

# Irrationality exponent and rational approximations with prescribed growth

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## 1 Introduction

In 1978, Apéry [2] proved the irrationality of  $\zeta(3)$  by constructing two explicit sequences of integers  $(u_n)_n$  and  $(v_n)_n$  such that  $0 \neq u_n \zeta(3) - v_n \rightarrow 0$  and  $u_n \rightarrow +\infty$ , both at geometric rates. He also deduced from this an upper bound for the irrationality exponent  $\mu(\zeta(3))$  of  $\zeta(3)$ . In general, the irrationality exponent  $\mu(\xi)$  of an irrational number  $\xi$  is defined as the infimum of all real numbers  $\mu$  such that the inequality

$$\left| \xi - \frac{p}{q} \right| > \frac{1}{q^\mu}$$

holds for all integers  $p, q$ , with  $q$  sufficiently large. It is well-known that  $\mu(\xi) \geq 2$  for any irrational number  $\xi$  and that it equals 2 for almost all irrational numbers. After Apéry, the following lemma has often been used to bound  $\mu(\xi)$  from above, for example for the numbers  $\log(2)$  and  $\zeta(2)$ . (Other lemmas can be used to bound the irrationality exponent of numbers of a different nature, like  $\exp(1)$ .)

**Lemma 1.** *Let  $\xi \in \mathbb{R} \setminus \mathbb{Q}$ , and  $\alpha, \beta$  be real numbers such that  $0 < \alpha < 1$  and  $\beta > 1$ . Assume there exist integer sequences  $(u_n)_{n \geq 1}$  and  $(v_n)_{n \geq 1}$  such that*

$$\lim_{n \rightarrow +\infty} |u_n \xi - v_n|^{1/n} = \alpha \text{ and } \limsup_{n \rightarrow +\infty} |u_n|^{1/n} \leq \beta. \quad (1.1)$$

*Then we have  $\mu(\xi) \leq 1 - \frac{\log \beta}{\log \alpha}$ .*

The proof of Lemma 1 is not difficult. Many variants of this result exist; a slightly more general version of Lemma 1 will be proved in §4.1. Another variant, proved in [6] (Proposition 3.1), asserts that Lemma 1 holds when (1.1) is replaced with

$$\limsup_{n \rightarrow +\infty} \frac{|u_{n+1} \xi - v_{n+1}|}{|u_n \xi - v_n|} \leq \alpha \text{ and } \limsup_{n \rightarrow +\infty} \frac{u_{n+1}}{u_n} \leq \beta.$$

In this text, we prove that Lemma 1 and these variants are best possible, by obtaining a very precise converse result:

**Theorem 1.** Let  $\xi \in \mathbb{R} \setminus \mathbb{Q}$ , and  $\alpha, \beta$  be real numbers such that  $0 < \alpha < 1$ ,  $\beta > 1$  and  $\mu(\xi) < 1 - \frac{\log \beta}{\log \alpha}$ . Then there exist integer sequences  $(u_n)_{n \geq 1}$  and  $(v_n)_{n \geq 1}$  such that

$$\lim_{n \rightarrow +\infty} \frac{u_n \xi - v_n}{\alpha^n} = \lim_{n \rightarrow +\infty} \frac{u_n}{\beta^n} = 1$$

and, consequently,

$$\lim_{n \rightarrow +\infty} \frac{|u_{n+1} \xi - v_{n+1}|}{|u_n \xi - v_n|} = \lim_{n \rightarrow +\infty} |u_n \xi - v_n|^{1/n} = \alpha \text{ and } \lim_{n \rightarrow +\infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow +\infty} |u_n|^{1/n} = \beta.$$

Theorem 1 answers completely all questions asked in [6], where the *density exponent* is defined (see §3 below).

An essential feature of Lemma 1, and all its variants, is that the sequences  $(u_n)$  and  $(u_n \xi - v_n)$  are assumed to have essentially geometrical behaviour. An assumption of this kind is necessary, since the convergents of the continued fraction expansion of  $\xi$  (for instance) always make up a sequence of approximants to  $\xi$  that are far more precise, but they don't have (in general) geometrical behaviour.

However, a geometrical behaviour is not necessary in Theorem 1, as the following generalization shows.

**Theorem 2.** Let  $\xi \in \mathbb{R} \setminus \mathbb{Q}$ , and let  $(Q_n)$  and  $(\varepsilon_n)$  be sequences of positive real numbers with

$$\lim_{n \rightarrow +\infty} Q_n = +\infty, \quad \lim_{n \rightarrow +\infty} \varepsilon_n = 0 \quad \text{and} \quad \varepsilon_n \geq Q_n^{-\frac{1}{\mu-1} + o(1)},$$

where  $\mu$  is a real number such that  $\mu > \mu(\xi)$ .

