Rational approximations of values of the Gamma function at rational points

Tanguy Rivoal

Institut Fourier, CNRS and Université Grenoble 1

Conference "Approximation et extrapolation des suites et des séries convergentes et divergentes", CIRM, 28/09–02/10/2009

Computation of a given a real number α.

1) **Numerical** point of view: find a sequence of rational numbers $(\frac{u_n}{v_n})_{n \geq 0}$ such that

$$\lim_{n \to +\infty} \frac{u_n}{v_n} = \alpha$$

with fast convergence and u_n/v_n reasonably easy to compute.

2) **Diophantine** point of view: find two sequences of integers $(u_n)_{n \geq 0}$ and $(v_n)_{n \geq 0}$ such that

$$\lim_{n \to +\infty} (v_n \alpha - u_n) = 0.$$

If furthermore $\forall n \ v_n \alpha - u_n \neq 0$, then $\alpha \notin \mathbb{Q}$.

Aim of this talk: to present rational approximations of the numbers $\Gamma(a/b)$, where $a/b \in \mathbb{Q}$ and Γ denotes the usual Gamma function.

A similar method enables us to get rational approximations of the numbers $\gamma + \log(x)$, where γ is Euler's constant and $x > 0, x \in \mathbb{Q}$.

None of these approximations are good enough to satisfy 2).

From the point of view of 1), they are new. The rational approximations are solutions of a **linear recurrence of finite order with polynomial coefficients**.

$\forall x > 0$ and $\forall \alpha \in \mathbb{C}$, consider the linear recurrence of order 3:

$$C_3(n,\alpha,x)\mathbf{U}_{n+3}+C_2(n,\alpha,x)\mathbf{U}_{n+2}+C_1(n,\alpha,x)\mathbf{U}_{n+1}+C_0(n,\alpha,x)\mathbf{U}_n = 0 \tag{1}$$

where the coefficients $C_j(n,\alpha,x)$, $j = 0,1,2,3$, are polynomials in n,α,x, of degree 16 in n, whose expressions are

$$C_3(n, \alpha, x) = -(n+3)^5(n+4)^2(8n^2+4\alpha n-3xn+38n-6x-\alpha x+10\alpha+44)$$
$$(n + 2)(8n^2 + 22n + 4\alpha n - 3xn + 6\alpha - 3x - \alpha x + 14)^2$$
$$(8n^2 + 38n + 4\alpha n - 3xn + 10\alpha - 6x - \alpha x + 44)$$

$$C_2(n, \alpha, x) =$$
$$(24n^5+7xn^4+28\alpha n^4+330n^4-6x^2n^3+91xn^3+1794n^3+310\alpha n^3+7\alpha xn^3$$
$$+70x\alpha n^2+13x\alpha^2 n^2-5x^2\alpha n^2-4\alpha^3 n^2+4824n^2-6\alpha^2 n^2-45x^2 n^2+418xn^2$$
$$+1272\alpha n^2+576x-25x^2\alpha n+218\alpha xn+79x\alpha^2 n-26\alpha^3 n+5\alpha^3 xn-4x^2\alpha^2 n$$
$$+816xn+2296\alpha n+6420n-111x^2 n-30\alpha^2 n+3384+1540\alpha+576x+216\alpha x$$
$$- \alpha^3 x^2 + 16\alpha^3 x - 10x^2\alpha^2 - 31x^2\alpha + 116x\alpha^2 - 40\alpha^3 - 90x^2 - 36\alpha^2)$$
$$(n + 3)^4(n + 2)(-8n^2\alpha x - 4\alpha n - 38n + 3xn6x - 44 - 10\alpha)$$
$$(-8n^2 - 22n - 4\alpha n + 3xn - 14 - 6\alpha + 3x + \alpha x)^2$$

