On Galochkin's characterization of hypergeometric G-functions

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Abstract

G-functions are power series in $\overline{\mathbb{Q}}[[z]]$ solutions of linear differential equations, and whose Taylor coefficients satisfy certain (non)-archimedean growth conditions. In 1929, Siegel proved that every generalized hypergeometric series ${}_{q+1}F_q$ with rational parameters are G-functions, but rationality of parameters is in fact not necessary for an hypergeometric series to be a G-function. In 1981, Galochkin found necessary and sufficient conditions on the parameters of a ${}_{q+1}F_q$ series to be a non polynomial G-function. His proof used specific tools in algebraic number theory to estimate the growth of the denominators of the Taylor coefficients of hypergeometric series with algebraic parameters. In this paper, we give a different proof using methods from the theory of arithmetic differential equations, in particular the André-Chudnovsky-Katz Theorem on the structure of the non-zero minimal differential equation satisfied by any given G-function, which is Fuchsian with rational exponents.

1 Introduction

Siegel [10] defined a G-function as any power series $F(z) = \sum_{n=0}^{\infty} A_n z^n \in \overline{\mathbb{Q}}[[z]]$ such that

(i) $F(z)$ is solution of a linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$;

(ii) there exists $C > 0$ such that for all $n \geq 0$, $\overline{|A_n|} \leq C^{n+1}$.

(iii) there exists $D > 0$ such that for all $n \geq 0$, $\mathrm{den}(A_0, A_1, \ldots, A_n) \leq D^{n+1}$.

Here, $\overline{|x|}$ denotes the maximum modulus of the Galoisian conjuguates of a non-zero algebraic x, and given m algebraic numbers $x_1, \ldots, x_m$, $\mathrm{den}(x_1, \ldots, x_m)$ is the smallest integer ≥ 1 such that $\mathrm{den}(x_1, \ldots, x_m)x_j$ is an algebraic integer for all $j \in \{1, \ldots, m\}$. Siegel also defined an E-function as a power series $\sum_{n=0}^{\infty} \frac{A_n}{n!} z^n \in \overline{\mathbb{Q}}[[z]]$ such that $\sum_{n=0}^{\infty} A_n z^n$ is a G-function. In fact, for E-functions, he considered weaker assumptions, with C^{n+1} and D^{n+1} both replaced by $n!^{\varepsilon}$ for any fixed $\varepsilon > 0$ provided n is large enough with respect to ε. It is believed that these two possible classes of E-functions are identical; see [2, p. 715].

G-functions form a subring of $\mathbb{C}[[z]]$, stable by $\frac{d}{dz}$ and $\int_0^z$, and in fact a differential $\overline{\mathbb{Q}}$-algebra. The first interesting examples of G-functions are algebraic functions over $\overline{\mathbb{Q}}(z)$, holomorphic at $z = 0$. Other important G-functions, like polylogarithms $\sum_{n=1}^{\infty} \frac{z^n}{n^s}$, $s \in \mathbb{Z}$, are obtained as specializations of the generalized hypergeometric series

$$
{}_pF_q \left(\begin{matrix} \alpha_1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_q \end{matrix} ; z \right) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(1)_n (\beta_1)_n \cdots (\beta_q)_n} \, z^n \tag{1.1}
$$

where $p, q \geq 0$, $\alpha_1, \ldots, \alpha_p \in \mathbb{C}$ and $\beta_1, \ldots, \beta_q \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. We also assume without loss of generality that $\alpha_j \neq \beta_k$ for all j, k, because if $\alpha_j = \beta_k$ we can simply replace $(\alpha_j)_n / (\beta_k)_n$ by 1; in turn, this assumption is important in Theorem 1 below, that would be false without it. Under these conditions, Siegel proved that if $p = q + 1$ and if the α's and β's are all rational numbers, then the hypergeometric series (1.1) is a G-function. Still when $p = q+1$, the converse is not true as the following example shows: for every $\alpha \in \overline{\mathbb{Q}} \setminus \mathbb{Z}_{\leq 0}$,

$$
{}_2F_1 \left(\begin{matrix} \alpha + 1, 1 \\ \alpha \end{matrix} ; z \right) = \sum_{n=0}^{\infty} \frac{(\alpha+1)_n (1)_n}{(1)_n (\alpha)_n} z^n = \sum_{n=0}^{\infty} \frac{\alpha + n}{\alpha} z^n = \frac{\alpha(1-z) + z}{\alpha(1-z)^2} \tag{1.2}
$$

is a G-function. Note that if $p \neq q + 1$, then the hypergeometric series (1.1) cannot be a G-function, unless it reduces to a polynomial in $\overline{\mathbb{Q}}[z]$, *ie.* when at least one of the α's is in $\mathbb{Z}_{\leq 0}$ and various simplifications occur between the Pochhammer symbols.

