ON THE ARITHMETIC NATURE OF THE VALUES OF THE GAMMA FUNCTION, EULER'S CONSTANT AND GOMPERTZ'S CONSTANT

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Abstract. We prove new results concerning the arithmetic nature values of the Gamma function Γ at algebraic points and Euler's constant γ. We prove that for any $\alpha \in \mathbb{Q}\backslash\mathbb{Z}$, $\alpha > 0$, at least one of the numbers $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}\mathrm{d}t$ and $\int_0^\infty (t+1)^{\alpha-1}e^{-t}\mathrm{d}t$ is an irrational number. Similarly, at least one of the numbers $\gamma = -\int_0^\infty \log(t)e^{-t}\mathrm{d}t$ and Gompertz's constant $\int_0^\infty e^{-t}/(1+t)\mathrm{d}t$ is an irrational number. Quantitative statements, obtained by means of Nesterenko's linear independence criterion, strengthen these irrationality assertions.

1. Introduction

In this article, we prove some results concerning the arithmetic nature of the values of the Gamma function Γ at rational or algebraic points, and for Euler's constant γ. A (completely open) conjecture of Rohrlich and Lang predicts that all polynomial relations between Gamma values over $\mathbb{Q}$ come from the functional equations satisfied by the Gamma function. This conjecture implies the transcendence over $\mathbb{Q}$ of $\Gamma(\alpha)$ at all algebraic non integral number. But, at present, the only known results are the transcendance of $\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(1/3)$ and $\Gamma(1/4)$ (each one of the last two being algebraically independent of π; see [5]). Using the well-known functional equations satisfied by Γ, we deduce the transcendence of other Gamma values like $\Gamma(1/6)$, but not of $\Gamma(1/5)$. Nonetheless, in [7, p. 52, Théorème 3.3.5], it is proved that the set $\{\pi, \Gamma(1/5), \Gamma(2/5)\}$ contains at least two algebraically independent numbers. In positive characteristic, all polynomial relations between values of the analogue of the Gamma function are known to come from the analogue of Rohrlich-Lang conjecture; see [1].

The results proved here are steps in the direction of transcendence results for the Gamma function. We start with a specific quantitative theorem and then prove more general results of qualitative nature. We define $\log(z)$ and z^α for $z \in \mathbb{C} \setminus (-\infty, 0]$ with the principal value of the argument $-\pi < \arg(z) < \pi$. An important function in the paper is the function

$$\mathcal{G}_\alpha(z) := z^{-\alpha} \int_0^\infty (t+z)^{\alpha-1} e^{-t}\mathrm{d}t.$$

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For any $\alpha \in \mathbb{C}$, it is an analytic function of z in $\mathbb{C} \setminus (-\infty, 0]$. When $\alpha = 0$ and $z = 1$, $\mathcal{G}_0(1)$ is known as Gompertz's constant (see [6]).

The main result of the paper is the following.

Theorem 1. *(i) For any rational number $\alpha \notin \mathbb{Z}$, any rational number $z > 0$ and any $\varepsilon > 0$, there exists a constant $c(\alpha, \varepsilon, z) > 0$ such that for any $p, q, r \in \mathbb{Z}$, $q \neq 0$, we have*

$$\left| \frac{\Gamma(\alpha)}{z^\alpha} - \frac{p}{q} \right| + \left| \mathcal{G}_\alpha(z) - \frac{r}{q} \right| \geq \frac{c(\alpha, \varepsilon, z)}{H^{3+\varepsilon}}, \tag{1.1}$$

where $H = \max(|p|, |q|, |r|)$. In particular, at least one of $\Gamma(\alpha)/z^\alpha$ and $\mathcal{G}_\alpha(z)$ is an irrational number.

(ii) For any rational number $z > 0$ and for any $\varepsilon > 0$, there exists a constant $d(\varepsilon, z) > 0$ such that for any $p, q, r \in \mathbb{Z}$, $q \neq 0$, we have

$$\left| \gamma + \log(z) - \frac{p}{q} \right| + \left| \mathcal{G}_0(z) - \frac{r}{q} \right| \geq \frac{d(\varepsilon, z)}{H^{3+\varepsilon}}. \tag{1.2}$$

In particular, at least one of $\gamma + \log(z)$ and $\mathcal{G}_0(z)$ is an irrational number.

Remarks. The constants $c(\alpha, \varepsilon, z)$ and $d(\varepsilon, z)$ could be explicited but this is not necessary here.

Aptekarev [2] was apparently the first to state explicitly that at least one of γ and $\mathcal{G}_0(1)$ is irrational. He constructed and studied precisely a sequence of linear form in $1, \gamma$ and $\mathcal{G}_0(1)$ with integers coefficients and tending to 0. The technique presented here is different but we show in Section 6 how to construct such linear forms using our approach. For other constructions of rational approximations for Gamma values, see [12, 13].

The proof of Theorem 1 is a consequence of the construction of Hermite-Padé type approximants to $1, \exp$ and a specific E-function (in Siegel's sense, for the definition see [10]). As almost always with Hermite-Padé approximants, they provide very precise diophantine estimates but at the cost of a lesser generality. In fact, using the much more general theorems of Shidlovskii on the algebraic independence of values of E-functions, we can obtain better qualitative results that we now explain. (Some of them are variations of results due to Mahler [8]). However, it is not clear to us that the precise irrationality measures in Theorem 1 could be obtained by Shidlovskii's methods.

Theorem 2. *(i) For any algebraic number $z \notin (-\infty, 0]$ and any algebraic number α, $\alpha \notin \mathbb{Z}$, the transcendence degree of the field generated by $e^z, \Gamma(\alpha)/z^\alpha$ and $\mathcal{G}_\alpha(z)$ is at least 2. In particular, at least one of the number $\Gamma(\alpha)/z^\alpha$ and $\mathcal{G}_\alpha(z)$ is transcendental.*

(ii) For any algebraic number $z \notin (-\infty, 0]$, the transcendence degree of the field generated by $\gamma + \log(z), e^z$ and $\mathcal{G}_0(z)$ is at least 2. In particular, at least one of $\gamma + \log(z)$ and $\mathcal{G}_0(z)$ is transcendental.

Since $\Gamma(1/2) = \sqrt{\pi}$, we have the following corollaries to Theorem 2(*i*), which is appealing because of the simultaneous occurences of the numbers π and e, whose algebraic independence over $\mathbb{Q}$ is still conjectural.