Then there exist integer sequences  $(u_n)$  and  $(v_n)$  such that

$$\lim_{n \rightarrow +\infty} \frac{u_n}{Q_n} = \lim_{n \rightarrow +\infty} \frac{u_n \xi - v_n}{\varepsilon_n} = 1.$$

The important point in this theorem is that our only assumption is that  $\varepsilon_n$  is not too small, namely

$$\limsup_{n \rightarrow +\infty} \frac{-\log \varepsilon_n}{\log Q_n} < \frac{1}{\mu(\xi) - 1}.$$

Theorem 2 answers the questions asked in §8 of [6].

The structure of this text is as follows. In Section 2, we prove Theorem 2 (and therefore, as a special case, Theorem 1). Then we recall (in §3) the definition [6] of the *density exponent*, and deduce from Theorem 1 that it is always 0 or  $\infty$ . In §4 we prove a slight generalization of Lemma 1 which enables us to prove a general statement (containing Lemma 1, Theorem 1 and Theorem 2) consisting in the equality of several exponents of Diophantine approximation.

Finally we partially generalize (in §4.3) this statement to the multivariate setting, where we consider simultaneously several real numbers  $\xi_0, \dots, \xi_r$  instead of just one  $\xi$ . We then explain the connection between Lemma 1 and Nesterenko's linear independence criterion [10], used in particular in the proof ([3], [11]) that  $\zeta(s)$  is irrational for infinitely many odd integers  $s \geq 3$ . This enables us to write down in a simple and elegant way the new proof of Nesterenko's criterion obtained in [7].

## 2 Proof of Theorem 2

The proof of Theorem 2 is based on the following lemma, which is proved inside the proof of Lemma 7.3 of [6] (p. 39) and is the main step in the proof [6] that almost all  $\xi$  (with respect to Lebesgue measure) have density exponent zero.

**Lemma 2.** *Let  $c, c', \varepsilon, Q$  be real numbers such that  $1 < c < c' < 2$ ,  $0 < \varepsilon < 1$ , and  $Q > 1$ .*

*Let  $\xi$  be an irrational number with  $0 < \xi < 1$ . Then (at least) one of the following assertions hold:*

(i) *There exist coprime integers  $u \geq 1$  and  $v \in \{0, \dots, u\}$  such that*

$$u < \frac{2c^2}{(c-1)(c'-c)} \frac{1}{\varepsilon}$$

*and*

$$\left| \xi - \frac{v}{u} \right| \leq \frac{2}{c-1} \left( 1 + \frac{c^2}{c'-c} \right) \frac{1}{uQ}.$$

(ii) *There exist integers  $p$  and  $q$  such that*

$$Q \leq q \leq cQ \text{ and } \frac{\varepsilon}{q} \leq \xi - \frac{p}{q} \leq \frac{c'\varepsilon}{q}.$$

This lemma is interesting when  $\varepsilon$  is much bigger than  $1/Q$ . It means that, unless  $\xi$  is very close to a real number with denominator essentially bounded by  $1/\varepsilon$ , it is possible to find a fraction  $p/q$  (which may not be in its lowest terms) such that  $q$  has essentially the size of  $Q$ , and  $q\xi - p$  that of  $\varepsilon$ . The interesting part, in proving Theorem 2, is that we obtain  $Q \leq q \leq cQ$  and  $\varepsilon \leq q\xi - p \leq c'\varepsilon$  where  $c$  and  $c'$  are constants that can be chosen arbitrarily close to 1. A variant of this lemma, in which one obtains only  $Q \leq q \leq 2Q$  and  $\varepsilon \leq q\xi - p \leq 3\varepsilon$ , is proved in [5] (Lemma 5). The proof uses the same ideas as the one of Lemma 2, but is fairly less complicated.

The proof [6] of Lemma 2 makes use of Farey fractions. We did not try to prove this lemma using continued fractions.

Let us deduce Theorem 2 now.

