HOLOMORPHIC SOLUTIONS OF E-OPERATORS

T. RIVOAL AND J. ROQUES

Abstract. We solve the problem of describing the solutions of E-operators of order $\mu \geq 1$ admitting at $z = 0$ a basis over $\mathbb{C}$ of local solutions which are all holomorphic at $z = 0$. We prove that the components of such a basis can be taken of the form $\sum_{j=1}^{\ell} P_j(z)e^{\beta_j z}$, where $\ell \leq \mu$, $\beta_1, \dots, \beta_\ell \in \overline{\mathbb{Q}}^{\times}$, and $P_1(z), \dots, P_\ell(z) \in \overline{\mathbb{Q}}[z]$.

1. Introduction

We fix an enbedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}$. An E-function is a power series

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n \in \overline{\mathbb{Q}}[[z]]$$

such that:

(1) $f(z)$ satisfies a non-zero linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$;
(2) there exists $C > 0$ such that
   (a) for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we have $|\sigma(a_n)| \leq C^{n+1}$;
   (b) there exists a sequence of positive integers d_n such that $d_n \leq C^{n+1}$ and $d_n a_m$ is an algebraic integer for all $m \leq n$.

A G-function at $z = 0$ is a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[[z]]$ such that $\sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$ is an E-function. Both classes of functions have been first introduced by Siegel ([1]). Much work has been devoted to the study of the Diophantine nature of the values of these functions, and the properties of the differential equations they satisfy proved to be crucial; we refer to [1, 4, 6, 10] for surveys of these results.

A theorem of Chudnovsky [5] says that if $\mathcal{L} \in \overline{\mathbb{Q}}[z, \frac{d}{dz}] \setminus \{0\}$ is the minimal differential operator annihilating a G-function at $z = 0$, then all its solutions at any point $\alpha \in \overline{\mathbb{Q}} \cup \{\infty\}$ are essentially G-functions of the variable $z - \alpha$ or $1/z$ if $\alpha = \infty$. Such a differential operator is called a G-operator. Throughout the paper, by "solution of a differential operator $\mathcal{L}$", it must be understood "solution of the differential equation $\mathcal{L}y(z) = 0$".

Recently, André [2, 3] defined an E-operator as a differential operator in $\overline{\mathbb{Q}}[z, \frac{d}{dz}]$ such that its Fourier-Laplace transform is a G-operator. Recall that the Fourier-Laplace transform $\widehat{\mathcal{L}} \in \overline{\mathbb{Q}}[z, \frac{d}{dz}]$ of an operator $\mathcal{L} \in \overline{\mathbb{Q}}[z, \frac{d}{dz}]$ is the image of $\mathcal{L}$ by the automorphism of the Weyl algebra $\overline{\mathbb{Q}}[z, \frac{d}{dz}]$ defined by $z \mapsto -\frac{d}{dz}$ and $\frac{d}{dz} \mapsto z$. Any E-function is solution of

Date: April 18, 2019.

[1]His definition was slightly less restrictive, but it is now believed that both definitions define the same class of functions.

an E-operator, which is not necessarily minimal for the degree in $\frac{d}{dz}$ but is minimal for the degree in z. André proved that the only possible singular points of an E-operator are 0 and ∞, 0 being a regular singularity. In general, ∞ is an irregular singularity but the slopes of the Newton polygon of $\mathcal{L}$ are in $\{0, 1\}$. We emphasize that the solutions of E-operators are not holomorphic at $z = 0$ in general, as they may have a non-trivial monodromy around 0. This note grew out from the observation that the solutions of "non-trivial" E-operators we have found in the literature have a non-trivial monodromy. Our main result shows that this is actually a general fact.