Corollary 1. *For any algebraic number $z \notin (-\infty, 0]$, the transcendence degree of the field generated by π, e^z and $\mathcal{G}_{1/2}(z)$ is at least* 2.

In particular for $z = 1$, we have the

Corollary 2. *The transcendence degree of the field generated by π, e and $\int_0^\infty \frac{e^{-t}\mathrm{d}t}{\sqrt{1+t}}$ is at least* 2.

It is easy to see that the asymptotic expansion

$$\mathcal{G}_\alpha(z) \sim \sum_{m=0}^{\infty} (-1)^m \frac{(1-\alpha)_m}{z^{m+1}}$$

holds as $|z| \to \infty$ in any open angular sector that does not contain $(-\infty, 0]$. Here, $(x)_m := x(x+1)\cdots(x+m-1)$ is Pochhammer symbol. The divergent asymptotic series on the right hand side is a Gevrey series of exact order 1. (A formal power series $\sum_{n\geq 0} a_n z^n$ with $a_n \in \mathbb{C}$ is a Gevrey series of order s, $s \in \mathbb{R}$, if the associated power series $\sum_{n\geq 0}(a_n/n!^s)z^n$ has a non-zero radius of convergence; it is of exact order s if the radius of convergence is finite non-zero.) The Taylor series for exp is a Gevrey series of exact order -1 and an E-function in the sense of Siegel, π is the sum of the series $4\sum_{m=0}^{\infty} \frac{z^m}{2m+1}$ at $z = -1$, which is a Gevrey series of exact order 0 and a G-function in the sense of Siegel, and the asymptotic expansion of $\mathcal{G}_{1/2}$ is a Gevrey series of exact order 1. Hence, Corollary 1 deals with three numbers at different levels in the hierarchy of Gevrey series. However, this is a kind of accident because the proof remains purely at the level of E-functions. It is still a very difficult open problem to find transcendence methods that would enable one to construct "good" auxiliary functions mixing E-functions and G-functions for example.

In Section 2, we prove the relation between $\Gamma(\alpha)/z^\alpha$, $\mathcal{G}_\alpha(z)$, respectively $\gamma + \log(z)$, $\mathcal{G}_0(z)$, and the E-functions mentioned above. In Section 3, we construct certain Hermite-Padé type approximants to these E-functions, which are needed for the proof of Theorem 1 in Section 4. In Section 5, we give the proofs of Theorem 2 and in the final section, we explain why Theorem 2 is implicit in a paper of Mahler [8].

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2. Some useful functional relations

In this section, we discuss the relations at the origin of Theorems 1 and 2. We define the function

$$\mathcal{E}_\alpha(z) := \sum_{m=0}^{\infty} \frac{z^m}{m!(m+\alpha+1)}$$

for any $z \in \mathbb{C}$ and any $\alpha \in \mathbb{C}$, $\alpha \neq -1, -2, \ldots$, and

$$\mathcal{E}(z) := \sum_{m=1}^{\infty} \frac{z^m}{m!m}$$

for any $z \in \mathbb{C}$. Both functions are E-functions discussed in Shidlovskii's book [10].

Proposition 1. (i) *For any* $z \in \mathbb{C} \setminus (-\infty, 0]$ *and any* $\alpha \in \mathbb{C}$, $\alpha \neq -1, -2, \ldots$, *we have*

$$\Gamma(\alpha+1)/z^{\alpha+1} = \mathcal{E}_\alpha(-z) + e^{-z}\mathcal{G}_{\alpha+1}(z), \tag{2.1}$$

(ii) *For any* $z \in \mathbb{C} \setminus (-\infty, 0]$, *we have*

$$\gamma + \log(z) = -\mathcal{E}(-z) - e^{-z}\mathcal{G}_0(z) \tag{2.2}$$

Proof. (i) We fix $z > 0$ and α such that $\Re(\alpha) > -1$, so that

$$\begin{aligned}\Gamma(\alpha+1) &= \int_0^z e^{-t}t^\alpha \mathrm{d}t + \int_z^\infty e^{-t}t^\alpha \mathrm{d}t \\ &= z^{\alpha+1}\int_0^1 e^{-tz}t^\alpha \mathrm{d}t + \int_0^\infty e^{-(t+z)}(t+z)^\alpha \mathrm{d}t \\ &= z^{\alpha+1}\int_0^1 e^{-tz}t^\alpha \mathrm{d}t + e^{-z}z^{\alpha+1}\mathcal{G}_{\alpha+1}(z).\end{aligned}$$

This identity can be analytically continued to any z such that $z \in \mathbb{C} \setminus (-\infty, 0]$ and any $\alpha \in \mathbb{C}$, $\alpha \neq -1, -2, \ldots$. This is nothing but (2.1) because

$$\int_0^1 e^{tz}t^\alpha \mathrm{d}t = \mathcal{E}_\alpha(z).$$

(ii) We use the same strategy as before. It is well-known that $\gamma = -\Gamma'(1)$. Hence, for any $z > 0$,

$$\begin{aligned}-\gamma &= \int_0^\infty e^{-t}\log(t)\mathrm{d}t = \int_0^z e^{-t}\log(t)\mathrm{d}t + \int_z^\infty e^{-t}\log(t)\mathrm{d}t \\ &= z\int_0^1 e^{-tz}\log(tz)\mathrm{d}t + \int_0^\infty e^{-(t+z)}\log(t+z)\mathrm{d}t \qquad (2.3) \\ &= \log(z) + z\int_0^1 e^{-tz}\log(t)\mathrm{d}t + e^{-z}\int_0^\infty \frac{e^{-t}}{t+z}\mathrm{d}t\end{aligned}$$

(after an integration by parts in the last integral of (2.3)). By analytic continuation, this holds for any $z \in \mathbb{C} \setminus (-\infty, 0]$, giving (2.2) because

$$z\int_0^1 e^{-tz}\log(t)\mathrm{d}t = \mathcal{E}(-z).$$

□

We conclude this section with an identity which is irrelevant for the questions considered in this paper, but which is interesting because it expresses $\mathcal{G}_\alpha(z)$ in term of a more natural integral, of Stieltjes type.