*Proof.* We may assume  $0 < \xi < 1$ . Let  $(\eta_n)$  be a sequence of positive real numbers such that  $\lim_{n \rightarrow +\infty} \eta_n = 0$  and  $\eta_n = \varepsilon_n^{o(1)}$ . We let  $\lambda_n = 1 + \eta_n$ ,  $\mu_n = 1 + 2\eta_n$ ,  $Q'_n = \frac{Q_n}{\sqrt{\lambda_n}}$  and  $\varepsilon'_n = \frac{\varepsilon_n}{\sqrt{\mu_n}}$ . For  $n$  sufficiently large, Lemma 2 applies with  $c = \lambda_n$ ,  $c' = \mu_n$ ,  $\varepsilon = \varepsilon'_n$ , and  $Q = Q'_n$ . If (i) holds in this lemma and  $n$  is sufficiently large, then we obtain integers  $u_n$  and  $v_n$  such that

$$\left| \xi - \frac{v_n}{u_n} \right| \leq \frac{2}{\eta_n} \left( 1 + \frac{\lambda_n^2}{\eta_n} \right) \frac{1}{u_n Q'_n} \leq \frac{20}{\eta_n^2 u_n Q_n}$$

and

$$u_n < \frac{2\lambda_n^2}{\eta_n^2} \frac{1}{\varepsilon'_n} \leq \frac{16}{\eta_n^2 \varepsilon_n}.$$

Since we have  $\eta_n = \varepsilon_n^{o(1)}$  and  $\varepsilon_n \geq Q_n^{-\frac{1}{\mu-1} + o(1)}$ , these inequalities yield

$$\left| \xi - \frac{v_n}{u_n} \right| \leq \frac{20}{\eta_n^2 u_n Q_n} \leq \frac{1}{u_n} \left( \frac{\varepsilon_n \eta_n^2}{16} \right)^{\mu-1+o(1)} \leq \frac{1}{u_n^{\mu+o(1)}}$$

which is possible only for finitely many values of  $n$  since  $\mu > \mu(\xi)$ . Therefore, as soon as  $n$  is sufficiently large, Assertion (ii) of Lemma 2 holds and provides integers  $p_n$  and  $q_n$  such that

$$\frac{Q_n}{\sqrt{\lambda_n}} \leq q_n \leq Q_n \sqrt{\lambda_n} \text{ and } \frac{\varepsilon_n}{\sqrt{\mu_n}} \leq q_n \xi - p_n \leq \varepsilon_n \sqrt{\mu_n}.$$

This concludes the proof of Theorem 2.  $\square$

### 3 Consequences for the density exponent

Let  $\xi \in \mathbb{R} \setminus \mathbb{Q}$ . For any non-decreasing sequence  $\mathbf{u} = (u_n)_n$  of positive integers, let us define

$$\alpha_\xi(\mathbf{u}) := \limsup_n \frac{|u_{n+1}\xi - v_{n+1}|}{|u_n\xi - v_n|}, \quad \beta(\mathbf{u}) := \limsup_n \frac{u_{n+1}}{u_n},$$

where  $v_n$  is the nearest integer to  $u_n\xi$ . We defined in [6] the density exponent  $\nu(\xi)$  of  $\xi$  as the infimum of the quantity  $\log \sqrt{\alpha_\xi(\mathbf{u})\beta(\mathbf{u})}$  when  $\mathbf{u}$  ranges through the non-decreasing sequences such that  $\alpha_\xi(\mathbf{u}) < 1$  and  $\beta(\mathbf{u}) < +\infty$  (with the convention  $\nu(\xi) = +\infty$  if there is no such  $\mathbf{u}$ ).

We proved in [6] that  $\nu(\xi) = +\infty$  when  $\xi$  is a Liouville number, i.e., when  $\mu(\xi) = +\infty$  (that is, when for any  $\mu > 0$ , there exists a rational number  $p/q$  such that  $|\xi - p/q| < 1/q^\mu$ ). Theorem 2 implies the converse statement, in a more precise form:

**Theorem 3.** *If  $\xi \in \mathbb{R} \setminus \mathbb{Q}$  is not a Liouville number, then  $\nu(\xi) = 0$ .*

Indeed, we may choose in Theorem 1 values of  $\alpha$  and  $\beta$  arbitrarily close to 1, so that the product  $\alpha\beta$  is also arbitrarily close to 1. In a sense, this annihilates the interest of  $\nu(\xi)$ , since it takes only two values (0 and  $+\infty$ ) and distinguishes only Liouville numbers

from the other irrational numbers. However, the ideas of [6] are at the base of the results presented in the present paper.

Let us precise here what we expected in [6]. In fact, we hoped to define a quantity that would enable us to distinguish between periods (in the sense of [8]) and other numbers. In particular, we computed upper bounds for  $\nu(\xi)$ , for many examples of  $\xi$  which are periods (see also [1]). But we did not really take into account another property of the approximations used for this: they all satisfy a linear recursion of finite order with polynomial coefficients of a special kind. Indeed, in all the examples of [6], the sequences  $(u_n)_n$  as well as  $(v_n)_n$  are such that the power series  $\sum_{n \geq 0} u_n z^n$  and  $\sum_{n \geq 0} v_n z^n$  are  $G$ -functions<sup>1</sup> satisfying the same minimal differential equation. This is a very strong property that is not satisfied (in general) by the sequences  $(u_n)_n$  and  $(v_n)_n$  constructed (by means of Lemma 2) to prove Theorem 1.

