ON THE COMPOSITIONAL INVERSE OF POLYLOGARITHMS

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We define the polylogarithm (of order s) as

$$\mathrm{Li}_s(z) := \sum_{m=1}^{\infty} \frac{z^m}{m^s}$$

where $s \geq 2$ is an integer and $|z| < 1$. Let $\ell_s(z) \in \mathbb{Q}[[z]]$ denote its compositional inverse, *ie* $\mathrm{Li}_s(\ell_s(z)) = z$. For instance, we have

$$\ell_2(z) = z - \frac{1}{4}z^2 + \frac{1}{72}z^3 - \frac{1}{576}z^4 - \frac{31}{86400}z^5 - \frac{149}{1036800}z^6 - \frac{18037}{304819200}x^7 - \cdots,$$

$$\ell_3(z) = z - \frac{1}{8}z^2 - \frac{5}{864}z^3 - \frac{31}{13824}z^4 - \frac{56039}{62208000}z^5 - \frac{628681}{1492992000}z^6 - \frac{800662417}{3687093043200}z^7 - \cdots.$$

The goal of this note is to prove the following result, where $[z^m](f)$ denotes the m-th Taylor coefficient of a formal power series $f \in \mathbb{C}[[z]]$.

Theorem 1. *As $m \to +\infty$, we have*

$$[z^m](\ell_2) \sim -\frac{\zeta(2)^{-m}}{m^2 \log(m)^2}$$

and, for every $s \geq 3$,

$$[z^m](\ell_s) \sim -\Big(\frac{\zeta(s)}{\zeta(s-1)}\Big)^{s-1} \cdot \frac{\zeta(s)^{-m}}{m^s}.$$

The difference between both cases is due to the difference of position of the term $(1-z)\log(1-z)$ in the singular expansions (4) (for $s=2$) and (7) (for $s \geq 3$) of $\mathrm{Li}_s(z)$ around $z=1$. For each integer $s \geq 2$, the quotients $[z^m](\ell_s)/[z^{m+1}](\ell_s)$ form a sequence of rational approximations to $\zeta(s)$, but they do not seem to be good enough to imply anything about the arithmetic nature of $\zeta(s)$.

The rest of this note is devoted to the proof of Theorem 1. For ease of reading, we shall set $\beta(s) := 1/\zeta(s)$ for all $s \geq 2$.

We shall need the following fact (cf. [1, p. 745, Theorem B.1]):

$$\frac{1}{2i\pi}\int_{\text{Hankel}} (-t)^{-s} e^{-t}\, dt = \frac{1}{\Gamma(s)} = \frac{\sin(\pi s)}{\pi}\Gamma(1-s).$$

Here "Hankel" is a countour that starts at ∞ below the real axis, turns around the origin clockwise, and proceeds towards ∞ above the real axis. In particular, the above integral vanishes whenever s is a non-positive integer. (This could also be derived from the fact

Date: February 10, 2025.

that, in that case, the integrand has no singularity in the interior of the Hankel contour.) As another corollary, by taking the derivative with respect to s on both sides, and then letting $s \to -n$, where n is a positive integer, we obtain

$$\frac{1}{2i\pi} \int_{\text{Hankel}} (-t)^n \log(-t) e^{-t}\, dt = (-1)^{n-1} n!\ . \tag{1}$$

By Lagrange's inversion formula (cf. [1, p. 732, Theorem A.2]), we can express the m-th Taylor coefficient of ℓ_s in the form

$$\frac{1}{2i\pi m} \int_{\mathcal{C}} \frac{dz}{\mathrm{Li}_s(z)^m}, \tag{2}$$

where $\mathcal{C}$ is a sufficiently small contour that encircles the origin once in positive (that is, counter-clockwise) direction.