The following characterization of non polynomial hypergeometric ${}_{q+1}F_q$ G-functions was obtained by Galochkin in [7, p. 8], and the goal of this paper is to give a new proof of his result ([1]).

Theorem 1 (Galochkin)**.** *Let $p = q + 1$, $q \geq 0$, $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_{q+1}) \in (\mathbb{C} \setminus \mathbb{Z}_{\leq 0})^{q+1}$ and $\boldsymbol{\beta} := (\beta_1, \ldots, \beta_q) \in (\mathbb{C} \setminus \mathbb{Z}_{\leq 0})^q$ be such that $\alpha_i \neq \beta_j$ for all i, j.*

Then, the hypergeometric series (1.1) *with parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is a G-function if and only if the following two conditions hold:*

(*i*) *The α's and β's are all in $\overline{\mathbb{Q}}$;*

(*ii*) *The α's and β's which are not rational (if any) can be grouped in $k \leq q$ pairs $(\alpha_{j_1}, \beta_{j_1}), \ldots, (\alpha_{j_k}, \beta_{j_k})$ such that $\alpha_{j_\ell} - \beta_{j_\ell} \in \mathbb{N}$.*

It follows that if $\alpha \notin \mathbb{Z}$, then

$$
{}_2F_1 \left(\begin{matrix} \alpha, 1 \\ \alpha + 1 \end{matrix} ; z \right) = \sum_{n=0}^{\infty} \frac{\alpha}{n + \alpha} z^n
$$

is not a G-function; compare with (1.2).

[1] Galochkin also assumed that $\alpha_1 = 1$ (in our notations) but this is not less general because he can recover the generalized hypergeometric series by taking $b_1 = 0$ (in his notations).

Observe that when at least one of the α's is in $\mathbb{Z}_{\leq 0}$ (in which case the hypergeometric series is a polynomial), then the characterization given by Theorem does not hold. Consider for instance

$$ {}_2F_1\left(\begin{matrix}-1,\alpha\\ \beta\end{matrix};z\right) = 1 - \frac{\alpha}{\beta}z. $$

This is a G-function when $\alpha/\beta \in \overline{\mathbb{Q}}$, and α and β need not necessarily be in $\overline{\mathbb{Q}}$ for this.

Galochkin's proof of Theorem 1 is *ad hoc* and not easy. He used certain of his previous results in [6] and in particular the "prime number theorem" for prime ideals in number fields to obtain precise estimates on the factors of the denominators of Taylor coefficients of hypergeometric series with algebraic parameters. We shall give a new proof of Theorem 1 using the theory of arithmetic differential equations. Our approach is quite different, hence it might be an alternative in other situations where his method would be difficult to adapt. But it is not fundamentally easier as it uses a result of Katz on the rationality of exponents of globally nilpotent differential operators in $\overline{\mathbb{Q}}(z)[\frac{d}{dz}]$. The latter result uses a consequence of Chebotarev's density theorem.

Galochkin also characterized in [7] non polynomial hypergeometric E-functions of the form ${}_pF_q(z^{q-p+1})$ in the case $q \geq p \geq 1$, for the paramaters of which he obtained necessary and sufficient conditions formally like in Theorem 1, *mutatis mutandis*. His method is strong enough to apply to hypergeometric E-functions with Siegel's original definition (see above). Even though Siegel did not consider G-functions in this extended sense, it is possible to do so and the methods of the present paper extend to prove Theorem 1 in this setting as well. Indeed, it is known that Theorem 2 below also holds for this more general notion of G-functions; see [3, pp. 746-747] and [8] for more details.

The proof of Theorem 1 is given in §3, after some preliminary results are stated in §2.