## 4 Exponents of Diophantine Approximation

In this section, we state the results of this paper in terms of exponents of Diophantine approximation. This enables us to explain the connection with Nesterenko's linear independence criterion [10] (and especially its new proof given in [7]), and ask some questions about multivariate generalizations of our results.

### 4.1 A generalization of Lemma 1

We start with a generalization of the usual Lemma 1. We do not write down the proof of this Proposition because it is a special case of the upper bound  $\tau_r(\xi) \leq \frac{1}{\omega_0(\xi)}$  proved in Theorem 5 below (see §4.3).

**Proposition 1.** *Let  $\xi \in \mathbb{R} \setminus \mathbb{Q}$  and  $\tau > 0$ . Assume there exist integer sequences  $(u_n)$  and  $(v_n)$  with  $u_n \neq 0$  for any  $n$ , and*

$$u_n \xi - v_n \rightarrow 0, \quad |u_{n+1} \xi - v_{n+1}| = |u_n \xi - v_n|^{1+o(1)}, \quad \text{and} \quad |u_n \xi - v_n| \leq |u_n|^{-\tau+o(1)}.$$

*Then we have  $\mu(\xi) \leq 1 + \frac{1}{\tau}$ .*

### 4.2 The univariate case

Let  $\xi$  be an irrational real number. Let us consider the following sets (in the definition of which we denote by  $o(1)$  any sequence that tends to 0 as  $n$  tends to infinity):

- $\mathcal{T}(\xi)$  is the set of all  $\tau > 0$  for which there exist integer sequences  $(u_n)$  and  $(v_n)$  with  $u_n \neq 0$  for any  $n$ , and

$$u_n \xi - v_n \rightarrow 0, \quad |u_{n+1} \xi - v_{n+1}| = |u_n \xi - v_n|^{1+o(1)}, \quad \text{and} \quad |u_n \xi - v_n| \leq |u_n|^{-\tau+o(1)}.$$

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<sup>1</sup>A power series  $\sum_{n \geq 0} a_n z^n \in \mathbb{Q}[[z]]$  is a  $G$ -function when: 1) it satisfies a linear differential equation, 2) it has a finite positive radius of convergence, 3) the least common multiple of the denominators of  $a_0, a_1, \dots, a_n$  is bounded by  $C^n$  for some  $C > 0$ .

- $\mathcal{T}'(\xi)$  is the set of all  $\tau > 0$  for which there exist integer sequences  $(u_n)$  and  $(v_n)$ , and  $0 < \alpha < 1 < \beta$ , with

$$|u_n\xi - v_n|^{1/n} \rightarrow \alpha, \quad \limsup_{n \rightarrow +\infty} |u_n|^{1/n} \leq \beta, \quad \text{and} \quad \tau = \frac{-\log \alpha}{\log \beta}.$$

- $\mathcal{T}''(\xi)$  is the set of all  $\tau > 0$  such that, for any increasing sequence  $(Q_n)$  of positive integers there exist integer sequences  $(u_n)$  and  $(v_n)$  with

$$|u_n| \leq Q_n^{1+o(1)} \quad \text{and} \quad |u_n\xi - v_n| = Q_n^{-\tau+o(1)}.$$

- $\mathcal{T}'''(\xi)$  is the set of all  $\tau > 0$  such that, for any sequences  $(Q_n)$  and  $(\varepsilon_n)$  of positive real numbers with

$$\lim_{n \rightarrow +\infty} Q_n = +\infty, \quad \lim_{n \rightarrow +\infty} \varepsilon_n = 0 \quad \text{and} \quad \varepsilon_n \geq Q_n^{-\tau+o(1)},$$

there exist integer sequences  $(u_n)$  and  $(v_n)$  with

$$\lim_{n \rightarrow +\infty} \frac{u_n}{Q_n} = \lim_{n \rightarrow +\infty} \frac{u_n\xi - v_n}{\varepsilon_n} = 1.$$

Theorem 4 below shows that  $\tau \leq 1$  for any  $\tau$  in  $\mathcal{T}(\xi)$  (resp.  $\mathcal{T}'(\xi)$ ,  $\mathcal{T}''(\xi)$ ,  $\mathcal{T}'''(\xi)$ ).