The case $s = 2$. We now deform $\mathcal{C}$ to the contour $\mathcal{H}(m)$ which consists of four parts. In the precise description of this contour, we need the function $f(m)$ defined as the (unique) solution to

$$\frac{\zeta(2) f(m)}{m} = \log(f(m)) + 1$$

between m and m^2 (for large m). It should be observed (this is easily derived by bootstrapping) that

$$f(m) = (1/\zeta(2)) m \log(m) + \mathcal{O}\big(m \log(\log(m))\big) \tag{3}$$

for $m \to \infty$. The precise description of the four parts of the contour then is

$$\mathcal{H}(m) = \mathcal{H}^{\circ}(m) \cup \mathcal{H}^{+}(m) \cup \mathcal{H}^{-}(m) \cup \mathcal{H}^{\infty}(m),$$

where

$$\begin{aligned}
\mathcal{H}^{\circ}(m) &= \{z = 1 - \tfrac{e^{i\theta}}{f(m)} : \theta \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]\}, \\
\mathcal{H}^{+}(m) &= \{z = 1 + \tfrac{i}{f(m)} + \tfrac{w}{f(m)} : 0 \le w \le \log^2(m)\}, \\
\mathcal{H}^{-}(m) &= \{z = 1 - \tfrac{i}{f(m)} + \tfrac{w}{f(m)} : 0 \le w \le \log^2(m)\},
\end{aligned}$$

and where $\mathcal{H}^{\infty}(m)$ connects the end points of $\mathcal{H}^{+}(m)$ and $\mathcal{H}^{-}(m)$ (the points corresponding to $w = \log^2(m)$) by a path that stays in the cut plane $\mathbb{C}\backslash[1, +\infty)$ in such a way that, along this path, we have always $|\,\mathrm{Li}_2(z)| \ge |\mathrm{Li}_2(1)| + \varepsilon = \zeta(2) + \varepsilon$, for some $\varepsilon > 0$. Such a path exists.

The contribution of $\mathcal{H}^{\infty}(m)$ is of the order of magnitude

$$\mathcal{O}((\zeta(2) + \varepsilon)^{-m}) = o((1/\zeta(2))^m m^{-2}),$$

and this will turn out to be negligible in comparison to the contributions of the other parts of the contour.

The singular expansion of $\mathrm{Li}_2(z)$ at $z = 1$ is given by

$$\mathrm{Li}_2(z) = \zeta(2) + (1 - z)\log(1 - z) - (1 - z) + \tfrac{1}{2}(1 - z)^2 \log(1 - z) + \mathcal{O}\big((1 - z)^2\big). \tag{4}$$

Hence,

$$\mathrm{Li}_2^m(z) = \zeta(2)^m \exp\Big(m\beta(2)(1-z)\log(1-z) - m\beta(2)(1-z) \\ + \mathcal{O}\big(m(1-z)^2\log(1-z)\big)\Big).$$

(We recall that $\beta(2) := 1/\zeta(2)$.) We substitute this in (2). Writing $\widetilde{\mathcal{H}}(m)$ for $\mathcal{H}^-(m) \cup \mathcal{H}^\circ(m) \cup \mathcal{H}^+(m)$, we obtain that the m-th Taylor coefficient of $\ell_2(z)$ is given by

$$\frac{\beta(2)^m}{2i\pi m}\int_{\widetilde{\mathcal{H}}(m)} \exp\Big(-m\beta(2)(1-z)\log(1-z) + m(\beta(2)(1-z) \\ + \mathcal{O}\big(m(1-z)^2\log(1-z)\big)\Big)\,dz + o\big(\beta(2)^m m^{-3}\big)$$