2 Preliminary results

Our method is based on the following theorem, due to the (independent) works of André, Chudnovsky and Katz. See [2, pp. 717-720] for stronger statements, and [3] or [4] for proofs.

Theorem 2. *Let $F(z) \neq 0$ be a G-function and $L \in \overline{\mathbb{Q}}(z)[\frac{d}{dz}] \setminus \{0\}$ be such that the differential equation $LF(z) = 0$ is of minimal order for $F(z)$. Then L is Fuchsian with rational exponents.*

We shall also use two lemmas, the proofs of which are included for the reader's convienience.

Lemma 1. *Let $L, M, N \in \mathbb{C}(z)[\frac{d}{dz}]$ be such that $L = MN$. We assume that L is Fuchsian. Then,*

(*i*) *M and N are Fuchsian.*

(ii) Let us fix $\xi \in \mathbb{C} \cup \{\infty\}$. Then the indicial polynomial of N at ξ divides the indicial polynomial of L at ξ.

(iii) Let us assume that the differential equation $Ly(z) = 0$ has a power series solution $y(z) = \sum_{n=0}^{\infty} U_n z^n$. Then there exist an integer $\ell \geq 0$ and some polynomials $Q_j(X) \in \mathbb{C}[X]$, $j = 0, \dots, \ell$, such that for any $n \geq \ell$,

$$\sum_{j=0}^{\ell} Q_j(n) U_{n-j} = 0, \tag{2.1}$$

where $Q_0(X) \not\equiv 0$ is the indicial polynomial of L at 0, and $Q_\ell(\ell - X) \not\equiv 0$ is the indicial polynomial of L at ∞.

Proof. (i) A differential operator in $\mathbb{C}(z)[\frac{d}{dz}]$ is Fuchsian at a given point $\xi \in \mathbb{C} \cup \{\infty\}$ if and only if 0 is its only slope at ξ. Moreover, the set of slopes of L at ξ is the union of the set of slopes of M and that of N at ξ (see [9, p. 92, Lemma 3.45]). The statement follows.

(ii) It is enough to prove the result for $\xi = 0$, because for $\xi \neq 0$, we can return to the case $\xi = 0$ by changing z to $z - \xi$ (if ξ is finite) and z to $1/z$ (if $\xi = \infty$) in L, M, N.

We first recall some general facts that apply to any Fuchsian differential operator $L \in \mathbb{C}(z)[\frac{d}{dz}]$, of order μ. Let $A(X) = X^a A_0(X)$ be any polynomial in $\mathbb{C}[X] \setminus \{0\}$ such that $A_0(0) \neq 0$, $a \in \mathbb{N}$ and $AL = \sum_{j=0}^{\mu} P_j(z)(\frac{d}{dz})^j \in \mathbb{C}[z][\frac{d}{dz}]$. Let $\alpha \neq 0$ be the leading coefficient of $A(X)$. We let $\delta = \max_j(\deg(P_j))$, $\omega = \mathrm{ord}_0(P_\mu)$ and $\ell = \delta - \omega$. Then,

$$AL = \sum_{j=0}^{\ell} z^{j+\omega-\mu} Q_j(\theta + j) \tag{2.2}$$

where $\theta = z\frac{d}{dz}$, $Q_j(X) \in \mathbb{C}[X]$ for every j and $\deg(Q_0) = \deg(Q_\ell) = \mu$ (because AL is Fuchsian, see [5, Lemma 1]). Given A and L, the representation (2.2) is unique. Moreover, $Q_0(X)$ depends on A only by the multiplicative factor $A_0(0) \neq 0$, while $Q_\ell(X)$ depends on A only by the multiplicative factor $\alpha \neq 0$. Hence, up to non-zero multiplicative constants, $Q_0(X)$ and $Q_\ell(X)$ depend uniquely on L. By definition, $Q_0(X)$ is the indicial polynomial of L at 0, while $Q_\ell(\ell - X)$ is the indicial polynomial of L at ∞.

