-1}+a_0$, A_0 by a_1 and A_1 by $a_{-1}+a_0+a_2$: after simplifications, we find that $\rho_n(q)$ is identically 0. This finishes the proof of Theorem 1.

2.3. **Proof of Kolmogorov's maximal inequality** (6)**.** Since $R_n(1/q)>0$ for $q>0$, we deduce from (9) that $R_n^*(q)\le(1+1/q)R_n(q)$ for $q>0$. Therefore, with $q=e^u$ and $u\ge 0$, we have $R_n^*(e^u)\le 2R_n(e^u)\le 2\exp(u^2/2)$ and the argument given in section 2.1 implies Kolmogorov's inequality $\mu(S_n^*\ge\alpha)\le 2e^{-\alpha^2/(2n)}$ for all $\alpha\ge 0$.

3. Conclusion

We conclude this note with a few comments.

Classically, the proof of Steinhaus' theorem follows from Bernstein's bound for $\mu(S_n\ge\alpha)$ by the Borel-Cantelli lemma. The proof of Khintchine's theorem follows from Kolmogorov's bound for $\mu(S_n^*\ge\alpha)$ (or at least this is true of the proof of (3) with ≤ 1 instead of $=1$ and ≥ -1 instead of $=-1$). The optimality in Khintchine's theorem (i.e, the fact that (3) is holds) uses the estimate $\mu(S_n\ge\alpha)\gg e^{-\alpha^2/(2n)}$ (for $\alpha\ge 0$, $n\ge 1$), which essentially amounts to proving the de Moivre-Laplace theorem (see [14, p. 18]). It also uses the converse of the Borel-Cantelli lemma which holds for independent events, a typical probability notion. The method presented here does not yield any simplifications of these two aspects. All this is very classical and we refer the reader to any one of the books [5, 6, 22] for example.

We also note that (9) with $q=e^{it}$ ($t\in\mathbb{R}$) provides a functional equation for the characteristic function of the sequence $(S_n^*)_{n\ge1}$. We can apply Lévy's theorem to this function and, since qR_n^* is a polynomial in q, we obtain easily the following well-known convergence in law: as $n\to+\infty$,

$$\mu(S_n^*\le\alpha\sqrt{n})\to\begin{cases}0 \text{ if } \alpha\le 0,\\ \sqrt{2/\pi}\displaystyle\int_0^\alpha e^{-t^2/2}\,\mathrm{d}t \text{ if } \alpha\ge 0.\end{cases}$$

The recursion method can be used to "compute" the integrals

$$\mathscr{S}_n(\alpha,\beta)=\int_0^1 q^{\max(\alpha,\max_{1\le k\le n}|S_k+\beta|)}\mathrm{d}x,$$
$$\mathscr{A}_n(\alpha)=\int_0^1 q^{\sum_{k=1}^n \mathbf{1}_{\{\alpha\}}(S_k)}\mathrm{d}x,$$

where $\mathbf{1}_A$ denotes the indicator function of a given set A. The reader will easily check that

$$2\mathscr{S}_{n+1}(\alpha,\beta)=\mathscr{S}_n(\max(\alpha,\beta-1),\beta-1)+\mathscr{S}_n(\max(\alpha,\beta+1),\beta+1),$$
$$2\mathscr{A}_{n+1}(\alpha)=q^{\delta_{\alpha,-1}}\mathscr{A}_n(\alpha+1)+q^{\delta_{\alpha,1}}\mathscr{A}_n(\alpha-1),$$

where δ is Kronecker's delta. These recursions enable us to compute easily (at least with a computer) the laws of the random variables $\max_{1\le k\le n}|S_k|$ and $\#\{1\le k\le n : S_k=0\}$

for small values of n. However, finding explicit expressions for these integrals does not seem easy by this method. In fact, apparently, the law of $\max_{1\le k\le n}|S_k|$ is not even known explicitly.

Finally, the method can be easily extended to prove, for example, that

$$\int_0^1\int_0^1 q^{4\sum_{j=0}^{n-1}\lfloor 2T^jx\rfloor\lfloor 2T^jy\rfloor - n}\,\mathrm{d}x\mathrm{d}y = \Big(\frac{q^3+3/q}{4}\Big)^n. \tag{14}$$

Like Borel's theorem, this equation has number theoretic consequences which are relevant to the notion of independent sequences (see [20] for definitions). Equation (14) can be extended to higher dimensions.

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