Proposition 2. *For any complex number* α *such that* $\Re(\alpha) < 1$ *and any* $z \in \mathbb{C} \setminus (-\infty, 0]$,

$$\mathcal{G}_\alpha(z) = \frac{1}{\Gamma(1-\alpha)}\int_0^\infty \frac{t^{-\alpha}e^{-t}}{t+z}\mathrm{d}t \tag{2.4}$$

Proof. With $x = 1/z > 0$ and $\alpha < 1$, it is enough to prove that
$$\Gamma(1-\alpha)\int_0^\infty \frac{e^{-t}}{(1+xt)^{1-\alpha}}\mathrm{d}t = \int_0^\infty \frac{t^{-\alpha}e^{-t}}{1+xt}\mathrm{d}t,$$
the complete result following by analytic continuation in x and α.

By definition of $\Gamma(1-\alpha)$, we have
$$\begin{aligned}\Gamma(1-\alpha)\int_0^\infty \frac{e^{-t}}{(1+xt)^{1-\alpha}}\mathrm{d}t &= \int_0^\infty \int_0^\infty \frac{e^{-(s+t)}s^{-\alpha}}{(1+xt)^{\alpha+1}}\mathrm{d}t\mathrm{d}s\\ &= \int_0^\infty e^{-u}\left(\int_0^u v^{1-\alpha}\frac{1+xu}{(1+xv)^2}\frac{1+xv}{v(1+xu)}\mathrm{d}v\right)\mathrm{d}u\\ &= \int_0^\infty e^{-u}\left(\int_0^u \frac{v^{-\alpha}}{1+xv}\mathrm{d}v\right)\mathrm{d}u\\ &= \int_0^\infty \frac{v^{-\alpha}}{1+xv}\left(\int_v^\infty e^{-u}\mathrm{d}u\right)\mathrm{d}v = \int_0^\infty e^{-v}\frac{v^{-\alpha}}{1+xv}\mathrm{d}v,\end{aligned}$$
which proves the expected identity. (We used the change of variables
$$\begin{cases} s = v\dfrac{1+xu}{1+xv}\\ t = \dfrac{u-v}{1+xv}\end{cases}$$
and the application of Fubini's theorem is licit by positivity.) □

3. Hermite-Padé type approximants of E-functions

In this section, we present the constructions of explicit Hermite-Padé type approximants of the functions $1, \exp, \mathcal{E}_\alpha$ on the one hand (Section 3.1), and $1, \exp, \mathcal{E}$ on the other hand (Section 3.2). In the latter case, the construction is an adaptation of the techniques in [14]. Propositions 3 and 4 are crucial ingredients in the proof of Theorem 1. Both are generalisations of a classical construction of diagonal Padé approximants of exp, based on the study of the integral
$$\frac{z^{2n+1}}{n!}\int_0^1 e^{tz}t^n(1-t)^n\mathrm{d}t \in \mathbb{Z}[z] + \mathbb{Z}[z]\exp(z).$$
See for example [3] for details.

3.1. Approximations to the functions $1, \exp$ and $\mathcal{E}_\alpha$.

Proposition 3. *Let us fix α such that $\Re(\alpha) > -1$ and $\alpha \notin \mathbb{Z}$. For any integer $n \geq 0$, there exist some polynomials A_n, C_n (of degree $\leq n$) and B_n (of degree $\leq n+1$) with coefficients in $\mathbb{Q}(\alpha)$ and such that*
$$\begin{aligned}R_{n,\alpha}(z) := \frac{z^{3n+1}}{n!^2}\int_0^1\int_0^1 e^{zuv}u^{2n+\alpha}(1-u)^n v^{2n}(1-v)^n\mathrm{d}u\mathrm{d}v\\ = A_n(z)e^z + B_n(z)\mathcal{E}_\alpha(z) + C_n(z). \qquad (3.1)\end{aligned}$$

The order at $z=0$ of $R_n(z)$ is $3n+1$.

Explicit expressions for the polynomials are provided by the proof. The condition that $\alpha \notin \mathbb{Z}$ is not necessary to define $R_{n,\alpha}(z)$ but the polynomials cannot be defined for $\alpha \in \mathbb{Z}$ in the explicit expressions. This is fixed in Section 3.2 in the case $\alpha = 0$.

This proposition fails to give a solution to the problem of finding the simultaneous Hermite-Padé approximants $[n; n+1; n]$ to the functions $1, \exp$ and $\mathcal{E}_\alpha$. But this is by a small margin because this would have been the case if the order at $z=0$ of $R_n(z)$ were $3n+3$.

To prove the Proposition, we need a lemma.

Lemma 1. *For any integers $k, j \geq 0$, any $z \in \mathbb{C}$ and any $\alpha \notin \mathbb{Z}$, $\Re(\alpha) > -1$, we have*

$$\int_0^1 \int_0^1 e^{zuv} u^{k+\alpha} v^j \mathrm{d}u\mathrm{d}v = \frac{1}{j-k+\alpha}\left(\frac{1}{z} M_{j,k,\alpha}\left(\frac{1}{z}\right) e^z + (-1)^k \frac{(\alpha+1)_k}{z^k} \mathcal{E}_\alpha(z) + (-1)^j \frac{j!}{z^{j+1}}\right),$$

where

$$M_{j,k,\alpha}(z) = \sum_{\ell=0}^{k-1} (k-\ell+\alpha+1)_\ell (-z)^\ell - \sum_{\ell=0}^{j} (j-\ell+1)_\ell (-z)^\ell.$$

Remark. The lemma does not hold when $\alpha \in \mathbb{Z}$, in which case it must be replaced by Lemma 2.

Proof of Lemma 1. Expanding $\exp(zuv)$ in series of powers of zuv, we get

$$\begin{aligned}\int_0^1 \int_0^1 e^{zuv} u^{k+\alpha} v^j \mathrm{d}u\mathrm{d}v &= \sum_{m=0}^{\infty} \frac{z^m}{m!} \cdot \frac{1}{(m+k+\alpha+1)(m+j+1)} \\ &= \frac{1}{j-k-\alpha}\left(\sum_{m=0}^{\infty} \frac{z^m}{m!(m+k+\alpha+1)} - \sum_{m=0}^{\infty} \frac{z^m}{m!(m+j+1)}\right).\end{aligned}$$