We now come back to the setting of the lemma, with $L, M, N \in \mathbb{C}(z)[\frac{d}{dz}]$ such that $L = MN$. Let $B(X) \in \mathbb{C}[X] \setminus \{0\}$ be such that $\widetilde{N} := BN \in \mathbb{C}[z][\frac{d}{dz}]$. The differential operator $M\frac{1}{B}$ can be written $\frac{1}{A}\widetilde{M}$ with $A(X) \in \mathbb{C}[X] \setminus \{0\}$ and $\widetilde{M} \in \mathbb{C}[z][\frac{d}{dz}]$. Therefore, $AL = \widetilde{M}\widetilde{N}$ and by (i) both $\widetilde{M}$ and $\widetilde{N}$ are Fuchsian because AL is. Moreover, by the discussion above, the indicial polynomial of L, respectively N, at 0 is the same as that of AL, respectively $\widetilde{N}$, at 0. Using obvious notations coherent with (2.2), we set

$$AL = \sum_{j=0}^{\ell} z^{j+\omega-\mu} Q_j(\theta + j), \quad \widetilde{M} = \sum_{i=0}^{m} z^{i+\widetilde{\omega}-\widetilde{\mu}} W_i(\theta + i), \quad \widetilde{N} = \sum_{k=0}^{n} z^{k+\widehat{\omega}-\widehat{\mu}} V_k(\theta + k)$$

where the Q's, V's and W's are all in $\mathbb{C}[X]$.

Let s be any integer. We shall apply the various differential operators to the function $z \mapsto z^s$. Since $\theta^r(z^s) = s^r z^s$, we have $AL(z^s) = \sum_{j=0}^{\ell} Q_j(s+j) z^{j+\omega-\mu+s}$ and

$$\widetilde{M}\widetilde{N}(z^s) = \sum_{i=0}^{m}\sum_{k=0}^{n} W_i(s+i+k+\widehat{\omega}-\widehat{\mu})V_k(s+k)z^{i+k+\widetilde{\omega}-\widetilde{\mu}+\widehat{\omega}-\widehat{\mu}+s}.$$

The equality $AL(z^s) = \widetilde{M}\widetilde{N}(z^s)$ is thus a Laurent polynomial identity in z (that depends on s). We now take s large enough so that it is a root of neither $Q_0(X)$ nor $W_0(X+\widehat{\omega}-\widehat{\mu})V_0(X)$. The monomials in z of lowest degree on both sides of $AL(z^s) = \widetilde{M}\widetilde{N}(z^s)$ (at $j = 0$ and $i = k = 0$ respectively) are thus $Q_0(s)z^{\omega-\mu+s}$ and $W_0(s+\widehat{\omega}-\widehat{\mu})V_0(s)z^{\widetilde{\omega}-\widetilde{\mu}+\widehat{\omega}-\widehat{\mu}+s}$ respectively. It follows that $\omega-\mu = \widetilde{\omega}-\widetilde{\mu}+\widehat{\omega}-\widehat{\mu}$ and that $Q_0(s) = W_0(s+\widehat{\omega}-\widehat{\mu})V_0(s)$. Since the integer s can be taken arbitrarily large, we must have the polynomial identity

$$Q_0(X) = W_0(X+\widehat{\omega}-\widehat{\mu})V_0(X).$$

This proves the claimed divisibility because $Q_0(X)$ and $V_0(X)$ are the indicial polynomials at 0 of L and N respectively.

(iii) Let $y(z) = \sum_{k=0}^{\infty} U_k z^k$ be such that $Ly(z) = 0$. We use the same notations as in (ii). Since $\theta^r(z^s) = s^r z^s$, we deduce from (2.2) that

$$\begin{aligned} 0 = z^{\mu-\omega}A(z)Ly(z) &= \sum_{j=0}^{\ell} z^j Q_j(\theta+j)y(z) \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{\ell} Q_j(k+j)U_k z^{k+j} = \sum_{n=0}^{\infty} z^n \sum_{\substack{\ell\ge j\ge 0, k\ge 0\\ k+j=n}} Q_j(k+j)U_k. \end{aligned}$$

Thus, for every $n \ge \ell$,

$$0 = \sum_{\substack{\ell\ge j\ge 0, k\ge 0\\ k+j=n}} Q_j(k+j)U_k = \sum_{j=0}^{\ell} Q_j(n)U_{n-j}.$$

This concludes the proof of the lemma. $\square$

Lemma 2. *Let the integers $p, q \ge 0$ be such that $p+q \ge 1$, and let $\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q \in \mathbb{C}\setminus\mathbb{Z}_{\le 0}$ be such that $\alpha_i \neq \beta_j$ for all i, j. Assume that for infinitely many $n \ge 0$,*

$$\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n} \in \overline{\mathbb{Q}}. \tag{2.3}$$

Then the α's and β's are in $\overline{\mathbb{Q}}$.