$$C_1(n, \alpha, x) =$$
$$-(24n^4 - 57xn^3 + 20\alpha n^3 + 186n^3 - 38x\alpha n^2 + 518n^2 + 4\alpha^2 n^2 + 26x^2 n^2 - 315xn^2$$
$$+120\alpha n^2 + 13x^2\alpha n - 148\alpha xn - 5x\alpha^2 n - 543xn + 232\alpha n + 610n + 85x^2 n + 14\alpha^2 n$$
$$-3x^3 n + 254 + 144\alpha - 285x - 138\alpha x - x^3\alpha + x^2\alpha^2 + 24x^2\alpha - 9x\alpha^2 + 59x^2 + 10\alpha^2$$
$$-3x^3)(n+3)^2(8n^2 + 22n + 4\alpha n - 3xn + 14 + 6\alpha - 3x - \alpha x)$$
$$(8n^2 + 4\alpha n + 54n - 3xn - \alpha x + 14\alpha - 9x + 90)(n - \alpha + 2)(n + 2 + \alpha)^2$$
$$(8n^2 - 3xn + 38\alpha xn + 4\alpha n + 10\alpha - 6x + 44)(n + 2)$$

$$C_0(n, \alpha, x) = (n - \alpha + 1)(n + 1 + \alpha)^2(8n^2 - 3xn + 4\alpha n + 38n\alpha x + 10\alpha - 6x + 44)^2$$
$$(n+3)^2(8n^2 + 22n + 4n\alpha - 3xn + 14 + 6\alpha - 3x - \alpha x)(n - \alpha + 2)(n + 2 + \alpha)^2$$
$$(8n^2 - 3xn + 54n + 4\alpha n + 14\alpha - \alpha x + 90 - 9x).$$

$(P_n(x,\alpha))_{n\geq 0}$ and $(Q_n(x,\alpha))_{n\geq 0}$ solutions of (1), with initial values:

$$P_0(\alpha,x) = x-\alpha-2, \; P_1(\alpha,x) = \frac{1}{4}\Big((1-\alpha)x^2+(6\alpha+2\alpha^2-4)x-\alpha^3-4-9\alpha-6\alpha^2\Big)$$

$$P_2(\alpha,x) = \frac{1}{36}\Big((-3\alpha+\alpha^2+2)x^3+(-3\alpha^3+30-9\alpha^2)x^2$$

$$+(-108+33\alpha^2+24\alpha^3+3\alpha^4-60\alpha)x-12\alpha^4-\alpha^5-24-90\alpha^2-49\alpha^3-76\alpha\Big)$$

$$Q_0(\alpha,x) = x-2, \quad Q_1(\alpha,x) = \frac{1}{4}\Big((1+\alpha^2+2\alpha)x^2+(-10\alpha-6\alpha^2-4)x-4+4\alpha^2\Big)$$

$$Q_2(\alpha,x) = \frac{1}{72}\Big((\alpha^4+12\alpha+13\alpha^2+4+6\alpha^3)x^3+(-60\alpha^3-12\alpha^4-96\alpha^2-48\alpha)x^2$$

$$+(-336\alpha-216+84\alpha^3+30\alpha^4-66\alpha^2)x+60\alpha^2-12\alpha^4-48\Big)$$

Theorem 1 (R, 2009). (i) $P_n(\alpha, x), Q_n(\alpha, x) \in \mathbb{Q}[\alpha, x]$.

(ii) $\forall \alpha = u/v \in \mathbb{Q}$ and $\forall x = a/b \in \mathbb{Q}$,

$$n!^2(n+1)!^2 v^{2n+1} b^{4n-1} P_n(\alpha, x) \in \mathbb{Z}, \quad n!^2(n+1)!^2 v^{3n+2} b^{4n-1} Q_n(\alpha, x) \in \mathbb{Z}.$$

(iii) $\forall x > 0$ and $\forall \alpha \in \mathbb{C} \setminus \{0\}$, $\mathsf{Re}(\alpha) > -1$, $\exists\, s(\alpha, x) \neq 0$ and $q(\alpha, x) \neq 0$ such that

$$\left| Q_n(\alpha, x) \Gamma(\alpha+1) - P_n(\alpha, x) x^\alpha \right| \leq \frac{s(\alpha, x)}{n^{2-2\mathsf{Re}(\alpha)/3}} \exp\left(-\frac{3}{2} x^{1/3} n^{2/3} + \frac{1}{2} x^{2/3} n^{1/3} \right)$$

and

$$Q_n(\alpha, x) \sim \frac{q(\alpha, x)}{n^{2-2\alpha/3}} \exp\left(3 x^{1/3} n^{2/3} - x^{2/3} n^{1/3} \right).$$