To evaluate both series, we remark that

$$\sum_{m=0}^{\infty} \frac{z^m}{m!(m+j+1)} = \int_0^1 e^{zt} t^j \mathrm{d}t$$

$$\sum_{m=0}^{\infty} \frac{z^m}{m!(m+k+\alpha+1)} = \int_0^1 e^{zt} t^{k+\alpha} \mathrm{d}t$$

and that, by repeated integrations by parts, we have

$$\int_0^1 e^{zt} t^j \mathrm{d}t = e^z \sum_{\ell=0}^{j} (-1)^\ell \frac{(j-\ell+1)_\ell}{z^{\ell+1}} + (-1)^{j+1} \frac{j!}{z^{j+1}} \tag{3.2}$$

and

$$\int_0^1 e^{zt} t^{k+\alpha} \mathrm{d}t = e^z \sum_{\ell=0}^{k-1} (-1)^\ell \frac{(k-\ell+\alpha+1)_\ell}{z^{\ell+1}} + (-1)^k \frac{(\alpha+1)_k}{z^k} \mathcal{E}_\alpha(z).$$

The lemma follows immediately. □

Proof of Proposition 3. We fix α such that $\Re(\alpha) > -1$. Set

$$P_n(t) = \frac{1}{n!}(t^n(1-t)^n)^{(n)} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{n}t^k \in \mathbb{Z}[t]$$
$$Q_{n,\alpha}(t) = \frac{1}{n!t^{n+\alpha}}(t^{2n+\alpha}(1-t)^n)^{(n)} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2n+k+\alpha}{n}t^k \in \mathbb{Z}[\alpha][t],$$

which are of degree n in t. Here, $\binom{2n+k+\alpha}{n} := \frac{(n+k+\alpha+1)_n}{n!}$ and it is standard that if $\alpha = a/b \in \mathbb{Q}$, with $a, b \in \mathbb{Z}$, then $b^{2n}\binom{2n+k+\alpha}{n} \in \mathbb{Z}$, so that $b^{2n}Q_{n,\alpha}(t) \in \mathbb{Z}[t]$ in this case.

Let us define

$$I_{n,\alpha}(z) = \int_0^1\int_0^1 e^{zuv}u^\alpha Q_{n,\alpha}(u)P_n(v)\mathrm{d}u\mathrm{d}v \tag{3.3}$$

for any $z \in \mathbb{C}$. For simplicity, we write

$$P_n(t) = \sum_{j=0}^{n} p_{j,n}t^j, \quad Q_{n,\alpha}(t) = \sum_{k=0}^{n} q_{k,n,\alpha}t^k.$$

Hence,

$$\begin{aligned} I_{n,\alpha}(z) &= \sum_{k=0}^{n}\sum_{j=0}^{n} q_{k,n,\alpha}p_{j,n}\int_0^1\int_0^1 e^{zuv}u^{k+\alpha}v^j\mathrm{d}u\mathrm{d}v \\ &= \sum_{k=0}^{n}\sum_{j=0}^{n}\frac{q_{k,n,\alpha}p_{j,n}}{j-k+\alpha}\left(\frac{1}{z}M_{j,k,\alpha}\left(\frac{1}{z}\right)e^z + \frac{(\alpha+1)_k}{z^k}\mathcal{E}_\alpha(z) - \frac{j!}{z^{j+1}}\right), \end{aligned} \tag{3.4}$$

by Lemma 1. Clearly, it follows that

$$z^{n+1}I_{n,\alpha}(z) = A_n(z)e^z + B_n(z)\mathcal{E}_\alpha(z) + C_n(z)$$

for some polynomials A_n, B_n and C_n as described in Proposition 3.

To conclude, it remains to prove that

$$z^{n+1}I_{n,\alpha}(z) = R_{n,\alpha}(z).$$

This is easily done as follows: in $z^{n+1}I_{n,\alpha}(z)$, we integrate n-times by parts in v, and then n-times by parts in u, which gives $R_{n,\alpha}(z)$. $\square$

3.2. **Approximations to the functions** $1, \exp$ **and** $\mathcal{E}$**.** In Proposition 3, the integral $R_{n,\alpha}(z)$ is well-defined for $\alpha = 0$, but its expansion as a linear form in $1, \exp(z)$ and $\mathcal{E}_0(z) = (\exp(z) - 1)/z$ does not hold because the polynomials A_n, B_n, C_n are not defined for $\alpha = 0$ (more precisely, because of the factor $1/(j-k-\alpha)$). However, this can be corrected.

Proposition 4. *For any integer $n \geq 0$, there exist some polynomials $\mathcal{A}_n$, $\mathcal{B}_n$, $\mathcal{C}_n$ (all of degree $\leq n$), with coefficients in $\mathbb{Q}$ and such that*

$$R_{n,0}(z) := \frac{z^{3n+1}}{n!^2} \int_0^1 \int_0^1 e^{zuv} u^{2n}(1-u)^n v^{2n}(1-v)^n \mathrm{d}u\mathrm{d}v$$
$$= \mathcal{A}_n(z)e^z + \mathcal{B}_n(z)\mathcal{E}(z) + \mathcal{C}_n(z). \tag{3.5}$$

The order at $z = 0$ of $R_{n,0}(z)$ is $3n+1$.

To prove the Proposition, we need an analogue of Lemma 1 in the case when $\alpha = 0$.

Lemma 2. *Fix any integers $k, j \geq 0$ and any $z \in \mathbb{C}$.*

If $k \neq j$, then

$$\int_0^1 \int_0^1 e^{zuv} u^k v^j \mathrm{d}u\mathrm{d}v = \frac{1}{j-k}\left(\frac{1}{z}\mathcal{M}_{j,k}\Big(\frac{1}{z}\Big)e^z + (-1)^{k+1}\frac{k!}{z^{k+1}} + (-1)^j \frac{j!}{z^{j+1}}\right),$$

where

$$\mathcal{M}_{j,k}(z) = \sum_{\ell=0}^{k} (k-\ell+1)_\ell (-z)^\ell - \sum_{\ell=0}^{j} (j-\ell+1)_\ell (-z)^\ell.$$

If $k = j$, then

$$\int_0^1 \int_0^1 e^{zuv} u^k v^k \mathrm{d}u\mathrm{d}v = \frac{1}{z}\mathcal{M}_k\Big(\frac{1}{z}\Big)e^z + (-1)^{k+1}\frac{k!}{z^{k+1}}\mathcal{E}(z) + \frac{(-1)^k k!}{z^{k+1}} \sum_{j=1}^{k} \frac{1}{j},$$

where

$$\mathcal{M}_k(z) = \sum_{\ell=1}^{k} \sum_{m=0}^{k-\ell} (-1)^{\ell+m+1} \frac{(k-\ell-m+1)_m k!}{(k-\ell+1)!} z^{\ell+m}.$$

Proof of Lemma 2. If $k \neq j$, we expand $\exp(zuv)$ in powers of zuv to get

$$\begin{aligned}\int_0^1 \int_0^1 e^{zuv} u^k v^j \mathrm{d}u\mathrm{d}v &= \sum_{m=0}^{\infty} \frac{z^m}{m!(m+k+1)(m+j+1)} \\ &= \frac{1}{j-k}\left(\sum_{m=0}^{\infty} \frac{z^m}{m!(m+k+1)} - \sum_{m=0}^{\infty} \frac{z^m}{m!(m+j+1)}\right) \\ &= \frac{1}{j-k}\big(I_k - I_j\big),\end{aligned}$$

where

$$I_k := \int_0^1 e^{zt} t^k \mathrm{d}t.$$

To conclude this case, we then use identity (3.2) which enables us to evaluate I_k and I_j.