We let

$$\tau(\xi) = \sup \mathcal{T}(\xi),$$

and in the same way  $\tau'(\xi) = \sup \mathcal{T}'(\xi)$ ,  $\tau''(\xi) = \sup \mathcal{T}''(\xi)$ ,  $\tau'''(\xi) = \sup \mathcal{T}'''(\xi)$ , with the convention  $\sup \emptyset = 0$ , so that each of  $\tau(\xi)$ ,  $\tau'(\xi)$ ,  $\tau''(\xi)$ ,  $\tau'''(\xi)$  belongs to  $[0, 1]$ .

If we have  $0 < \tau < \tau'$  and  $\tau' \in \mathcal{T}(\xi)$ , then  $\tau \in \mathcal{T}(\xi)$  so that  $\mathcal{T}(\xi)$  is  $\emptyset$ ,  $(0, \tau(\xi)]$  or  $(0, \tau(\xi))$ . The same holds for  $\mathcal{T}'(\xi)$ ,  $\mathcal{T}''(\xi)$ ,  $\mathcal{T}'''(\xi)$ .

Moreover the inclusions  $\mathcal{T}'''(\xi) \subset \mathcal{T}''(\xi) \subset \mathcal{T}'(\xi) \subset \mathcal{T}(\xi)$  hold trivially, so that we have

$$\tau'''(\xi) \leq \tau''(\xi) \leq \tau'(\xi) \leq \tau(\xi). \quad (4.1)$$

The main result of this section is the following equality, which summarizes Lemma 1, Theorem 1 and Theorem 2.

**Theorem 4.** *For any  $\xi \in \mathbb{R} \setminus \mathbb{Q}$  we have*

$$\tau'''(\xi) = \tau''(\xi) = \tau'(\xi) = \tau(\xi) = \frac{1}{\mu(\xi) - 1} \in [0, 1].$$

*In particular the following assertions are equivalent:  $\tau'''(\xi) = 0$ ;  $\tau''(\xi) = 0$ ;  $\tau'(\xi) = 0$ ;  $\tau(\xi) = 0$ ;  $\xi$  is a Liouville number.*

As a corollary, we have  $\tau'''(\xi) = \tau''(\xi) = \tau'(\xi) = \tau(\xi) = 1$  for almost all  $\xi$  with respect to Lebesgue measure.

**Proof:** Since  $2 \leq \mu(\xi) \leq +\infty$  for any  $\xi \in \mathbb{R} \setminus \mathbb{Q}$ , we have  $\frac{1}{\mu(\xi)-1} \in [0, 1]$ .

For any  $\tau \in \mathcal{T}(\xi)$ , Proposition 1 yields  $\mu(\xi) \leq 1 + \frac{1}{\tau}$ , that is  $\tau \leq \frac{1}{\mu(\xi)-1}$ . If  $\mathcal{T}(\xi) \neq \emptyset$  this gives  $\mu(\xi) < \infty$  and  $\tau(\xi) \leq \frac{1}{\mu(\xi)-1}$ ; this upper bound holds trivially if  $\mathcal{T}(\xi) = \emptyset$ .

Thanks to Eq. (4.1), we just have to prove that  $\frac{1}{\mu(\xi)-1} \leq \tau'''(\xi)$  in order to conclude the proof of Theorem 4. This is trivial if  $\mu(\xi) = +\infty$ . Otherwise, for any  $\mu > \mu(\xi)$ , Theorem 2 gives  $\frac{1}{\mu-1} \in \mathcal{T}'''(\xi)$  so that  $\frac{1}{\mu(\xi)-1} \leq \tau'''(\xi)$ . This concludes the proof of Theorem 4.

### 4.3 The multivariate case

Let  $\xi_0, \dots, \xi_r$  be real numbers, with  $r \geq 1$ . Throughout this section we assume

$$\dim_{\mathbb{Q}} \text{Span}_{\mathbb{Q}}(\xi_0, \dots, \xi_r) \geq 2$$

so that non-vanishing linear forms in  $\xi_0, \dots, \xi_r$  with integer coefficients can be arbitrarily small. We consider linear forms  $L = \ell_0 X_0 + \dots + \ell_r X_r$  with integer coefficients  $\ell_i$ , and we let  $H(L) = \max_{0 \leq i \leq r} |\ell_i|$  and  $L(\xi) = \ell_0 \xi_0 + \dots + \ell_r \xi_r$ , where  $\xi$  stands for the point  $(\xi_0, \dots, \xi_r)$  in  $\mathbb{R}^{r+1}$ .