Theorem 1. *Consider an E-operator $\mathcal{L} \in \overline{\mathbb{Q}}[z, \frac{d}{dz}]$ of order μ having a basis over $\mathbb{C}$ of holomorphic solutions at $z = 0$. Then, $\mathcal{L}$ has a basis over $\mathbb{C}$ of solutions of the form*

$$P_1(z)e^{\beta_1 z} + \cdots + P_\ell(z)e^{\beta_\ell z} \tag{1.1}$$

for some integer $\ell \leq \mu$, some $\beta_1, \ldots, \beta_\ell \in \overline{\mathbb{Q}}^\times$, and some $P_1(z), \ldots, P_\ell(z) \in \overline{\mathbb{Q}}[z]$.

Remark. In [4], Bertrand gives two new proofs of the Siegel-Shidlovsky Theorem using Laurent's interpolation determinants method. The second proof given in Section 5 of [4] works under the assumption that the solutions of the underlying E-operator are holomorphic at $z = 0$. He then observes that "*cette hypothèse n'est en pratique vérifiée que dans le cas du théorème de Lindemann-Weierstrass*" ([2]). Theorem 1 shows that Bertrand's observation is the general situation: his second proof is indeed only a new proof of the Lindemann-Weierstrass Theorem, as it does not cover any other E-function than those of the form (1.1).

As a by-product of the proof of Theorem 1, we also obtain the following result.

Theorem 2. *Given an E-operator $\mathcal{L} \in \overline{\mathbb{Q}}[z, \frac{d}{dz}]$ of order μ, let us assume the equation $\mathcal{L}y(z) = 0$ has a non-zero local solution $F(z - \alpha)$ at some $\alpha \in \overline{\mathbb{Q}}^\times$ such that $F(z)$ is an E-function. Then $F(z)$ is of the form $Q_1(z)e^{\kappa_1 z} + \cdots + Q_\ell(z)e^{\kappa_\ell z}$ for some integer $\ell \leq \mu$, some $\kappa_1, \ldots, \kappa_\ell \in \overline{\mathbb{Q}}^\times$, and some $Q_1(z), \ldots, Q_\ell(z) \in \overline{\mathbb{Q}}[z]$.*

One of the steps in the proof of Theorem 1 is the following result, which is of independent interest.

Proposition 1. *Consider a differential operator $\mathcal{L} = \sum_{j=0}^{\mu} A_j(z)\left(\frac{d}{dz}\right)^j$ with $A_j(z) \in \overline{\mathbb{Q}}[z]$, and $A_\mu(z) \neq 0$. Let*

$$F(z - \alpha) = \sum_{n=0}^{\infty} \frac{a_n}{n!}(z - \alpha)^n \in \overline{\mathbb{Q}}[[z - \alpha]]$$

be a local solution of $\mathcal{L}$ at an algebraic point α such that $A_\mu(\alpha) \neq 0$. Let d_n denote the smallest positive integer such that $d_n a_0, d_n a_1, \ldots, d_n a_n$ are algebraic integers. Then, there exists a positive integer C such that, for all $n \geq 0$, d_n divides C^{n+1}.

Moreover, if $F^\sigma(z) := \sum_{n=0}^{\infty} \frac{\sigma(a_n)}{n!} z^n$ is an entire function for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and if the slopes of the Newton polygon of $\mathcal{L}$ at ∞ are ≤ 1, then $F(z)$ is an E-function.

Acknowledgements. We would like to thank the referee for her or his careful reading.

[2]in practice, this assumption is satisfied only in the case of the Lindemann-Weierstrass Theorem.

2. Proof of Proposition 1

Up to replacing $\mathcal{L}$ by $m\mathcal{L}$ for some non zero integer m, we can and will assume that $A_0(z), \dots, A_\mu(z)$ are polynomials with algebraic integers coefficients. The vector function $Y(z) = {}^t\big(F(z-\alpha), F'(z-\alpha), \dots, F^{(\mu-1)}(z-\alpha)\big)$ satisfies the differential system $Y' = MY$ where

$$M = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -\frac{A_0}{A_\mu} & -\frac{A_1}{A_\mu} & \cdots & \cdots & -\frac{A_{\mu-1}}{A_\mu} \end{pmatrix} \in \mathrm{M}_\mu(\overline{\mathbb{Q}}(z)).$$