As a next step we perform the substitution $z = 1 + \frac{t}{f(m)}$. Taking into account (3), we obtain

$$\frac{\beta(2)^m}{2i\pi m f(m)}\int_{\widehat{\mathcal{H}}(m)} \exp\Big(-t - \tfrac{1}{\log(f(m))+1}(-t)\log(-t) + \mathcal{O}\big(m^{-1}\big)\Big)\,dt + o\big(\beta(2)^m m^{-3}\big),$$

where $\widehat{\mathcal{H}}(m)$ is a Hankel contour around 0 restricted to $\Re(z) \le \log^2(m)$ (and the latter condition is the only dependence on m). By expanding the exponential, this becomes

$$\begin{aligned}
&\frac{\beta(2)^m}{2i\pi m f(m)}\int_{\widehat{\mathcal{H}}(m)} e^{-t}\Big(1 - \tfrac{1}{\log(f(m))+1}(-t)\log(-t) + \mathcal{O}\left(\tfrac{(-t)^2\log^2(-t)}{(\log(f(m))+1)^2}\right)\Big)\,dt + o\big(\beta(2)^m m^{-3}\big) \\
&\quad = \frac{\beta(2)^m}{2i\pi m f(m)}\int_{\widehat{\mathcal{H}}(m)} e^{-t}\,dt - \frac{\beta(2)^m}{2i\pi m f(m)}\int_{\widehat{\mathcal{H}}(m)} e^{-t}\tfrac{1}{\log(f(m))+1}(-t)\log(-t)\,dt \\
&\qquad + \frac{\beta(2)^m}{2i\pi m^2\log^3(m)}\int_{\widehat{\mathcal{H}}(m)} \mathcal{O}\big(e^{-t}(-t)^2\log^2(-t)\big)\,dt + o\big(\beta(2)^m m^{-3}\big), \qquad (5)
\end{aligned}$$

In the last expression, the integral

$$\int_{\widehat{\mathcal{H}}(m)} \mathcal{O}\big(e^{-t}(-t)^2\log^2(-t)\big)\,dt$$

can be bounded as follows: we cut the Hankel contour into the part turning around the origin, say the part with $\Re(t) \le 1$, and the remaining two horizontal lines, for which we have $1 \le \Re(t) \le \log^2(m)$. The first part contributes a constant to the integral, while for the latter parts we have $\log(-t) = \mathcal{O}(t)$. Putting this together, we obtain

$$\begin{aligned}
\int_{\widehat{\mathcal{H}}(m)} \mathcal{O}\big(e^{-t}(-t)^2\log^2(-t)\big)\,dt &= \mathcal{O}(1) + \int_{\widehat{\mathcal{H}}(m)} \mathcal{O}\big(e^{-t}t^4\big)\,dt \\
&= \mathcal{O}(1) + \int_0^\infty \mathcal{O}\big(e^{-t}t^4\big)\,dt \\
&= \mathcal{O}(1).
\end{aligned}$$

If we use this in (5), then we arrive at

$$\frac{\beta(2)^m}{2i\pi m f(m)}\int_{\widehat{\mathcal{H}}(m)} e^{-t}\,dt - \frac{\beta(2)^m}{2i\pi m f(m)}\int_{\widehat{\mathcal{H}}(m)} e^{-t}\frac{1}{\log(f(m))+1}(-t)\log(-t)\,dt + o\big(\beta(2)^m m^{-2}\log^{-3}(m)\big),$$

Now we may extend $\widehat{\mathcal{H}}(m)$ to a full Hankel contour around 0 at the cost of making a negligible error. Thus, we obtain

$$\begin{aligned}\frac{\beta(2)^m}{2i\pi m f(m)}\int_{\text{Hankel}} e^{-t}\,dt - \frac{\beta(2)^m}{2i\pi m f(m)}\int_{\text{Hankel}} e^{-t}\frac{1}{\log(f(m))+1}(-t)\log(-t)\,dt\\ + o\big(\beta(2)^m m^{-2}\log^{-3}(m)\big)\\ = 0 - \frac{\beta(2)^m}{m f(m)(\log(f(m))+1)} + o\big(\beta(2)^m m^{-2}\log^{-3}(m)\big)\\ = -\frac{\beta(2)^m}{m^2\log^2(m)}\left(1+o\left(\frac{\log(\log(m))}{\log(m)}\right)\right),\end{aligned}$$

where we used (1), and again (3).

