Values of G-functions

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G-functions

We fix an embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}$.

Definition 1

A (formal) power series $F(z) = \sum_{n\geq 0} a_n z^n \in \mathbb{C}[[z]]$ is a G-function if there exists $C > 0$ such that

(i) $a_n \in \overline{\mathbb{Q}}$ for all $n \geq 0$.

(ii) The maximum of the moduli of the algebraic conjuguates of a_n is bounded by C^{n+1}.

(iii) There exist $D_n \in \mathbb{Z}$ such that $D_n a_j$ is an algebraic integer for all $j \leq n$ and $|D_n| \leq C^{n+1}$.

(iv) F is holonomic over $\overline{\mathbb{Q}}(z)$, i.e., F satisfies a homogeneous linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$.

Examples

- Algebraic functions over $\overline{\mathbb{Q}}(z)$, holomorphic at $z = 0$, like

$$f(z) = \frac{1}{\sqrt{1-4z}} = \sum_{n\geq 0} \binom{2n}{n} z^n.$$

This is a consequence of Eisenstein's theorem.

- Hypergeometric series

$$\sum_{n\geq 0} \frac{(a_1)_n \cdots (a_p)_n}{(1)_n (b_1)_n \cdots (b_{p-1})_n} z^n$$

where $(x)_m := x(x+1)\cdots(x+m-1)$, $p \geq 1$ and $a_j, b_j \in \mathbb{Q}$.

In particular,

$$-\log(1-z) = \sum_{n\geq 1} \frac{z^n}{n}, \quad \mathrm{Li}_s(z) = \sum_{n\geq 1} \frac{z^n}{n^s} \quad (s \in \mathbb{Z}).$$

• "Periods", i.e. solutions of Picard-Fuchs differential equations. Grosso modo, these are functions defined as integrals of algebraic forms over cycles in families of algebraic varieties over $\overline{\mathbb{Q}}$.

A famous conjecture of Bombieri and Dwork predicts that G-functions should coincide with periods (in a suitable sense). André proved that periods "are" G-functions.

Non-examples

• E-functions, like

$$\exp(z) = \sum_{n\geq 0} \frac{z^n}{n!}.$$

• anti$E$-functions, like

$$\sum_{n\geq 0} n! z^n.$$

• Mahler type-functions, like

$$\sum_{n\geq 0} z^{2^n}.$$

Properties of G-functions

• The set of G-functions is a ring (for the usual addition and Cauchy product of series), stable by differentiation, integration and Hadamard product.

• André: the units of the ring of G-functions are exactly the holomorphic algebraic functions that do not vanish at $z = 0$.

• A G-function can be analytically continued to $\mathbb{C}$, minus a finite number of cuts.

Much more is true.

André-Chudnovski-Katz Theorem

Given a G-function $F(z)$, consider the minimal linear differential equation $Ly = 0$ of order μ and with coefficients in $\overline{\mathbb{Q}}(z)$, of which $F(z)$ is a solution. Let $\xi_1, \ldots, \xi_p$ denote the singularities of L at finite distance. Then,

• L is globally fuchsian, with rational exponents at each ξ_j and at ∞.

• For all $\xi \in \mathbb{C}$ minus (fixed) cuts with the $\xi_j' s$ for origin (but $\xi = \xi_j$ is ok), L has a local basis of solutions $G_1(z), \ldots, G_\mu(z)$ at $z = \xi$ such that, for any $k = 1, \ldots, \mu$,

$$G_k(z) = \sum_{s \in S_k} \sum_{t \in T_k} \log(z-\xi)^s (z-\xi)^t F_{s,t,k}(z-\xi)$$

where

$S_k \subset \mathbb{N}$ and $T_k \subset \mathbb{Q}$ are finite, and if $\xi \neq \xi_k$, $S_k = T_k = \{0\}$.

$F_{s,t,k}(z)$ are G-functions.

• If $\xi = \infty$, the same result holds provided we replace $z - \xi$ by $1/z$ everywhere.

Diophantine motivation

• Apéry proved that $\zeta(3) \notin \mathbb{Q}$ by constructing two sequences a_n and b_n such that

$$a_n \in \mathbb{Z}, \quad \mathrm{lcm}(1,2,\dots,n)^3 b_n \in \mathbb{Z}$$

$$0 \neq \mathrm{lcm}(1,2,\dots,n)^3 (a_n \zeta(3) - b_n) \longrightarrow 0.$$

Beukers and Dwork observed that $\sum_{n\geq 0} a_n z^n$ and $\sum_{n\geq 0} b_n z^n$ are G-functions (not with the same minimal equation).

• It is a difficult problem to find interesting real numbers that can be proved irrational by Apéry's method.

Can we at least say what "interesting" means?