If $k = j$, we have

$$\int_0^1 \int_0^1 e^{zuv} u^k v^k \mathrm{d}u\mathrm{d}v = -\int_0^1 e^{zt} t^k \log(t) \mathrm{d}t =: -J_k.$$

We have $J_0 = z^{-1}\mathcal{E}(z)$. For $k \geq 1$, by integration by parts, we get

$$J_k = -\frac{1}{z} I_{k-1} - \frac{k}{z} J_{k-1},$$

which we iterate to obtain

$$\begin{aligned} J_k &= \sum_{\ell=1}^{k} \frac{(-1)^\ell}{z^\ell} \cdot \frac{k!}{(k-\ell+1)!} I_{k-\ell} + (-1)^k \frac{k!}{z^k} J_0 \\ &= -\frac{1}{z}\mathcal{M}_k\Big(\frac{1}{z}\Big)e^z + (-1)^k \frac{k!}{z^{k+1}}\mathcal{E}(z) + \frac{(-1)^{k+1}k!}{z^{k+1}} \sum_{j=1}^{k} \frac{1}{j}. \end{aligned}$$

This concludes the proof of the lemma. $\square$

Proof of Proposition 4. We start from the integral

$$I_{n,0}(z) := \int_0^1 \int_0^1 e^{zuv} Q_{n,0}(u) P_n(v) \mathrm{d}u\mathrm{d}v.$$

Expanding the polynomials $Q_{n,0}$ and P_n and using Lemma 2, we see that

$$\begin{aligned} I_{n,0}(z) &= \sum_{k=0}^{n}\sum_{j=0}^{n} q_{k,n,0}p_{j,n} \int_0^1 \int_0^1 e^{zuv} u^k v^j \mathrm{d}u\mathrm{d}v \\ &= \sum_{\substack{j,k=0 \\ j\neq k}}^{n} \frac{q_{k,n,0}p_{j,n}}{j-k} \Big(\frac{1}{z}\mathcal{M}_{j,k}\Big(\frac{1}{z}\Big)e^z + (-1)^{k+1}\frac{k!}{z^{k+1}} + (-1)^j \frac{j!}{z^{j+1}} \Big) \\ &\qquad + \sum_{k=0}^{n} q_{k,n,0}p_{k,n} \Big(\frac{1}{z}\mathcal{M}_k\Big(\frac{1}{z}\Big)e^z + (-1)^{k+1}\frac{k!}{z^{k+1}}\mathcal{E}(z) + \frac{(-1)^k k!}{z^{k+1}} \sum_{j=1}^{k}\frac{1}{j} \Big). \end{aligned}$$

It follows that $z^{n+1}I_{n,0}(z) = \mathcal{A}_n(z)e^z + \mathcal{B}_n(z)\mathcal{E}(z) + \mathcal{C}_n(z)$, where the polynomials $\mathcal{A}_n, \mathcal{B}_n$ and $\mathcal{C}_n$ are as described in Proposition 4. To prove that $z^{n+1}I_{n,0}(z) = R_{n,0}(z)$, we integrate n-times by parts in v, and then n-times by parts in u. $\square$

4. Proof of Theorem 1

The Hermite-Padé approximants constructed in Section 3.1 provide good *functional* simultaneous approximations to the functions $\exp(z)$ and $\mathcal{E}_\alpha(z)$, and, as usual, it is natural to expect that they also provide good *numerical* simultaneous approximations to the values of both functions. In our situation, the transfer is operated by means of Nesterenko's criterion for linear independence of real numbers, that we first recall.

Proposition 5 (Nesterenko's criterion [9]). *Let $\xi_1, \ldots, \xi_N$ denote N real numbers such that there exist N sequences of integers $(p_{j,n})_{n\geq 0}$, $j = 1, \ldots, N$, four positive real numbers τ_1, τ_2, c_1, c_2 and a monotonically increasing function σ (defined on $\mathbb{R}$) satisfying the* *following properties:*

$$(i)\qquad \lim_{t\to+\infty}\sigma(t)=+\infty \quad \textit{and} \quad \limsup_{t\to+\infty}\frac{\sigma(t+1)}{\sigma(t)}=1;$$

$$(ii)\qquad \max_{j=1,\dots,N}|p_{j,n}|\le e^{\sigma(n)};$$

$$(iii)\qquad c_1e^{-\tau_1\sigma(n)}\le\Big|\sum_{j=1}^{N}p_{j,N}\xi_j\Big|\le c_2e^{-\tau_2\sigma(n)}.$$

Then the dimension of the vector space spanned over $\mathbb{Q}$ by $\xi_1,\dots,\xi_N$ is at least $\frac{\tau_1+1}{1+\tau_1-\tau_2}$.

We will also use a quantitative version of the criterion when $\tau_1=\tau_2=N-1$. In that case the dimension is maximal equal to N and for any $\varepsilon>0$, there exists a constant $\eta_\varepsilon>0$ such that for any $(a_1,\dots,a_N)\in\mathbb{Z}^N\setminus\{(0,\dots,0)\}$, we have

$$\Big|\sum_{j=1}^{N}a_j\xi_j\Big|\ge\frac{\eta_\varepsilon}{\max\limits_{j=1,\dots,N}|a_j|^{N-1+\varepsilon}}. \tag{4.1}$$

This is a consequence of the theorem stated on page 72 of [9], which in fact encompasses Proposition 5.

To apply the proposition and (4.1), we need two lemmas. The first one is used for case (i) of Theorem 1 whereas the second is used in case (ii). Set $d_n:=\mathrm{lcm}(1,2,\dots,n)$.