Example. Define $(p_n)_{n\geq 0}$ and $(q_n)_{n\geq 0}$ to be the solutions of

$$64(8n+17)(n+4)^2(8n+9)(n+3)^3(n+2)U_{n+3}$$

$$-16(8n+9)(n+2)(24n^2+123n+155)(2n+7)^2(n+3)^2U_{n+2}$$

$$+4(2n+7)(n+2)(8n+25)(48n^3+158n^2+147n+32)(2n+5)^2U_{n+1}$$

$$-(8n+25)(8n+17)(2n+1)(2n+7)(2n+5)^2(2n+3)^2U_n = 0$$

with $p_0 = \frac{3}{2}$, $p_1 = \frac{81}{32}$, $p_2 = \frac{2185}{384}$, $q_0 = 1$, $q_1 = \frac{45}{16}$, $q_2 = \frac{825}{128}$. Then

$$\lim_{n\to+\infty} \frac{p_n}{q_n} = \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

Idea of the proof.

- Euler's functions: For $z \notin [0, +\infty)$ and $\mathrm{Re}(\beta) > -1$, set

$$\mathscr{F}_\beta(z) := \int_0^\infty \frac{t^\beta e^{-t}}{z - t}\, \mathrm{d}t \sim \sum_{k=1}^\infty \frac{\Gamma(\beta + k)}{z^k}.$$

- Laguerre type polynomials of degree $2n$ in x:

$$A_{n,\alpha}(x) := \frac{1}{n!^2}\, e^x \left(x^{n-\alpha} (e^{-x} x^{n+\alpha})^{(n)} \right)^{(n)} \in \mathbb{Q}[\alpha, x].$$

They are orthogonal on $(0, +\infty)$ for the **two** weights e^{-x} and $x^\alpha e^{-x}$. $(\alpha \neq 0)$

Simultaneous type II Padé approximants at $z = \infty$ **to** $\mathscr{F}_0(z)$ **and** $\mathscr{F}_\alpha(z)$**.**

Lemma 1. *For* $\beta \in \{0, \alpha\}$, $\alpha \in \mathbb{C} \backslash \{0\}$, $\mathrm{Re}(\alpha) > -1$ *and* $z \in \mathbb{C} \backslash [0, +\infty)$,

$$R_{n,\alpha,\beta}(z) := \int_0^\infty \frac{A_{n,\alpha}(t)}{z-t} t^\beta e^{-t}\, \mathrm{d}t = A_{n,\alpha}(z)\mathscr{F}_\beta(z) - B_{n,\alpha,\beta}(z)$$

$$\sim \sum_{k=1}^\infty \frac{(k-\beta-n)_n(k-\alpha+\beta-n)_n}{n!^2} \cdot \frac{\Gamma(\beta+k)}{z^k} = \mathcal{O}\Big(\frac{1}{z^{n+1}}\Big).$$

Here,

$$B_{n,\alpha,\beta}(z) := \int_0^\infty \frac{A_{n,\alpha}(z) - A_{n,\alpha}(t)}{z-t} t^\beta e^{-t}\, \mathrm{d}t \in \Gamma(1+\beta)\mathbb{Q}[\alpha, z]$$

is of degree at most $2n - 1$ in z.

Crucial fact: $B_{n,\alpha,0}(z) \in \mathbb{Q}[\alpha, z]$ and $B_{n,\alpha,\alpha}(z) = \Gamma(1 + \alpha)C_{n,\alpha}(z)$ for some $C_{n,\alpha}(z) \in \mathbb{Q}[\alpha, z]$.

For $\beta \in \mathbb{C} \setminus \{0\}$, take z^β to be given by its principal value whenever $-\pi < \arg(z) < \pi$.

Set

$$\mathscr{F}(z) := z^\alpha \mathscr{F}_0(z) - \mathscr{F}_\alpha(z) = \int_0^\infty \frac{z^\alpha - t^\alpha}{z - t}\, e^{-t}\, \mathrm{d}t,$$

$$R_{n,\alpha}(z) := z^\alpha R_{n,\alpha,0}(z) - R_{n,\alpha,\alpha}(z) = \int_0^\infty \frac{z^\alpha - t^\alpha}{z - t}\, A_{n,\alpha}(t)e^{-t}\, \mathrm{d}t,$$

which are both analytic on $\mathbb{C} \setminus (-\infty, 0]$.