If $p = 0$, resp. $q = 0$, its must be understood that the lemma applies to

$$\frac{1}{(\beta_1)_n \cdots (\beta_q)_n}, \quad \text{resp. } (\alpha_1)_n \cdots (\alpha_p)_n,$$

and the condition "$\alpha_i \neq \beta_j$ for all i, j" is dropped. Note that the conclusion does not necessarily hold if one of the α's is in $\mathbb{Z}_{\leq 0}$. ([2])

Proof. The proof is a slight generalization of that of [7, Lemma 1], where the case $p \leq q$ was considered. Let

$$V_n := \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n}.$$

Since the α's and β's are in $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, $V_n \neq 0$ for all $n \geq 0$ and V_n/V_{n-1} is a well defined non-zero algebraic number for all $n \geq 1$. Let $R(X) := \prod_{j=1}^{p}(X + \alpha_j - 1) = \sum_{\ell=0}^{p} r_\ell X^\ell$ and $S(X) := \prod_{j=1}^{q}(X + \beta_j - 1) = \sum_{\ell=0}^{q} s_\ell X^\ell$, with $r_p = s_q = 1$. Let $\omega_1, \omega_2, \ldots, \omega_t$ be a basis of the $\overline{\mathbb{Q}}$-vector space generated by the r_ℓ's and s_ℓ's, with $\omega_1 := 1$ and $t \leq p + q + 2$. We write

$$R(X) = \sum_{\ell=1}^{t} R_\ell(X)\omega_\ell, \quad S(X) = \sum_{\ell=1}^{t} S_\ell(X)\omega_\ell$$

where $R_\ell(X), S_\ell(X) \in \overline{\mathbb{Q}}[X]$ for each $\ell \in \{1, 2, \ldots, t\}$. Since $\omega_1 = 1$ and $r_p = s_q = 1$, we have $\deg(R_1) = p$ and $\deg(S_1) = q$, and both polynomials have leading coefficient 1. The identity $S(n)\frac{V_n}{V_{n-1}} - R(n) = 0$ (for all $n \in \mathcal{N}$, an infinite set by assumption) becomes

$$\sum_{\ell=1}^{t} \Big(S_\ell(n)\frac{V_n}{V_{n-1}} - R_\ell(n)\Big)\omega_\ell = 0, \quad \forall n \in \mathcal{N}.$$

Since $S_1(X) \not\equiv 0$, we have $S_1(n) \neq 0$ if $n \in \mathcal{N}$ is large enough, say $n \geq N$. By independence of the ω's, it follows that $\frac{R(n)}{S(n)} = \frac{R_1(n)}{S_1(n)}$ for every $n \in \mathcal{N}$ such that $n \geq N$. This must then be an equality of rational fractions, ie $\frac{R(X)}{S(X)} \equiv \frac{R_1(X)}{S_1(X)}$. Now the assumption that $\alpha_j \neq \beta_k$ for all j, k implies that R and S are coprime. Hence, comparing the degrees and leading coefficients, it follows that $R(X) \equiv R_1(X)$ and $S(X) \equiv S_1(X)$. $\square$

3 Proof of Theorem 1

3.1 Sufficiency

We first prove that every hypergeometric series with parameters not in $\mathbb{Z}_{\leq 0}$ and satisfying (i) and (ii) of Theorem 1 is a G-function. If $k = 0$, then all the α's and β's are rational numbers, and Siegel's result applies directly. We now assume that $k \geq 1$. Reordering the

[2] Given $\alpha, \beta \in \mathbb{C}$, $\beta \notin \mathbb{Z}_{\leq 0}$, consider for instance $\frac{(-1)_n(\alpha)_n}{(\beta)_n}$, equal to 1 for $n = 0$, to $-\frac{\alpha}{\beta}$ for $n = 1$ and to 0 for $n \geq 2$: the assumption $\frac{(-1)_n(\alpha)_n}{(\beta)_n} \in \overline{\mathbb{Q}}$ for all $n \geq 0$ only implies that $\frac{\alpha}{\beta} \in \overline{\mathbb{Q}}$.