Lemma 3. *Let $\alpha=a/b\in\mathbb{Q}\setminus\mathbb{Z}$, $\alpha>-1$, $b\ge1$ $z=u/v\in\mathbb{Q}^*$.*

(i) The numbers

$$b^{3n}v^nd_{bn+|a|}A_n(z),\ b^{3n}v^{n+1}d_{bn+|a|}B_n(z),\ b^{3n}v^nd_{bn+|a|}C_n(z)$$

are integers.

(ii) For all large enough n, we have $\max(|A_n(z)|,|B_n(z)|,|C_n(z)|)\le c_3^nn!$, for some $c_3>0$ that depends on α and z.

(iii) We have $R_{n,\alpha}(z)=c_4^{n(1+o(1))}/n!^2$, where $c_4:=16z^3/81$.

Lemma 4. *Set $z=u/v\in\mathbb{Q}^*$.*

(i) The numbers

$$v^nd_n\mathcal{A}_n(z),\ v^{n+1}\mathcal{B}_n(z),\ v^nd_n\mathcal{C}_n(z)$$

are integers.

(ii) For all large enough n, we have $\max(|\mathcal{A}_n(z)|,|\mathcal{B}_n(z)|,|\mathcal{C}_n(z)|)\le c_5^nn!$, for some $c_5>0$ that depends on z.

(iii) We have $R_{n,0}(z)=c_4^{n(1+o(1))}/n!^2$, where $c_4:=16z^3/81$.

We give only the proof of Lemma 3, that of Lemma 4 being totally similar.

Proof of Lemma 3. (*i*) This is immediate using the expression (3.4) for $I_{n,\alpha}(z)$, and $p_{k,n} \in \mathbb{Z}$ and $v^{2n} q_{k,n} \in \mathbb{Z}$ (the latter because $v^{2n}\binom{2n+k+u/v}{n} \in \mathbb{Z}$).

(*ii*) Again, this is immediate using the expression (3.4). Indeed, the coefficients $p_{k,n}$ and $q_{k,n,\alpha}$ of the polynomials P_n and Q_n are uniformely bounded (for $k = 0, \ldots, n$) by c_6^n for some constant c_6 that depends only on α.

(*iii*) An application of Laplace's method to the integral expression (3.1) for $R_n(z)$ shows that

$$\lim_{n\to+\infty} (n!^2 R_n(z))^{1/n} = z^3 \max_{(u,v)\in[0,1]^2} \big(u^2(1-u)v^2(1-v)\big) = \frac{16z^3}{81}.$$

(The fact that α and z are real is used here.) □

Proof of Theorem 1. We only prove (*i*), since (*ii*) is proved in a similar fashion. First, we remark that the restriction that $\alpha > -1$ in Lemma 3 is inessential: we can remove it, provided we assume n is large enough, say $n \geq N(\alpha)$, which is of course possible in the lemma and in Proposition 5.

For $n \geq N(\alpha)$, we construct a sequence of linear forms

$$\ell_n = a_n e^z + b_n \mathcal{E}_\alpha(z) + c_n$$

with $a_n, b_n, c_n \in \mathbb{Z}$ by setting

$$\ell_n = b^{3n} v^{n+1} d_{bn+|a|} R_n(z), \quad a_n = b^{3n} v^{n+1} d_{bn+|a|} A_n(z)$$
$$b_n = b^{3n} v^{n+1} d_{bn+|a|} B_n(z), \quad c_n = b^{3n} v^{n+1} d_{bn+|a|} C_n(z).$$

Since $d_n = e^{n(1+o(1))}$, the various estimates in Lemma 3 show that we can apply Proposition 5 with $\sigma(n) = \log(n!) = n\log(n)(1+o(1))$ and $\tau_1 = \tau_2 = 2$. (The exact values of $c_1, c_2 > 0$ are not important.) It follows that the dimension of the vector space spanned over $\mathbb{Q}$ by 1, e^z and $\mathcal{E}_\alpha(z)$ is exactly 3.

Recall (2.1),i.e., that

$$\Gamma(\alpha+1)/z^{\alpha+1} = \mathcal{E}_\alpha(-z) + e^{-z}\mathcal{G}_{\alpha+1}(z).$$

Since $\mathcal{E}_\alpha(-z)$ and e^{-z} are $\mathbb{Q}$-linearly independent, at least one $\Gamma(\alpha+1)/z^{\alpha+1}$ and $\mathcal{G}_{\alpha+1}(z)$ is irrational for any $z \in \mathbb{Q}^*$, $z > 0$ and any $\alpha \in \mathbb{Q}\setminus\mathbb{Z}$. We now prove a quantitative version of this statement. (We change α to $\alpha - 1$ for simplicity.) Indeed, we are in a situation where we can use the linear independence measure (4.1): for any integers p, q, r not all zero and any $\varepsilon > 0$, we have

$$\big|p + qe^{-z} + r\mathcal{E}_{\alpha-1}(-z)\big| \geq \frac{c_7}{H^{2+\varepsilon}}, \tag{4.2}$$

where $H = \max(|p|, |q|, |r|)$ and c_7 depends on α, ε, z.

We claim this implies that, for any integers p, q, r not all zero and any $\varepsilon > 0$,

$$|q\Gamma(\alpha)/z^\alpha - p| + |q\mathcal{G}_\alpha(z) - r| \geq \frac{c_8}{H^{2+\varepsilon}}, \tag{4.3}$$

where $c_8 = c_7/(1+e^{-z})$. To get a contradiction, let us assume we can find some integers p', q', r' not all zero and an $\varepsilon > 0$ such that

$$|q'\Gamma(\alpha)/z^\alpha - p'| + |q'\mathcal{G}_\alpha(z) - r'| < \frac{c_8}{\widetilde{H}^{2+\varepsilon}},$$

where $\widetilde{H} = \max(|p'|, |q'|, |r'|)$. Hence

$$|q'\Gamma(\alpha)/z^\alpha - p'| < \frac{c_8}{\widetilde{H}^{2+\varepsilon}}$$

and

$$|q'e^{-z}\mathcal{G}_\alpha(z) - r'e^{-z}| < \frac{c_8 e^{-z}}{\widetilde{H}^{2+\varepsilon}}.$$

On the other hand, by (4.2),

$$\begin{aligned}\frac{c_7}{\widetilde{H}^{2+\varepsilon}} &\leq \left|-p' + r'e^{-z} + q'\mathcal{E}_{\alpha-1}(-z)\right| = \left|-p' + r'e^{-z} + q'\big(\Gamma(\alpha)/z^\alpha - \mathcal{G}_\alpha(z)\big)\right| \\ &\leq |q'\Gamma(\alpha)/z^\alpha - p'| + e^{-z}\,|q'\mathcal{G}_\alpha(z) - r'| < \frac{c_8(1+e^{-z})}{\widetilde{H}^{2+\varepsilon}} = \frac{c_7}{\widetilde{H}^{2+\varepsilon}}.\end{aligned}$$

This is a contradiction, and thus (4.3) holds, which is the inequality (1.1) in disguise with $c(\alpha, \varepsilon, z) = c_8$.