We define the sequence of square matrices $(M_n)_{n\geq 0}$ by $M_0 = I_\mu$ and, for all $n \geq 0$,

$$M_{n+1} = M_n M + M_n'.$$

Then, we have

$$Y(z) = \left(\sum_{n=0}^{\infty} \frac{M_n(\alpha)}{n!}(z-\alpha)^n\right) Y(\alpha).$$

(See [8, p. 93].) By induction on n, we see that M_n is of the form $A_\mu^{-n}\widetilde{M}_n$ for some $\widetilde{M}_n \in \mathrm{M}_\mu(\overline{\mathbb{Q}}[z])$ whose entries are polynomials with algebraic integers coefficients, and that there exists $B > 0$ such that the degree of each entry of $\widetilde{M}_n$ is $\leq Bn$.

Let α be an algebraic number such that $A_\mu(\alpha) \neq 0$. Let u and v be non-zero integers such that $u\alpha$ and $v/A_\mu(\alpha)$ are algebraic integers. Then, for any $n \geq 0$, the entries of the matrix $u^{Bn}v^n M_n(\alpha) = u^{Bn}\widetilde{M}_n(\alpha)(v/A_\mu(\alpha))^n$ are algebraic integers. Let also w be a non-zero integer such that the vector $wY(\alpha)$ has algebraic integers components. Then the n-th Taylor coefficient of each component of $Y(z)$ becomes an algebraic integer after multiplication by $w(u^B v)^n n!$. This proves the first part of the proposition, and in fact a more precise result.

Let us now assume that, for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, $F^\sigma(z)$ is an entire function and that the slopes of the Newton polygon of $\mathcal{L}$ at ∞ are ≤ 1. The function $F^\sigma(z - \sigma(\alpha))$ is a solution of the differential operator $\mathcal{L}^\sigma$ whose polynomial coefficients are images by σ of those of $\mathcal{L}$. Then, for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the slopes of the Newton polygon of $\mathcal{L}^\sigma$ at ∞ are ≤ 1 as well, and, hence, the solutions of $\mathcal{L}^\sigma$ have at most exponential growth at ∞ in any sector of bounded aperture. In particular, $F^\sigma(z-\sigma(\alpha))$ and, hence, $F^\sigma(z)$ have at most exponential growth of order 1 at ∞, i.e. $|F^\sigma(z)| \ll e^{\kappa|z|}$ for some $\kappa > 0$ as $z \to \infty$. Using the Cauchy formula

$$\sigma(a_n) = \frac{n!}{2i\pi} \int_{|z|=n} \frac{F^\sigma(z)}{z^{n+1}} dz$$

and Stirling's formula, we then deduce that

$$\limsup_{n\to+\infty} |\sigma(a_n)|^{1/n} \leq e^{\kappa-1} < \infty.$$

Together with the above bound for the denominator of the Taylor coefficients of $F(z)$, this implies that $F(z)$ is an E-function.

Remark. The proof shows the following. If the local solution $F(z-\alpha)=\sum_{n=0}^{\infty}\frac{a_n}{n!}(z-\alpha)^n$ is entire and of order of growth ≤ 1, then it is a *quasi* E-function in the following sense: $F(z)$ satisfies conditions (1), (2b) and a weak version of (2a) for $\sigma=\mathrm{id}_{\overline{\mathbb{Q}}}$ only (*a priori*), where these labels refer to the definition of an E-function in the Introduction. The assumption on $F^{\sigma}(z)$ in the second part of Proposition 1 forces (2a) for all $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. We don't know if a quasi E-function is automatically an E-function. A similar problem can be formulated for G-functions.