Let us define the following sets:

- $\mathcal{T}_r(\xi)$  is the set of all  $\tau > 0$  for which there exists a sequence  $(L_n)$  of linear forms with  $L_n(\xi) \neq 0$  for any  $n$ , and

$$L_n(\xi) \rightarrow 0, \quad |L_{n+1}(\xi)| = |L_n(\xi)|^{1+o(1)}, \quad \text{and} \quad |L_n(\xi)| \leq H(L_n)^{-\tau+o(1)}.$$

- $\mathcal{T}'_r(\xi)$  is the set of all  $\tau > 0$  for which there exists a sequence  $(L_n)$  of linear forms, and  $0 < \alpha < 1 < \beta$ , with

$$|L_n(\xi)|^{1/n} \rightarrow \alpha, \quad \limsup_{n \rightarrow +\infty} H(L_n)^{1/n} \leq \beta, \quad \text{and} \quad \tau = \frac{-\log \alpha}{\log \beta}.$$

- $\mathcal{T}''_r(\xi)$  is the set of all  $\tau > 0$  such that, for any increasing sequence  $(Q_n)$  of positive integers, there exists a sequence  $(L_n)$  of linear forms with

$$H(L_n) \leq Q_n^{1+o(1)} \quad \text{and} \quad |L_n(\xi)| = Q_n^{-\tau+o(1)}.$$

- $\mathcal{T}'''_r(\xi)$  is the set of all  $\tau > 0$  such that, for any sequences  $(Q_n)$  and  $(\varepsilon_n)$  of positive real numbers with

$$\lim_{n \rightarrow +\infty} Q_n = +\infty, \quad \lim_{n \rightarrow +\infty} \varepsilon_n = 0 \quad \text{and} \quad \varepsilon_n \geq Q_n^{-\tau+o(1)},$$

there exists a sequence  $(L_n)$  of linear forms with

$$\lim_{n \rightarrow +\infty} \frac{H(L_n)}{Q_n} = \lim_{n \rightarrow +\infty} \frac{L_n(\xi)}{\varepsilon_n} = 1.$$

Theorem 5 below shows that  $\tau \leq s$  for any  $\tau$  in  $\mathcal{T}_r(\xi)$ , with  $s = \dim_{\mathbb{Q}} \text{Span}_{\mathbb{Q}}(\xi_0, \dots, \xi_r) - 1$  (and the same holds for  $\mathcal{T}'_r(\xi)$ ,  $\mathcal{T}''_r(\xi)$  and  $\mathcal{T}'''_r(\xi)$ ).

We let

$$\tau_r(\xi) = \sup \mathcal{T}_r(\xi),$$

and in the same way  $\tau'_r(\xi) = \sup \mathcal{T}'_r(\xi)$ ,  $\tau''_r(\xi) = \sup \mathcal{T}''_r(\xi)$ ,  $\tau'''_r(\xi) = \sup \mathcal{T}'''_r(\xi)$ , with the convention  $\sup \emptyset = 0$ , so that each of  $\tau_r(\xi)$ ,  $\tau'_r(\xi)$ ,  $\tau''_r(\xi)$ ,  $\tau'''_r(\xi)$  belongs to  $[0, s]$ .

If we have  $0 < \tau < \tau'$  and  $\tau' \in \mathcal{T}_r(\xi)$ , then  $\tau \in \mathcal{T}_r(\xi)$  so that  $\mathcal{T}_r(\xi)$  is  $\emptyset$ ,  $(0, \tau_r(\xi)]$  or  $(0, \tau_r(\xi))$ . The same holds for  $\mathcal{T}'_r(\xi)$ ,  $\mathcal{T}''_r(\xi)$ ,  $\mathcal{T}'''_r(\xi)$ .

Moreover the inclusions  $\mathcal{T}'''_r(\xi) \subset \mathcal{T}''_r(\xi) \subset \mathcal{T}'_r(\xi) \subset \mathcal{T}_r(\xi)$  hold trivially, so that we have

$$\tau'''_r(\xi) \leq \tau''_r(\xi) \leq \tau'_r(\xi) \leq \tau_r(\xi). \quad (4.2)$$

Let  $\omega_0(\xi)$  be the supremum of the set of all  $\omega > 0$  such that there exists infinitely many tuples  $(q_0, \dots, q_r) \in \mathbb{Z}^{r+1}$  with

$$|q_i \xi_j - q_j \xi_i| \leq \max(|q_0|, \dots, |q_r|)^{-\omega} \text{ for any } 1 \leq i < j \leq r. \quad (4.3)$$

Up to renumbering  $\xi_0, \dots, \xi_r$ , we may assume  $\xi_0 \neq 0$  and in this case we can replace (4.3) with

$$\left| \frac{\xi_j}{\xi_0} - \frac{q_j}{q_0} \right| \leq |q_0|^{-\omega-1} \text{ for any } i \in \{1, \dots, r\} \quad (4.4)$$

so that  $\omega_0(\xi)$  measure the quality of simultaneous approximations to  $(\xi_1/\xi_0, \dots, \xi_r/\xi_0)$  by rational numbers with the same denominator.