Inequality (4.3) quantifies the assertion "at least one of $\Gamma(\alpha)/z^\alpha$ and $\mathcal{G}_\alpha(z)$ is irrational". Indeed, if $\Gamma(\alpha)/z^\alpha$ or $\mathcal{G}_\alpha(z)/z$ is rational, say $\Gamma(\alpha)/z^\alpha = p_0/q_0 \in \mathbb{Q}^*$ to simplify, we set $p = p_0 q'$, $q = q_0 q'$ and $r = q_0 p'$ in Inequality (4.3) for any integers $p', q' \neq 0, r'$. In particular,

$$\left|\frac{\Gamma(\alpha)}{z^\alpha} - \frac{p}{q}\right| = 0.$$

Consequently,

$$\frac{c_8}{H^{3+\varepsilon}} \leq \left|\frac{\Gamma(\alpha)}{z^\alpha} - \frac{p}{q}\right| + \left|\mathcal{G}_\alpha(z) - \frac{r}{q}\right| = \left|\mathcal{G}_\alpha(z) - \frac{p'}{q'}\right|$$

and $H := \max(|p|, |q|, |r|) = \max(|p_0 q'|, |q_0 q'|, |q_0 p'|) \leq \max(|p_0|, |q_0|) \cdot \widehat{H}$, where $\widehat{H} := \max(|p'|, |q'|)$. Hence, setting $c_9 = c_8 \max(|p_0|, |q_0|)^{3+\varepsilon}$, for any integers $p', q' \neq 0$, we have

$$\left|\mathcal{G}_\alpha(z) - \frac{p'}{q'}\right| \geq \frac{c_9}{\widehat{H}^{3+\varepsilon}},$$

which shows that $\mathcal{G}_\alpha(z)$ is an irrational number (and even a non-Liouville number). □

5. Proof of Theorem 2

(*i*) For any $\alpha \notin \mathbb{Z}$, the function $\exp(z)$ and $\mathcal{E}_\alpha(z)$ are algebraically independent over $\mathbb{C}(z)$ ([10, p. 191, Lemma 7]) and both functions satisfy the homogeneous linear differential system:

$$\begin{cases} y_1' = y_1 \\ y_2' = \dfrac{1}{z}y_1 - \dfrac{\alpha+1}{z}y_2. \end{cases} \tag{5.1}$$

If α is an algebraic and non-integer, Shidlovskii's classical theorem on E-functions ([10, p. 192, Theorem 3]) yields that for any algebraic number $z \neq 0$, the numbers $\mathcal{E}_\alpha(-z)$ and $\exp(-z)$ are algebraically independent over $\mathbb{Q}$.

We now use identity (2.1) to deduce that, for any $\alpha \in \overline{\mathbb{Q}} \backslash \mathbb{Z}$ and any $z \in \overline{\mathbb{Q}}^*$, $z \notin (-\infty, 0]$. the field generated over $\mathbb{Q}$ by the numbers

$$\Gamma(\alpha)/z^\alpha, \quad e^z, \quad \mathcal{G}_\alpha(z)$$

has transcendence degree at least 2. This is the content of Theorem 2(i).

(ii) Although this is not proved in [10], the functions $\exp(z)$ and $\mathcal{E}(z)$ are algebraically independent over $\mathbb{Q}(z)$. Since they satisty the inhomogeneous linear differential system

$$\begin{cases} y_1' = y_1 \\ y_2' = \frac{1}{z} y_1 - \frac{1}{z} \end{cases}$$

we can thus apply Shidlovskii's Second Fundamental Theorem ([10, p. 123]) to deduce that for any $z \in \overline{\mathbb{Q}}^*$, the numbers $\mathcal{E}(-z)$ and $\exp(-z)$ are algebraically independent over $\mathbb{Q}$. Together with identity (2.2), this immediately implies Theorem 2(ii).

6. A sequence of linear forms in $1, \Gamma(\alpha)/z^\alpha$ and $\mathcal{G}_\alpha(z)$

In this section, we construct an explicit sequence of linear forms

$$L_n(\alpha, z) \in \mathbb{Z} + \mathbb{Z}\Gamma(\alpha)/z^\alpha + \mathbb{Z}\mathcal{G}_\alpha(z)$$

that tends to 0 as $n \to +\infty$ under the assumptions that $z \in \mathbb{Q}^*$, $z > 0$, and $\alpha \in \mathbb{Q} \setminus \mathbb{Z}$.

The principle of the construction is simple and was already used in [11, 12] (for a different purpose however). We consider simultaneously $R_{n,\alpha}(-z)$ and $R_{n+1,\alpha}(-z)$ and define the five determinants

$$S_n(z) = \begin{vmatrix} A_n(-z) & R_{n,\alpha}(-z) \\ A_{n+1}(-z) & R_{n+1,\alpha}(-z) \end{vmatrix}, \quad T_n(z) = \begin{vmatrix} R_{n,\alpha}(-z) & B_n(-z) \\ R_{n+1,\alpha}(-z) & B_{n+1}(-z) \end{vmatrix},$$

$$U_n(z) = \begin{vmatrix} A_n(-z) & C_n(-z) \\ A_{n+1}(-z) & C_{n+1}(-z) \end{vmatrix}, \quad V_n(z) = \begin{vmatrix} A_n(-z) & B_n(-z) \\ A_{n+1}(-z) & B_{n+1}(-z) \end{vmatrix}$$

and

$$W_n(z) = \begin{vmatrix} C_n(-z) & B_n(-z) \\ C_{n+1}(-z) & B_{n+1}(-z) \end{vmatrix}$$

Clearly, U_n, V_n, W_n are polynomials in z of degree at most $2n+2$, with coefficients in $\mathbb{Q}(\alpha)$. Furthermore, we have the relations

$$\begin{cases} V_n(z)\mathcal{E}_\alpha(-z) + U_n(z) = S_n(z) = \mathcal{O}(z^{3n+1}) \\ \\ V_n(z)e^{-z} + W_n(z) = T_n(z) = \mathcal{O}(z^{3n+1}). \end{cases}$$

(These functional approximations almost provide the diagonal simultaneous Padé approximants of type II for the functions $\exp(z)$ and $\mathcal{E}_\alpha(z)$.)