parameters if necessary, we assume without loss of generality that $j_\ell = \ell$ for $\ell = 1, \ldots, k$, and let $m_\ell := \alpha_\ell - \beta_\ell$, which is in $\mathbb{N}$ by assumption. We then have for every $n \geq 0$:

$$\frac{(\alpha_{j_\ell})_n}{(\beta_{j_\ell})_n} = \frac{(\beta_\ell + m_\ell)_n}{(\beta_\ell)_n} = \frac{(\beta_\ell + n)_{m_\ell}}{(\beta_\ell)_{m_\ell}} =: P_\ell(n) \in \overline{\mathbb{Q}}[n].$$

Hence,

$$F(z) := {}_{q+1}F_q\left(\begin{matrix}\alpha_1, \ldots, \alpha_{q+1}\\ \beta_1, \ldots, \beta_q\end{matrix}; z\right) = \sum_{n=0}^{\infty}\left(\prod_{\ell=1}^{k} P_\ell(n)\right)\frac{(\alpha_{k+1})_n \cdots (\alpha_{q+1})_n}{(1)_n(\beta_{k+1})_n \cdots (\beta_q)_n}\, z^n.$$

Writing $\prod_{\ell=1}^{k} P_\ell(n) = \sum_{j=0}^{d} q_j n^j$ with $q_j \in \overline{\mathbb{Q}}$, we have

$$F(z) = \sum_{j=0}^{d} q_j \theta^j \left({}_{q+1-k}F_{q-k}\left(\begin{matrix}\alpha_{k+1}, \ldots, \alpha_{q+1}\\ \beta_{k+1}, \ldots, \beta_q\end{matrix}; z\right)\right) \tag{3.1}$$

where $\theta := z\frac{d}{dz}$. Since $\alpha_j, \beta_j \in \mathbb{Q}$ for every $j \geq k+1$, each hypergeometric function on the righ-hand side of (3.1) is a G-function (again, by Siegel). Thus $F(z)$ is a G-function.

3.2 Necessity

We set

$$U_n := \frac{(\alpha_1)_n \cdots (\alpha_{q+1})_n}{(1)_n(\beta_1)_n \cdots (\beta_q)_n},$$

the n-th Taylor coefficient of the hypergeometric series. The U_n's are defined and not equal to 0 for every $n \geq 0$ because the $\alpha's$ and β's are not in $\mathbb{Z}_{\leq 0}$. Since $F(z) := \sum_{n=0}^{\infty} U_n z^n$ is a G-function, we also have that $U_n \in \overline{\mathbb{Q}}$ for all $n \geq 0$. This is equivalent to the requirement that $\frac{(\alpha_1)_n \cdots (\alpha_{q+1})_n}{(\beta_1)_n \cdots (\beta_q)_n} \in \overline{\mathbb{Q}}$ for all $n \geq 0$. The assumptions of Theorem 1 enable us to apply Lemma 2 with $p = q+1$, so that the α's and β's are in fact algebraic numbers, that is (i) in Theorem 1 holds.

We now turn our attention to the proof of (ii). The classical differential equation satisfied by the hypergeometric series (1.1) when $p = q+1$ is $Ly(z) = 0$ with

$$L := \theta(\theta + \beta_1 - 1) \cdots (\theta + \beta_q - 1) - z(\theta + \alpha_1) \cdots (\theta + \alpha_{q+1}) \in \mathbb{C}[z][\frac{d}{dz}].$$

It reflects the fact that the sequence $(U_n)_{n\geq 0}$ satisfies the linear recurrence $B(n)U_n - A(n)U_{n-1} = 0$ $(n \geq 1)$, where

$$A(X) = \prod_{j=1}^{q+1}(X + \alpha_j - 1), \quad B(X) = \prod_{j=1}^{q+1}(X + \beta_j - 1)$$

are both in $\overline{\mathbb{Q}}[X]$, with $\beta_{q+1} := 1$. In particular, the indicial polynomial of L at 0 is $B(X)$ and the indicial polynomial of L at ∞ is $A(1-X)$; their roots are in $\overline{\mathbb{Q}}$.