When  $r = 1$  and  $\xi_0 \neq 0$ , we have  $\tau_1(\xi_0, \xi_1) = \tau(\xi_1/\xi_0)$  (and the analogous equalities for  $\tau'_1(\xi_0, \xi_1)$ ,  $\tau''_1(\xi_0, \xi_1)$  and  $\tau'''_1(\xi_0, \xi_1)$ ), and  $\omega_0(\xi_0, \xi_1) = \mu(\xi_1/\xi_0) - 1$ . This explains why the following result is a partial generalization of Theorem 4:

**Theorem 5.** *Let  $\xi_0, \dots, \xi_r \in \mathbb{R}$ , with  $r \geq 1$ . Then we have*

$$\tau'''_r(\xi) \leq \tau''_r(\xi) \leq \tau'_r(\xi) \leq \tau_r(\xi) \leq \frac{1}{\omega_0(\xi)} \leq s,$$

*under the assumption that  $s = \dim_{\mathbb{Q}} \text{Span}_{\mathbb{Q}}(\xi_0, \dots, \xi_r) - 1$  is positive.*

In the rest of this section, we prove this theorem and make some comments. The upper bounds  $\tau'''_r(\xi) \leq \tau''_r(\xi) \leq \tau'_r(\xi) \leq \tau_r(\xi)$  hold trivially, and have been observed before in Eq. (4.2).

Let us prove that  $\tau_r(\xi) \leq \frac{1}{\omega_0(\xi)}$ . As above, we may assume that  $\xi_0 = 1$ . If  $\tau_r(\xi) = 0$ , this result is trivial. Otherwise, let  $0 < \tau < \tau_r(\xi)$  and  $0 < \omega < \omega_0(\xi)$  (where  $\omega_0(\xi)$  could be  $+\infty$ ). Let  $(L_n)$  be a sequence of linear forms such that  $L_n(\xi) \neq 0$  for any  $n$ , and

$$L_n(\xi) \rightarrow 0, \quad |L_{n+1}(\xi)| = |L_n(\xi)|^{1+o(1)}, \quad \text{and} \quad |L_n(\xi)| \leq H(L_n)^{-\tau+o(1)}. \quad (4.5)$$



There exists integers  $q_0, \dots, q_r$ , with  $|q_0|$  arbitrarily large, such that (4.4) holds. Let  $n$  be the least positive integer such that  $|q_0 L_n(\xi)| \leq 1/2$ . Taking  $|q_0|$  sufficiently large ensure that  $n$  can be made arbitrarily large; we denote by  $o(1)$  any sequence that tends to 0 as  $n$  tends to infinity, so that  $o(1)$  can be made arbitrarily small.

By minimality of  $n$  we have  $1/2 < |q_0| |L_{n+1}(\xi)| = |q_0| |L_n(\xi)|^{1+o(1)}$  so that

$$|q_0| = |L_n(\xi)|^{-1+o(1)}. \quad (4.6)$$

Now we have (since  $\xi_0 = 1$ )

$$L_n(q_0, \dots, q_r) = q_0 L_n(\xi) + L_n(0, q_1 - q_0 \xi_1, \dots, q_r - q_0 \xi_r).$$

In the right handside, the first term has absolute value less than or equal to  $1/2$ , by choice of  $n$ . If the second term has absolute value less than the first one, then the integer  $L_n(q_0, \dots, q_r)$  has absolute value less than 1 so that it vanishes, and both terms in the right handside have the same absolute value, thereby contradicting the assumption.

So we have, using also (4.6), (4.5) and (4.4):

$$\begin{aligned} |L_n(\xi)|^{o(1)} = |q_0 L_n(\xi)| &\leq |L_n(0, q_1 - q_0 \xi_1, \dots, q_r - q_0 \xi_r)| \\ &\leq r H(L_n) \max_{1 \leq i \leq r} |q_i - q_0 \xi_i| \\ &\leq |L_n(\xi)|^{-\frac{1}{r}+o(1)} |q_0|^{-\omega} = |L_n(\xi)|^{\omega - \frac{1}{r}+o(1)}. \end{aligned}$$

Since  $\lim_{n \rightarrow +\infty} |L_n(\xi)| = 0$  and  $n$  can be chosen arbitrarily large, this implies  $\omega \leq \frac{1}{r}$ , thereby concluding the proof that  $\tau_r(\xi) \leq \frac{1}{\omega_0(\xi)}$ .