Can we characterise the real numbers ξ such that there exist $p_n \in \mathbb{Q}$ and $q_n \in \mathbb{Q}$ such that

$$\frac{p_n}{q_n} \longrightarrow \xi$$

and $\sum_{n\geq 0} p_n z^n$, $\sum_{n\geq 0} q_n z^n$ are G-functions?

G-values

With Stéphane Fischler, *On the values of G-functions*, to appear.

Definition 2

The set $\mathbb{G}$ of G-values is defined as the set of all values taken by any analytic continuation of any G-function at any algebraic point.

$\mathbb{G}$ is a countable set.

There is currently no known algorithm to decide whether a number is in $\mathbb{G}$ or not.

Theorem 1

A number ξ is in $\mathbb{G}$ iff $\xi = F(1)$ where

(i) F is a G-function with coefficients in $\mathbb{Q}(i)$.

(ii) The radius of convergence of F can be as large as a priori wished.

Given $\xi \in \mathbb{G}$, it seems difficult to describe explicitly F from our proof.

Theorem 1 for $\overline{\mathbb{Q}}$ and $\log(\overline{\mathbb{Q}}^*)$

Let $\alpha \in \overline{\mathbb{Q}}$, $Q(X) \in \mathbb{Q}[X]$ such that α is a simple root of Q. Let $u \in \mathbb{Q}(i)$ such that $Q'(u) \neq 0$. Consider

$$\Phi_u(z) := u + \sum_{n\geq 1} (-1)^n \frac{Q(u)^n}{n!} \cdot \frac{\partial^{n-1}}{\partial x^{n-1}} \left(\left(\frac{x-u}{Q(x)-Q(u)} \right)^n \right)_{x=u} z^n.$$

Then

(i) $\Phi_u(z)$ is algebraic over $\overline{\mathbb{Q}}(z)$, with coefficients in $\mathbb{Q}(i)$:

$$Q(\Phi_u(z)) = (1-z)Q(u).$$

(ii) For any $R > 1$, we can choose u close enough to α such that the radius of convergence of Φ_u is $\geq R$ and $\Phi_u(1) = \alpha$.

- Proof based on Lagrange's inversion formula.

- Similarly, we have explicit G-functions F as in Theorem 1 such that $F(1) = \log(\alpha)$ for any non-zero algebraic number and any prescribed branch of the logarithm.

From Theorem 1, we deduce

Corollary 1

$\mathbb{G}$ is a subring of $\mathbb{C}$.

$\mathbb{G}$ is presumably not a field.

Proposition 1

The group of units of $\mathbb{G}$ contains $\overline{\mathbb{Q}}^*$ and the values $B(a,b)$, $a,b \in \mathbb{Q}$, where

$$B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

provided $B(a,b)$ is defined and non-zero.

The numbers $\Gamma(a/b)^b$, $a/b \in \mathbb{Q} \setminus \{0,-1,-2,\dots\}$, are units.

For instance, $\pi = \Gamma(1/2)^2$ is a unit. Proof:

$$\pi = \sum_{n\geq 0} \frac{4(-1)^n}{2n+1}, \qquad \frac{1}{\pi} = \sum_{n\geq 0} \frac{(42n+5)\binom{2n}{n}^3}{2^{12n+4}}.$$

• The proof of Theorem 1 is long and technically complicated.

The ACK theorem is crucial.

We also use the fact that the theorem holds for algebraic numbers and logarithms of algebraic numbers (previous slide).

• We need the following result, of independent interest.

Theorem 2

Let $F(z)$ be a G-function solution of the minimal differential equation $Ly = 0$. For any given $\xi \in \mathbb{C} \cup \{\infty\}$ (minus cuts), let $G_1(z), \dots, G_\mu(z)$ be a basis of local solutions of $Ly = 0$ around $z = \xi$. We have

$$F(z) = \sum_{k=1}^{\mu} \omega_k G_k(z)$$

for any z in the (multi)cut plane.

Then for all k, the connection constants $\omega_k \in \mathbb{G}$.

Answer to the characterisation question

Theorem 3

Let $\xi \in \mathbb{R}^*$. The following statements are equivalent.

(i) There exist two sequences of rational numbers a_n and b_n such that $\sum_{n\geq 0} a_n z^n$, $\sum_{n\geq 0} b_n z^n$ are G-functions, with $b_n \neq 0$ for all $n \gg 1$ and

$$\frac{a_n}{b_n} \longrightarrow \xi.$$

(ii) $\xi \in \mathrm{Frac}(\mathbb{G}) \cap \mathbb{R} = \mathrm{Frac}(\mathbb{G} \cap \mathbb{R}) =$ *interesting numbers.*

(iii) For any given $R \geq 1$, there exist two G-functions $A(z) = \sum_{n\geq 0} a_n z^n$ and $B(z) = \sum_{n\geq 0} b_n z^n$, with $a_n, b_n \in \mathbb{Q}$, both with radius of convergence 1, and such that $A(z) - \xi B(z)$ has radius of convergence $\rho > R$. In particular,

$$a_n - b_n\xi = \mathcal{O}(\rho^{-n}).$$

The proof uses Theorems 1-2 and Singularity Analysis à la Flajolet-Sedgewick.