The case $s\ge 3$. By Lagrange's inversion formula again, we can express the m-th Taylor coefficient of $\ell_s(z)$ in the form

$$\frac{1}{2i\pi m}\int_{\mathcal{C}}\frac{dz}{\operatorname{Li}_s(z)^m}, \tag{6}$$

where $\mathcal{C}$ is a sufficiently small contour that encircles the origin once in positive direction. We now deform $\mathcal{C}$ to the contour $\mathcal{H}(m)$ which consists of four parts. To be precise,

$$\mathcal{H}(m) = \mathcal{H}^\circ(m)\cup\mathcal{H}^+(m)\cup\mathcal{H}^-(m)\cup\mathcal{H}^\infty(m),$$

where

$$\begin{aligned}\mathcal{H}^\circ(m) &= \{z = 1 - \tfrac{e^{i\theta}}{m} : \theta\in[-\tfrac{\pi}{2},\tfrac{\pi}{2}]\},\\ \mathcal{H}^+(m) &= \{z = 1 + \tfrac{i}{m} + \tfrac{w}{m} : 0\le w\le \log^2(m)\},\\ \mathcal{H}^-(m) &= \{z = 1 - \tfrac{i}{m} + \tfrac{w}{m} : 0\le w\le \log^2(m)\},\end{aligned}$$

and where $\mathcal{H}^\infty(m)$ connects the end points of $\mathcal{H}^+(m)$ and $\mathcal{H}^-(m)$ (the points corresponding to $w=\log^2(m)$) by a path that stays in the cut plane $\mathbb{C}\backslash[1,+\infty)$ in such a way that, along this path, we have always $|\operatorname{Li}_s(z)|\ge|\operatorname{Li}_s(1)|+\varepsilon=\zeta(s)+\varepsilon$, for some $\varepsilon>0$. Such a path exists.

The contribution of $\mathcal{H}^\infty(m)$ is of the order of magnitude

$$\mathcal{O}((\zeta(s)+\varepsilon)^{-m}) = o((1/\zeta(s))^m m^{-s}),$$

and this will turn out to be negligible in comparison to the contributions of the other parts of the contour.

The singular expansion of $\mathrm{Li}_s(z)$ at $z = 1$ is given by

$$\mathrm{Li}_s(z) = \zeta(s) - \zeta(s-1)(1-z) + \cdots + \frac{(-1)^s}{(s-1)!}(1-z)^{s-1}\log(1-z) \\ + \mathcal{O}\big((1-z)^{s-1}\big). \quad (7)$$

Notice the presence in (7) of the term $\zeta(s-1)(1-z)$ which in (4) (for $s = 2$) becomes $(1-z)\log(1-z) - (1-z)$. This explain the different Hankel contours of integration for $s = 2$ and $s \geq 3$.

Hence,

$$\mathrm{Li}_s(z)^m = \zeta(s)^m \exp\Big(- m\tfrac{\zeta(s-1)}{\zeta(s)}(1-z) + \cdots + m\tfrac{(-1)^s}{\zeta(s)\,(s-1)!}(1-z)^{s-1}\log(1-z) \\ + \mathcal{O}\big(m(1-z)^{s-1}\big)\Big).$$

We substitute this in (6). Writing $\widetilde{\mathcal{H}}(m)$ for $\mathcal{H}^-(m) \cup \mathcal{H}^\circ(m) \cup \mathcal{H}^+(m)$, we obtain that the m-th Taylor coefficient of $\ell_s(z)$ is given by

$$\frac{\beta(s)^m}{2i\pi m}\int_{\widetilde{\mathcal{H}}(m)} \exp\Big(m\tfrac{\zeta(s-1)}{\zeta(s)}(1-z) + \cdots - m\tfrac{(-1)^s}{\zeta(s)\,(s-1)!}(1-z)^{s-1}\log(1-z) \\ + \mathcal{O}\big(m(1-z)^s\big)\Big)\,dz + o\big(\beta(s)^m m^{-s-1}\big).$$

(We recall that $\beta(s) := 1/\zeta(s)$.)