Let $N \in \mathbb{C}[z][\frac{d}{dz}] \setminus \{0\}$ be such that $NF(z) = 0$ and is of minimal order for $F(z)$. Then N is a right factor of L. Since $NF(z) = 0$, by Lemma 1(iii), there exist an integer $\ell \geq 1$ and polynomials $C_j(X)$, $j = 1, \ldots, \ell$, such that

$$\sum_{j=0}^{\ell} C_j(n) U_{n-j} = 0 \tag{3.2}$$

for all $n \geq \ell$, and $C_0(X)$ and $C_\ell(X - \ell)$ are the respective indicial polynomials of N at 0 and ∞.

Below, we shall consider the multiset $R(P)$ of the roots of a polynomial P; each element of $R(P)$ appears as many times as its multiplicity as a root of P. We denote a multiset by $\{\!\{\cdot\}\!\}$ to distinguish it from a set $\{\cdot\}$.

Informations coming from the indicial polynomials of L and N at 0. Recall that the indicial polynomial of L at 0 is $B(X)$. By Lemma 1(ii), there exists $D_0(X) \in \mathbb{C}[X]$ such that

$$B(X) = C_0(X) D_0(X).$$

By Theorem 2, the roots of C_0 are in $\mathbb{Q}$, so that those of D_0 are in $\overline{\mathbb{Q}}$. Since $U_n \neq 0$ for all $n \geq 0$, we can rewrite the recurrence (3.2) as

$$C_0(n) \frac{U_n}{U_{n-\ell}} = -\sum_{j=1}^{\ell} C_j(n) \frac{U_{n-j}}{U_{n-\ell}}, \quad \forall n \geq \ell. \tag{3.3}$$

Now,

$$\frac{U_{n-j}}{U_{n-\ell}} = \prod_{k=j}^{\ell-1} \frac{A(n-k)}{B(n-k)}$$

so that after clearing the denominators, (3.3) yields

$$C_0(n) \prod_{k=0}^{\ell-1} A(n-k) = -\sum_{j=1}^{\ell} C_j(n) \prod_{k=0}^{j-1} B(n-k) \prod_{k=j}^{\ell-1} A(n-k), \quad \forall n \geq \ell.$$

This is a polynomial identity for infinitely many values of the integer n, hence a genuine polynomial identity

$$C_0(X) \prod_{k=0}^{\ell-1} A(X-k) = -\sum_{j=1}^{\ell} C_j(X) \prod_{k=0}^{j-1} B(X-k) \prod_{k=j}^{\ell-1} A(X-k).$$

We observe that $B(X)$ is a factor of each summand on the right hand side. Hence $B(X)$ divides $C_0(X) \prod_{k=0}^{\ell-1} A(X-k)$, so that $D_0(X)$ divides $\prod_{k=0}^{\ell-1} A(X-k)$.

Since $B(X) = C_0(X) D_0(X)$, we have

$$R(C_0) = \{\!\{1-\beta_{j_1}, \ldots, 1-\beta_{j_\kappa}\}\!\}, \quad R(D_0) = \{\!\{1-\beta_{j_{\kappa+1}}, \ldots, 1-\beta_{j_{q+1}}\}\!\}$$

for some $\kappa \in \{0, \dots q+1\}$, and where $\{j_m : m = 1, \dots, q+1\} = \{1, 2, \dots q+1\}$. With $\widehat{A}(X) := \prod_{k=0}^{\ell-1} A(X-k)$, we have

$$R(\widehat{A}) = \{\!\{1-\alpha_1, \dots, 1-\alpha_{q+1}, 2-\alpha_1, \dots, 2-\alpha_{q+1}, \dots, \ell-\alpha_1, \dots, \ell-\alpha_{q+1}\}\!\}.$$

and $R(D_0) \subset R(\widehat{A})$.

Now, let $1-\beta_j \in R(B)$.

- If $1-\beta_j \in R(C_0)$, then $\beta_j \in \mathbb{Q}$.
- If $1-\beta_j \in R(D_0)$, then $1-\beta_j \in R(\widehat{A})$. Hence, $1-\beta_j = k-\alpha_i$ for some integers k, i such that $1 \le k \le \ell$ and $1 \le i \le q+1$. It follows that $\alpha_i - \beta_j = k-1 \ge 0$. (If $\beta_j \in \mathbb{Q}$, $\alpha_i \in \mathbb{Q}$ as well).