Moral consequences of Theorem 3

• It is believed that Euler's constant $\gamma = \lim_n \sum_{k=1}^n \frac{1}{k} - \log(n)$ is irrational. Why? Because if $\gamma = p/q \in \mathbb{Q}$ with $(p, q) = 1$, then

$$|q| \geq 10^{242080}.$$

It is also believed that $\gamma \notin \mathbb{G}$. Why? Because Euler and Ramanujan would have found various formulas proving this fact.

It is also plausible that γ is not even in $\mathrm{Frac}(\mathbb{G})$.

Then Theorem 3 rules out the possibility to prove the irrationality of γ à la Apéry, i.e., with sequences generated by G-functions.

• Aptekarev has showed the existence of sequences of rational numbers p_n and q_n such that $p_n/q_n \to \gamma$ and such that $\sum_{n\geq 0} p_n z^n$, $\sum_n q_n z^n$ are *holonomic functions*.

But these series are not G-functions.

• Similar considerations apply to the number $\exp(1)$, except of course that we already know that it is irrational.

Proof of Proposition 1

- For $x, y \in \mathbb{Q} \cap (0,1]$,

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t = \int_0^1 \sum_{n\geq 0}(-1)^n \binom{y-1}{n} t^{n+x-1}\mathrm{d}t$$
$$= \sum_{n\geq 0}(-1)^n \frac{\binom{y-1}{n}}{n+x} \in \mathbb{G}.$$

- Extension to suitable $x, y \in \mathbb{Q}$ by means of

$$B(x,y) = \frac{x+y}{x}B(x+1,y), \qquad B(x,y) = \frac{x+y}{y}B(x,y+1).$$

- For suitable $x, y \in \mathbb{Q}$,

$$\frac{1}{B(x,y)} = \frac{\sin(\pi x)\sin(\pi y)}{\sin\pi(x+y)} \cdot \frac{1-x-y}{\pi} \cdot B(1-x,1-y),$$

hence $1/B(x,y) \in \mathbb{G}$.

- Finally, $\Gamma(a/b)^b = \Gamma(a)\prod_{j=1}^b B(a/b, ja/b)$ is a unit of $\mathbb{G}$.

Sketch of proof of Theorem 2

- We start from the relation

$$F(z) = \sum_{j=1}^{\mu} \omega_j G_j(z)$$

that we assume to hold in an neighborhood V containing ξ (maybe on its boundary) where the power series for $F(z)$ and those in the expressions of the G_j (given by the ACK Theorem) are absolutely convergent.

- For any $k = 0, \ldots, \mu - 1$,

$$F^{(k)}(z) = \sum_{j=1}^{\mu} \omega_j G_j^{(k)}(z).$$

Hence, for any $z \in V$,

$$\omega_j = \frac{1}{W(z)} \begin{vmatrix} G_1(z) & \cdots & G_{j-1}(z) & F(z) & G_{j+1}(z) & \cdots & G_\mu(z) \\ G_1^{(1)}(z) & \cdots & G_{j-1}^{(1)}(z) & F^{(1)}(z) & G_{j+1}^{(1)}(z) & \cdots & G_\mu^{(1)}(z) \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ G_1^{(\mu-1)}(z) & \cdots & G_{j-1}^{(\mu-1)}(z) & F^{(\mu-1)}(z) & G_{j+1}^{(\mu-1)}(z) & \cdots & G_\mu^{(\mu-1)}(z) \end{vmatrix}.$$

Here,

$$W(z) = \begin{vmatrix} G_1(z) & \cdots & G_\mu(z) \\ G_1^{(1)}(z) & \cdots & G_\mu^{(1)}(z) \\ \vdots & \cdots & \vdots \\ G_1^{(\mu-1)}(z) & \cdots & G_\mu^{(\mu-1)}(z) \end{vmatrix}.$$

is a wronskien of the differential equation L.

• We can choose $\beta \in V \cap \overline{\mathbb{Q}}$ such that all the values $F^{(k)}(\beta)$, $G_j^{(k)}(\beta)$ are in $\mathbb{G}$.

This is clear for $F^{(k)}(\beta)$.

For $G_j^{(k)}(\beta)$, we use the ACK Theorem around $z = \xi$, as well as the fact that algebraic numbers and logarithms of algebraic numbers are in $\mathbb{G}$.

• Since L has only algebraic singularities with rational exponents, $W(z)$ is an algebraic function over $\overline{\mathbb{Q}}(z)$. We can also assume that $W(\beta) \neq 0$.

Hence $1/W(\beta) \in \mathbb{G}$.

• In the formula for ω_j as a quotient of the two determinants, we set $z = \beta$ and the above arguments prove that $\omega_j \in \mathbb{G}$.