As a next step we perform the substitution $z = 1 + \frac{\zeta(s)}{\zeta(s-1)}\frac{t}{m}$. In this manner, we obtain

$$\frac{\zeta(s)^{1-m}}{2i\pi\zeta(s-1)m^2}\int_{\widehat{\mathcal{H}}(m)} \exp\Big(- t + \cdots - \tfrac{1}{m^{s-2}}\tfrac{(-1)^s}{(s-1)!}\tfrac{\zeta(s)^{s-2}}{\zeta(s-1)^{s-1}}(-t)^{s-1}\log(-t) \\ + \mathcal{O}\big(m^{1-s}\big)\Big)\,dt + o\big(\beta(s)^m m^{-s-1}\big),$$

where $\widehat{\mathcal{H}}(m)$ is a Hankel contour around 0 restricted to $\Re(z) \leq \mathcal{O}\big(\log^2(m)\big)$ (and the latter condition is the only dependence on m). By expanding the exponential, this becomes

$$\frac{\zeta(s)^{1-m}}{2i\pi m^2}\int_{\widehat{\mathcal{H}}(m)} e^{-t}\Big(1 + \tfrac{c_2}{m}(-t)^2 + \cdots - \tfrac{1}{m^{s-2}}\tfrac{(-1)^s}{(s-1)!}\tfrac{\zeta(s)^{s-2}}{\zeta(s-1)^{s-1}}(-t)^{s-1}\log(-t) \\ + \mathcal{O}\big(m^{1-s}\big)\Big)\,dt + o\big(\beta(s)^m m^{-s-1}\big) \\ = \frac{\zeta(s)^{1-m}}{2i\pi m^2}\int_{\widehat{\mathcal{H}}(m)} e^{-t}\Big(1 + \tfrac{c_2}{m}(-t)^2 + \cdots - \tfrac{1}{m^{s-2}}\tfrac{(-1)^s}{(s-1)!}\tfrac{\zeta(s)^{s-2}}{\zeta(s-1)^{s-1}}(-t)^{s-1}\log(-t)\Big)\,dt \\ + \mathcal{O}\big(\beta(s)^m m^{-s-1}\log^2(m)\big),$$

since the length of the truncated Hankel contour $\widehat{\mathcal{H}}(m)$ is of order of magnitude $\mathcal{O}(\log^2(m))$. Now we may extend $\widehat{\mathcal{H}}(m)$ to a full Hankel contour around 0 at the cost of making a negligible error. Thus, we obtain

$$\frac{\zeta(s)^{1-m}}{2i\pi m^2}\int_{\text{Hankel}} e^{-t}\Big(1+\tfrac{c_2}{m}(-t)^2+\cdots-\tfrac{1}{m^{s-2}}\tfrac{(-1)^s}{(s-1)!}\tfrac{\zeta(s)^{s-2}}{\zeta(s-1)^{s-1}}(-t)^{s-1}\log(-t)\Big)\,dt$$
$$+\,\mathcal{O}\big((1/\zeta(s))^m m^{-2}\log^{-s-1}(m)\log^2(m)\big)$$
$$=\frac{\zeta(s)^{1-m}}{m^2}\Big(0+0+\cdots-\tfrac{1}{m^{s-2}}\tfrac{\zeta(s)^{s-2}}{\zeta(s-1)^{s-1}}\Big)+\mathcal{O}\big(\beta(s)^m m^{-s-1}\log^2(m)\big)$$
$$=-\frac{\zeta(s)^{s-1}}{\zeta(s-1)^{s-1}}\frac{\beta(s)^m}{m^s}\left(1+\mathcal{O}\left(\frac{\log^2(m)}{m}\right)\right),$$

where we used again (1). This completes the proof of Theorem 1.

Bibliography

[1] P. Flajolet and R. Sedgewick, *Analytic combinatorics*, Cambridge University Press, Cambridge, 2009.