This completely determines the nature of the parameters β in accordance with (ii) in Theorem 1. However, if $\kappa \ge 1$, then $\deg(A) > \deg(D_0)$ and thus there exists at least one parameter α which is not associated to a parameter β such that $1-\beta_j \in R(D_0)$. It might even be the case that $\kappa = q+1$ and $\deg(D_0) = 0$, so that this argument says in fact nothing on the α's. We shall now give explain how to determine the nature ot the α's.

Informations coming from the indicial polynomials of L and N at ∞. Recall that the indicial polynomial of L at ∞ is $A(1-X)$. By Lemma 1(ii), there exists $D_\ell(X) \in \mathbb{C}[X]$ such that

$$A(X-\ell+1) = C_\ell(X) D_\ell(X).$$

By Theorem 2, the roots of C_ℓ are in $\mathbb{Q}$, so that those of D_ℓ are in $\overline{\mathbb{Q}}$. We can rewrite the recurrence (3.2) as

$$C_\ell(n) \frac{U_{n-\ell}}{U_n} = -\sum_{j=0}^{\ell-1} C_j(n) \frac{U_{n-j}}{U_n}, \quad \forall n \ge \ell. \tag{3.4}$$

Now,

$$\frac{U_{n-j}}{U_n} = \prod_{k=0}^{j-1} \frac{B(n-k)}{A(n-k)}$$

so that after clearing the denominators, (3.4) yields

$$C_\ell(n) \prod_{k=0}^{\ell-1} B(n-k) = -\sum_{j=0}^{\ell-1} C_j(n) \prod_{k=0}^{j-1} B(n-k) \prod_{k=j}^{\ell-1} A(n-k), \quad \forall n \ge \ell.$$

This is a polynomial identity for infinitely many values of the integer n, hence a genuine polynomial identity

$$C_\ell(X) \prod_{k=0}^{\ell-1} B(X-k) = -\sum_{j=0}^{\ell-1} C_j(X) \prod_{k=0}^{j-1} B(X-k) \prod_{k=j}^{\ell-1} A(X-k).$$

We observe that $\widetilde{A}(X) := A(X-\ell-1)$ is a factor of each summand on the right hand side. Hence $\widetilde{A}(X)$ divides $C_\ell(X) \prod_{k=0}^{\ell-1} B(X-k)$, so that $D_\ell(X)$ divides $\prod_{k=0}^{\ell-1} B(X-k)$.

Since $\widetilde{A}(X) = C_\ell(X)D_\ell(X)$, we have

$$R(C_\ell) = \{\!\{\ell - \alpha_{j_1}, \dots, \ell - \alpha_{j_\omega}\}\!\}, \quad R(D_\ell) = \{\!\{\ell - \alpha_{j_{\omega+1}}, \dots, \ell - \alpha_{k_{q+1}}\}\!\}$$

for some $\omega \in \{0, \dots q+1\}$, and where $\{j_m : m = 1, \dots, q+1\} = \{1, 2, \dots q+1\}$. With $\widetilde{B}(X) := \prod_{k=0}^{\ell-1} B(X-k)$, we have

$$R(\widetilde{B}) = \{\!\{1 - \beta_1, \dots, 1 - \beta_{q+1}, 2 - \beta_1, \dots, 2 - \beta_{q+1}, \dots, \ell - \beta_1, \dots, \ell - \beta_{q+1}\}\!\}.$$

and $R(D_\ell) \subset R(\widetilde{B})$.

Now, let $\ell - \alpha_i \in R(\widetilde{A})$.

• If $\ell - \alpha_i \in R(C_\ell)$, then $\alpha_i \in \mathbb{Q}$.

• If $\ell - \alpha_i \in R(D_\ell)$, then $\ell - \alpha_i \in R(\widetilde{B})$. Hence, $\ell - \alpha_i = k - \beta_j$ for some integers k, j such that $1 \le k \le \ell$ and $1 \le j \le q+1$. It follows that $\alpha_i - \beta_j = \ell - k \ge 0$. (If $\alpha_i \in \mathbb{Q}$, $\beta_j \in \mathbb{Q}$ as well).

This completely determines the nature of the parameters α in accordance with (ii) in Theorem 1, the proof of which is now complete.

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