Obviously,

$$R_{n,\alpha}(z) = A_{n,\alpha}(z)\mathscr{F}(z) + \Gamma(1+\alpha)C_{n,\alpha}(z) - z^{\alpha}B_{n,\alpha,0}(z).$$

To "remove" the term $A_{n,\alpha}(z)\mathscr{F}(z)$, set

$$P_n(\alpha, z) = \begin{vmatrix} A_{n,\alpha}(z) & B_{n,\alpha,0}(z) \\ A_{n+1,\alpha}(z) & B_{n+1,\alpha,0}(z) \end{vmatrix} \in \mathbb{Q}[\alpha, z]$$

$$Q_n(\alpha, z) = \begin{vmatrix} A_{n,\alpha}(z) & C_{n,\alpha}(z) \\ A_{n+1,\alpha}(z) & C_{n+1,\alpha}(z) \end{vmatrix} \in \mathbb{Q}[\alpha, z],$$

$$S_n(\alpha, z) = \begin{vmatrix} A_{n,\alpha}(z) & R_{n,\alpha}(z) \\ A_{n+1,\alpha}(z) & R_{n+1,\alpha}(z) \end{vmatrix}.$$

Lemma 2. (i) *For any* $z \in \mathbb{C} \setminus (-\infty, 0]$, *we have*

$$S_n(\alpha, z) = Q_n(\alpha, z)\Gamma(1 + \alpha) - z^\alpha P_n(\alpha, z)$$

where

$$n!^2(n+1)!^2 P_n(\alpha, z) \in \mathbb{Z}[\alpha, z], \qquad n!^2(n+1)!^2 Q_n(\alpha, z) \in \mathbb{Z}[\alpha, z].$$

(ii) $P_n(\alpha, z)$, $Q_n(\alpha, z)$ *and* $S_n(\alpha, z)$ *are solutions of the linear recurrence* (1).

(iii) *The degrees of* Q_n *and* P_n *in* z *are at most* $4n - 1$, *those in* α *at most* $3n + 2$ *and* $2n + 1$ *respectively.*

(i) and (iii) are "easy". (ii) is consequence of the facts

$-$ that the sequences $A_{n,\alpha}$, $B_{n,\alpha,0}$, $B_{n,\alpha,\alpha}$, $R_{n,\alpha,0}$, $R_{n,\alpha,\alpha}$ (hence $R_{n,\alpha}$) are all solutions of the same linear recurrence of order 3, say $\mathcal{R}$, obtained explicitly using Zeilberger's algorithm EKHAD.

$-$ that if a_n and b_n are solutions of a recurrence of order 3:

$$U_{n+3} = p_n U_{n+2} + q_n U_{n+1} + r_n U_n,$$

then the sequence of determinants $a_n b_{n+1} - a_{n+1} b_n$ is solution of the linear recurrence also of order 3:

$$U_{n+3} = -q_{n+1} U_{n+2} - p_n r_{n+1} U_{n+1} + r_n r_{n+1} U_n.$$

Final steps.

Lemma 3. (i) $\forall x > 0$, *the modulus of* $A_{n,\alpha}$, $B_{n,\alpha,0}$, $B_{n,\alpha,\alpha}$ *are bounded by*

$$\frac{u(x,\alpha)}{n^{1-\mathrm{Re}(\alpha)/3}} \exp(3/2 \cdot x^{1/3}n^{2/3} - 1/2 \cdot x^{2/3}n^{1/3}),$$

where $u(x,\alpha)$ *depends on the sequence.*

(ii) $\forall x > 0$, $\exists r(x,\alpha) \neq 0$ *s.t.*

$$R_{n,\alpha}(x) \sim \frac{r(x,\alpha)}{n^{1-\alpha/3}} \exp(-3x^{1/3}n^{2/3} + x^{2/3}n^{1/3}).$$

(i) follows from Birkhoff-Trjitzinsky theory applied to the recurrence $\mathcal{R}$.

(ii) is a consequence of the identity

$$R_{n,\alpha}(x) = \alpha x^\alpha \frac{(1-\alpha)_n (1+\alpha)_n}{n!^2} \int_0^\infty \int_0^\infty \frac{u^n t^{n+\alpha} e^{-t}}{(1+u)^{n+1-\alpha}(x+ut)^{n+1+\alpha}} \, \mathrm{d}t \mathrm{d}u$$

which implies that $R_{n,\alpha}(x) \to 0$ when $n \to 0$, and then we apply Birkhoff-Trjitzinsky theory again to $\mathcal{R}$.