Let us prove that  $\frac{1}{\omega_0(\xi)} \leq s$ , that is  $\omega_0(\xi) \geq 1/s$ . Renumbering  $\xi_0, \dots, \xi_r$  if necessary, we may assume that  $\xi_0, \dots, \xi_s$  are linearly independent over the rationals; then  $\xi_{s+1}, \dots, \xi_r$  are linear combinations over  $\mathbb{Q}$  of these numbers, and it is easy to check that  $\omega_0(\xi_0, \dots, \xi_r) = \omega_0(\xi_0, \dots, \xi_s)$ . Now the lower bound  $\omega_0(\xi_0, \dots, \xi_s) \geq 1/s$  is a classical consequence of Dirichlet's pigeonhole principle, as follows. For any positive integer  $Q$ , consider the  $Q^s + 1$  points  $(\{q_0 \xi_1 / \xi_0\}, \dots, \{q_0 \xi_s / \xi_0\}) \in [0, 1]^s$ , for  $0 \leq q_0 \leq Q^s$  (here  $\{x\}$  denotes the fractional part of a real number  $x$ ), and the  $Q^s$  cubes  $\mathcal{C}_{i_1, \dots, i_s}$  defined, for  $0 \leq i_1, \dots, i_s < Q$ , by the inequalities  $\frac{i_1}{Q} \leq x_1 < \frac{i_1+1}{Q}, \dots, \frac{i_s}{Q} \leq x_s < \frac{i_s+1}{Q}$ . At least two of these points, given (say) by  $q'_0$  and  $q''_0$ , lie in the same cube. Letting  $q_0 = |q'_0 - q''_0|$  and denoting by  $q_j$  the nearest integer to  $q_0 \xi_j / \xi_0$  for  $j \in \{1, \dots, s\}$ , we obtain (4.4) with  $\omega = 1/s$ . Since a given tuple  $(q_0, \dots, q_s)$  is obtained in this way from only finitely many integers  $Q$  (because  $\xi_0, \dots, \xi_s$  are  $\mathbb{Q}$ -linearly independent), we obtain infinitely many tuples satisfying (4.4) so that  $\omega_0(\xi_0, \dots, \xi_r) = \omega_0(\xi_0, \dots, \xi_s) \geq 1/s$ . This concludes the proof of Theorem 5.

A consequence of Theorem 5 is the inequality  $\tau_r(\xi) \leq s$ , which amounts to the following statement, known as Nesterenko's linear independence criterion [10]:

Assume there exists a sequence  $(L_n)$  of linear forms with  $L_n(\xi) \neq 0$  for any  $n$  and

$$L_n(\xi) \rightarrow 0, \quad |L_{n+1}(\xi)| = |L_n(\xi)|^{1+o(1)}, \quad \text{and} \quad |L_n(\xi)| \leq H(L_n)^{-\tau+o(1)}$$

for some  $\tau > 0$ . Then we have

$$\dim_{\mathbb{Q}} \text{Span}_{\mathbb{Q}}(\xi_0, \dots, \xi_r) \geq \tau + 1.$$

The above arguments provide a simple proof of this criterion, based on Dirichlet's box principle and the upper bound  $\tau_r(\xi) \leq \frac{1}{\omega_0(\xi)}$  (which is also proved, essentially in the same way, as the first step in Nesterenko's inductive proof [10]). This proof is essentially a special case of the one of [7].

In the one-dimensional case, the upper bound  $\tau_r(\xi) \leq \frac{1}{\omega_0(\xi)}$  corresponds to Proposition 1, while  $\frac{1}{\omega_0(\xi)} \leq s$  simply means  $\mu(\xi) \geq 2$  (and one way to prove this fact is to use Dirichlet's box principle like in the multivariate setting).

It would be very interesting to investigate on Theorem 5, for instance to know for which real numbers  $\xi$  equality holds (as in the univariate case of Theorem 4). Is it the case for almost all  $\xi$  with respect to Lebesgue measure? It is well known that  $\frac{1}{\omega_0(\xi)} = s = r$  for almost all  $\xi$ .

Another question worth studying is the connection between  $\tau_r(\xi)$ ,  $\tau_r'(\xi)$ ,  $\tau_r''(\xi)$ ,  $\tau_r'''(\xi)$ , and the exponent  $\omega_k(\xi)$  that measure the distance of  $\xi = (\xi_0, \dots, \xi_r)$  to subspaces of dimension  $k + 1$  of  $\mathbb{R}^{r+1}$  defined over  $\mathbb{Q}$ , for  $k < s$  (see [12], [9], [4] and [10]).

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