3. Proof of Theorem 1

Using Theorem 4.3 (*iii*) in [2], we see that $\mathcal{L}$ has a basis over $\mathbb{C}$ of solutions $E_1,\ldots,E_\mu$ which consists of E-functions. Then, for any $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the E-functions $E_1^{\sigma},\ldots,E_\mu^{\sigma}$ form a basis over $\mathbb{C}$ of solutions of $\mathcal{L}^{\sigma}$. Indeed, if $E_1^{\sigma},\ldots,E_\mu^{\sigma}$ were $\mathbb{C}$-linearly dependent, they would also be $\overline{\mathbb{Q}}$-linearly dependent, and using σ^{-1} on the induced $\overline{\mathbb{Q}}$-relations of Taylor coefficients, this would imply that $E_1,\ldots,E_\mu$ are $\overline{\mathbb{Q}}$-linearly dependent. In particular, for any $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, any solution of $\mathcal{L}^{\sigma}$ is an entire function.

Let us consider $\alpha\in\overline{\mathbb{Q}}^{\times}$. Theorem 4.3 (*iv*) of [2] ensures that the slopes of the Newton polygon of $\mathcal{L}$ at ∞ are ≤ 1. Using Proposition 1, we infer that the differential operator $\mathcal{L}$ admits at $z=\alpha$ a basis of solutions of the form $F_1(z-\alpha),\ldots,F_\mu(z-\alpha)$, where each $F_j(z)$ is an E-function. We now fix j and we set $F(z):=F_j(z)$. We have

$$\mathcal{L}_\alpha F(z)=0$$

where $\mathcal{L}_\alpha$ is the shifted operator (of order μ) obtained by changing z to $z+\alpha$ in $\mathcal{L}$.

We recall that the Laplace transform of $F(z)=\sum_{n=0}^{\infty}\frac{a_n}{n!}z^n$ is defined by $g(z):=\int_0^{\infty}e^{-zt}F(t)dt$. Locally around $z=\infty$, we have $g(z)=\sum_{n=0}^{\infty}\frac{a_n}{z^{n+1}}$, which is a G-function of the variable $1/z$. Denoting by $\widehat{\mathcal{L}}_\alpha$ the Fourier-Laplace transform of $\mathcal{L}_\alpha$, we have

$$\left(\frac{d}{dz}\right)^{\mu}\widehat{\mathcal{L}}_\alpha\big(g(z)\big)=0.$$

(This is a general property, see [2, p. 716].) Since $\widehat{\mathcal{L}}_\alpha=e^{\alpha z}\widehat{\mathcal{L}}e^{-\alpha z}$, the above equality can be rewritten as follows:

$$\widehat{\mathcal{L}}\big(e^{-\alpha z}g(z)\big)=e^{-\alpha z}P(z) \tag{3.1}$$

for some $P(z)\in\overline{\mathbb{Q}}[z]$ of degree $\leq\mu-1$. Let S be the (finite) set of singularities of g over $\mathbb{P}^1(\mathbb{C})$. Consider $[\gamma]\in\pi_1(\mathbb{P}^1(\mathbb{C})\setminus S,z_0)$ (where $z_0\in\mathbb{P}^1(\mathbb{C})\setminus S$ is arbitrary) and denote by $[\gamma]g$ the analytic continuation of g along $[\gamma]$. We set $\mathrm{var}_{[\gamma]}(g)=[\gamma]g-g$. Then, we deduce from (3.1) that

$$\widehat{\mathcal{L}}\big(e^{-\alpha z}\mathrm{var}_{[\gamma]}(g)\big)=0.$$

Therefore, we have

$$\mathrm{var}_{[\gamma]}(g)(z)=e^{\alpha z}\varphi(z)$$

for some solution φ of $\widehat{\mathcal{L}}$. We claim that $\varphi = 0$. Indeed, assume at the contrary that $\varphi \neq 0$. Note that $\widehat{\mathcal{L}}$ is a G-operator and, hence, a fuchsian operator. Thus, φ has moderate growth at ∞, so that $e^{\alpha z}\varphi(z)$ has exponential growth at ∞ along some direction. But, g is a G-function hence it has moderate growth near each $s \in S$ along any sector with finite aperture. It follows that $\mathrm{var}_{[\gamma]}(g)$ has moderate growth at ∞ along any sector with finite aperture. Whence a contradiction.