We now use Eq. (2.1) in the form

$$\mathcal{E}_\alpha(-z) = \Gamma(\alpha+1)/z^{\alpha+1} - e^{-z}\mathcal{G}_{\alpha+1}(z)$$

so that

$$\begin{cases} S_n(z) = V_n(z)\Gamma(\alpha+1)/z^{\alpha+1} - V_n(z)e^{-z}\mathcal{G}_{\alpha+1}(z) + U_n(z) \\ \\ T_n(z) = V_n(z)e^{-z} + W_n(z), \end{cases}$$

from which we finally obtain that

$$S_n(z) + \mathcal{G}_{\alpha+1}(z)T_n(z) = V_n(z)\Gamma(\alpha+1)/z^{\alpha+1} + W_n(z)\mathcal{G}_{\alpha+1}(z) + U_n(z). \tag{6.1}$$

The estimates given in Lemma 3 show that there exist some constants c_{10} and c_{11} (depending on α and z) such that

$$|S_n(z) + \mathcal{G}_{\alpha+1}(z)T_n(z)| \leq \frac{c_{10}^n}{n!},$$

and, when $z > 0$ and $\alpha \notin \mathbb{Z}$ are rational numbers, the common denominator D_n of the coefficients of $V_n(z)$, $W_n(z)$ and $U_n(z)$ is bounded by c_{11}^n. Hence

$$L_n(\alpha+1, z) := D_n\big(S_n(z) + \mathcal{G}_{\alpha+1}(z)T_n(z)\big) \in \mathbb{Z} + \mathbb{Z}\Gamma(\alpha+1)/z^{\alpha+1} + \mathbb{Z}\mathcal{G}_{\alpha+1}(z)$$

tends to 0 essentially as fast as $1/n!$ (up to some factor with exponential growth in n). To conclude that at least one of $\Gamma(\alpha+1)/z^{\alpha+1}$ and $\mathcal{G}_{\alpha+1}(z)$ is irrational, it remains to prove that $L_n(\alpha+1, z) \neq 0$ for infinitely many n. As seen in Section 4, this is a consequence of the linear independence of the numbers $\exp(z)$ and $\mathcal{E}_\alpha(z)$ over $\mathbb{Q}$. This is not an easy task if we don't want to remember this fact. In principal, we could explictly compute the recurrence satisfied by A_n, B_n, C_n, R_n, then deduce it is satisfied by S_n, T_n, U_n, V_n, W_n and find the exact asymptotic behavior of $zS_n(-z) + \mathcal{G}_{\alpha+1}(z)T_n(-z)$ by means of Birkhoff-Trjitzinski theory. A similar construction of sequences of linear forms in $\gamma + \log(z)$ and $\mathcal{G}_0(z)$ can be done.

7. Connexion with Mahler's paper [8]

In the Introduction, we mentioned that Theorem 2 is related with Mahler's article [8], where he says: "the results proved in this paper are quite trivial consequences of Shidlovski's work, and they do not even imply the irrationality of γ or of $\zeta(3)$. However, they deserve perhaps a little interest because, up to now, nothing was known about the arithmetic of these constants". Mahler's comment refers to his remark that the number $\frac{\pi Y_0(2)}{2J_0(2)} - \gamma$ and other similar numbers are transcendental, but it could certainly be applied to our Theorem 2. Note that [8] was published in 1967, many years before Apéry's proof of the irrationality of $\zeta(3)$.

On the last five lines of [8], he mentions without proof the following theorem: *For $z \in \overline{\mathbb{Q}}^*$, integer $k \geq 0$ and rational number $\alpha > -1$, any finite number of integrals*

$$\int_0^1 t^\alpha \log(t)^k e^{-zt}\mathrm{d}t \tag{7.1}$$

are algebraically independent over $\mathbb{Q}$. Clearly, this contains as particular case the algebraic independence over $\mathbb{Q}$ of the numbers $\exp(z)$ and $\mathcal{E}_\alpha(z)$, respectively of the numbers $\exp(z)$ and $\mathcal{E}(z)$ in the above conditions. Although Mahler did not give a proof, it is clear that is was based on the observation that the integral in (7.1) is an E-function (of the variable z) very similar to $\mathcal{E}_\alpha$ and $\mathcal{E}$.

As an application of Mahler's result, we mention a generalisation of Theorem 2(ii): *For any* $z \in \overline{\mathbb{Q}}$, $z \notin (-\infty, 0]$, *and any integer* $s \geq 1$, *the transcendence degree of the field generated over* $\mathbb{Q}$ *by*

$$\Gamma^{(s)}(1),\ \log(z),\ e^z,\ \int_0^\infty \log(t+z)^s e^{-t}\mathrm{d}t$$

is at least 2. *In particular, at least one of* $\Gamma^{(s)}(1) = \int_0^\infty \log(t)^s e^{-t}\mathrm{d}t$ *and* $\int_0^\infty \log(1+t)^s e^{-t}\mathrm{d}t$ *is transcendental.*

The proof amounts to the observation that

$$\begin{aligned}\Gamma^{(s)}(1) &= z\int_0^1 \log(tz)^s e^{-zt}\mathrm{d}t + \int_z^\infty \log(t)^s e^{-t}\mathrm{d}t \\ &= z\sum_{j=0}^{s}\binom{s}{j}\log(z)^{s-j}\int_0^1 \log(t)^j e^{-tz}\mathrm{d}t + e^{-z}\int_0^\infty \log(t+z)^s e^{-t}\mathrm{d}t\end{aligned}$$

for any $z \in \mathbb{C} \setminus (-\infty, 0]$, at which point we can use Mahler's result. We conclude by mentioning that, for any integer $s \geq 1$, $\Gamma^{(s)}(1)$ can be expressed as a polynomial in $\gamma, \zeta(2), \zeta(3), \ldots, \zeta(s)$ with rational coefficients (see [11, eq. (3.1)] for a precise statement). For example, $\Gamma'(1) = -\gamma$, $\Gamma''(1) = \zeta(2) + \gamma^2$, $\Gamma'''(1) = -2\zeta(3) - 3\gamma\zeta(2) - \gamma^3$.

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