Lemma 3 (both (i) and (ii)) implies that $S_n(\alpha, x) \to 0$ when $n \to +\infty$.

Apply again Birkhoff-Trjitzinsky theory to recurrence (1) to get the bounds in Theorem 1, ie., $\forall x > 0$,

$$\left| Q_n(\alpha, x)\Gamma(\alpha+1) - P_n(\alpha, x)x^\alpha \right| \leq \frac{s(\alpha, x)}{n^{2-2\mathrm{Re}(\alpha)/3}} \exp\left(-\frac{3}{2}x^{1/3}n^{2/3} + \frac{1}{2}x^{2/3}n^{1/3}\right),$$

$$Q_n(\alpha, x) \sim \frac{q(\alpha, x)}{n^{2-2\alpha/3}} \exp\left(3x^{1/3}n^{2/3} - x^{2/3}n^{1/3}\right).$$

Results related to Euler's constant γ.

Theorem 2 (R, 2008, Transactions AMS). *There exists two polynomial solutions $P_n(z)$ and $Q_n(z)$ of a linear recurrence of order 3, of degree at most $n+1$ such that*

(i) $n!^2 P_n(z)$ *and* $n!^2 Q_n(z)$ *belong to* $\mathbb{Z}[z]$.

(ii) $\forall x > 0$, $\exists s(x) \neq 0$ *and* $q(x) \neq 0$ *s.t.*

$$|Q_n(x)\big(\ln(x) + \gamma\big) - P_n(x)| \leq \frac{s(x)}{n^2} \exp(-\frac{3}{2}x^{1/3}n^{2/3} + \frac{1}{2}x^{2/3}n^{1/3}),$$

$$Q_n(x) \sim \frac{q(x)}{n^2} \exp(3x^{1/3}n^{2/3} - x^{2/3}n^{1/3}).$$

Examples

The recurrence

$$(n+3)^2(8n+11)(8n+19)U_{n+3}$$

$$= (24n^2 + 145n + 215)(8n+11)U_{n+2}$$

$$- (24n^3 + 105n^2 + 124n + 25)(8n+27)U_{n+1}$$

$$+ (n+2)^2(8n+19)(8n+27)U_n$$

provides two sequences of rational numbers $(p_n)_{n \geq 0}$ and $(q_n)_{n \geq 0}$ with $p_0 = -1$, $p_1 = 4$, $p_2 = 77/4$ and $q_0 = 1, q_1 = 7, q_2 = 65/2$ such that

$$\frac{p_n}{q_n} \to \gamma.$$

The first such recurrence for γ was obtained by Aptekarev in 2007.

The recurrence

$$(n+1)(n+2)(n+3)U_{n+3} = (3n^2 + 19n + 29)(n+1)U_{n+2}$$

$$- (3n^3 + 6n^2 - 7n - 13)U_{n+1} + (n+2)^3 U_n$$

provides two sequences of rational numbers $(p_n)_{n \geq 0}$ and $(q_n)_{n \geq 0}$ with $p_0 = -1$, $p_1 = 11$, $p_2 = 71$ and $q_0 = 0, q_1 = 8, q_2 = 56$ such that

$$\frac{p_n}{q_n} \to \ln(2) + \gamma.$$

The construction is based on

- Euler type functions

$$\int_0^\infty \frac{\log(t)^s e^{-t}}{z-t}\, \mathrm{d}t \sim \sum_{k=1}^\infty \frac{\Gamma^{(s)}(k)}{z^k}.$$

- Laguerre type polynomials with $\alpha = 0$:

$$A_{n,0}(x) = \frac{1}{n!^2}\, e^x \Big(x^n (e^{-x} x^n)^{(n)} \Big)^{(n)} \in \mathbb{Q}[x],$$

orthogonal on $(0, +\infty)$ for the weigths e^{-x} and $\log(x)e^{-x}$. (**Remember that the case $\alpha = 0$ was excluded.**)

This construction can be viewed as a limiting case of our results for $\Gamma(1 + \alpha)$ because

$$\lim_{\alpha \to 0} \frac{\Gamma(1 + \alpha) - 1}{\alpha} = \Gamma'(1) = -\gamma.$$