Therefore, we have $\mathrm{var}_{[\gamma]}(g) = 0$ *i.e.* $[\gamma]g = g$. In other words, the monodromy of g is trivial. Since g has moderate growth at each $s \in S$, we get $g \in \overline{\mathbb{Q}}(z)$.

To recover $F(z)$ from $g(z)$, we use the inverse Laplace transform formula. Let δ be any real number larger than 0 and than all the real parts of the singularities of $g(z)$. Then, by [7, p. 61, Théorème 1], we have

$$\lim_{T\to+\infty} \frac{1}{2i\pi} \int_{\delta-iT}^{\delta+iT} \frac{g(x)}{x} e^{xz} dx = \begin{cases} \int_0^z F(u)du & \text{if } z > 0 \\ 0 & \text{if } z < 0. \end{cases}$$

Now, since $g(z)$ is a rational function, the integral

$$\lim_{T\to+\infty} \frac{1}{2i\pi} \int_{\delta-iT}^{\delta+iT} \frac{g(x)}{x} e^{xz} dx$$

is easily computed by the residues theorem; see [9] for a similar computation. It is equal to $\sum_\rho Q_\rho(z)e^{\rho z}$ when $z > 0$, where the summation is over the poles of $g(x)/x$ and where each $Q_\rho(z) \in \overline{\mathbb{Q}}[z]$. Hence, by differentiation, for $z > 0$, we have

$$F(z) = \sum_\rho \big(Q'_\rho(z) + \rho Q_\rho(z)\big)e^{\rho z}. \tag{3.2}$$

By analytic continuation, this identity holds for any $z \in \mathbb{C}$. Since $F(z - \alpha)$ represents any local solution of $\mathcal{L}y(z) = 0$ at $z = \alpha$, it follows that $\mathcal{L}$ has a basis of solutions of the form (3.2). This completes the proof of the theorem.

Bibliography

[1] Y. André, *G-functions and Geometry*, Aspects of Mathematics, **E13**. Friedr. Vieweg & Sohn, Braunschweig, 1989.

[2] Y. André, *Séries Gevrey de type arithmétique I. Théorèmes de pureté et de dualité*, Annals of Math. **151** (2000), 705–740.

[3] Y. André, *Séries Gevrey de type arithmétique II. Transcendance sans transcendance*, Annals of Math. **151** (2000), 741–756.

[4] D. Bertrand, *Le théorème de Siegel-Shidlovsky revisité*, in Number Theory, Analysis and Geometry: In Memory of Serge Lang, Springer-Verlag New-York, 2012, 51–67.

[5] G. V. Chudnovsky, *On applications of diophantine approximations*, Proc. Natl. Acad. Sci. USA **81** (1984), 7261–7265.

[6] D. V. Chudnovsky, G. V. Chudnovsky, *Applications of Padé approximations to Diophantine inequalities in values of G-functions*, Number theory (New York, 1983/84), Lecture Notes in Math. **1135**, Springer, Berlin, 1985, 9–51.

[7] V. Ditkine, A. Proudnikov, *Calcul Opérationnel*, Éditions Mir, 1979, traduction du russe.

[8] B. Dwork, G. Gerrotto, F. J. Sullivan, *An introduction to G-functions*, Annals of Mathematical Studies **133**, 1994.

[9] S. Fischler, T. Rivoal, *Arithmetic theory of E-operators*, Journal de l'École polytechnique – Mathématiques **3** (2016), 31–65.

[10] A. B. Shidlovsky, *Transcendental Numbers*, de Gruyter Studies in Mathematics **12**, 1989.

T. Rivoal, Institut Fourier, Université Grenoble 1, CNRS UMR 5582, 100 rue des Maths, BP 74, 38402 Saint-Martin d'Hères cedex, France.
Tanguy.Rivoal@univ-grenoble-alpes.fr

J. Roques, Institut Fourier, Université Grenoble 1, CNRS UMR 5582, 100 rue des Maths, BP 74, 38402 Saint-Martin d'Hères cedex, France.
Julien.Roques@ujf-grenoble.fr