Polynomial continued fractions for $\exp(\pi)$

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Abstract

We present two (inequivalent) polynomial continued fraction representations of the number e^π with all their elements in $\mathbb{Q}$; no such representation was seemingly known before. More generally, a similar result for $e^{r\pi}$ is obtained for every $r \in \mathbb{R}$ such that $r^2 \in \mathbb{Q}$. The proof uses a classical polynomial continued fraction representation of α^β, for $|\arg(\alpha)| < \pi$ and $\beta \in \mathbb{C} \setminus \mathbb{Z}$, of which we present a new proof that enables us to obtain the exact rate of convergence of the convergents of the continued fraction for e^π. We also deduce some consequences of arithmetic interest concerning the elements of certain polynomial continued fraction representations of the (transcendental) Gel'fond-Schneider numbers α^β, where $\alpha \in \overline{\mathbb{Q}} \setminus \{0, 1\}$ and $\beta \in \overline{\mathbb{Q}} \setminus \mathbb{Q}$, where $\overline{\mathbb{Q}}$ is the field of algebraic numbers, embedded into $\mathbb{C}$.

1 Introduction

To prove the irrationality of some classical constant $\xi \in \mathbb{C}$, a standard method is to construct two sequences of integers $(p_n)_{n\geq 0}$ and $(q_n)_{n\geq 0}$ such that $0 \neq q_n\xi - p_n \to 0$ as $n \to +\infty$. One of the most celebrated example is Apéry's explicit construction [1] of two sequences of rational numbers $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ with common denominator $D_n := \mathrm{lcm}(1, 2, \dots, n)^3$ and such that $0 \neq D_n(a_n\zeta(3) - b_n) \to 0$, which implies the irrationality of $\zeta(3)$. (Note that proving only the convergence $a_n/b_n \to \zeta(3)$ is not enough to prove the irrationality.) Apéry's sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are P-recursive of order 2: they are both solutions of the linear recurrence $(n+1)^3u_{n+1} - (34n^3+51n^2+27n+5)u_n + n^3u_{n-1} = 0$. A sequence $(u_n)_{n\geq 0}$ is said to be P-recursive of order d when it satisfies a linear recurrence relation of the form $\sum_{j=0}^{d} f_j(n)u_{n+j} = 0$, where $f_j(x) \in \mathbb{C}[x]$ and $f_d(x)$ not identically 0. Let $\overline{\mathbb{Q}}$ denote the field of algebraic numbers embedded into $\mathbb{C}$. We shall say that a number $\xi \in \mathbb{C}$ is a PA-number if there exist two P-recursive sequences $(p_n)_{n\geq 0} \in \overline{\mathbb{Q}}^{\mathbb{N}}$ and $(q_n)_{n\geq 0} \in \overline{\mathbb{Q}}^{\mathbb{N}}$ (not necessarily of the same order) such that $p_n/q_n \to \xi$ as $n \to +\infty$ and whose underlying recurrences have coefficients in $\overline{\mathbb{Q}}[x]$. This notion is related to the 'Apéry limits' studied in [3]. Notice that given two P-recursive sequences solutions of linear recurrences R_1 and R_2, it is always possible to assume that $R_1 = R_2$, by taking the least common left multiple of R_1 and R_2 in the non-commutative ring $\overline{\mathbb{Q}}(n)[S]$, where S is the usual 'shift by +1' (Ore property, see [21, §10]); this procedure increases the order of the recurrence in general.

The termwise sum and product of P-recursive sequences are P-recursive. Hence PA-numbers form a countable subfield of $\mathbb{C}$ we shall denote by $P(\overline{\mathbb{Q}})$: indeed, if $p_n/q_n \to \xi_1$ and $\widetilde{p}_n/\widetilde{q}_n \to \xi_2$, we have $(p_n\widetilde{p}_n)/(q_n\widetilde{q}_n) \to \xi_1\xi_2$, $(p_n\widetilde{q}_n + q_n\widetilde{p}_n)/(q_n\widetilde{q}_n) \to \xi_1 + \xi_2$ and $q_n/p_n \to 1/\xi_1$. Given B a subset of $\overline{\mathbb{Q}}$, the subset $P(B)$ of $P(\overline{\mathbb{Q}})$ consists of PA-numbers for which the underlying linear recurrences can be found with coefficients in $B[x]$, and where their solutions are considered in $\overline{\mathbb{Q}}^{\mathbb{N}}$ and not necessarily in $B^{\mathbb{N}}$. We also consider the sets $P_d(B)$ of PA-numbers for which the two underlying recurrences are exactly of order $d \geq 1$ with coefficients in $\mathbb{B}[x]$, and where again their solutions are considered in $\overline{\mathbb{Q}}^{\mathbb{N}}$ and not necessarily in $B^{\mathbb{N}}$. $P(\overline{\mathbb{Q}})$ contains two important subrings introduced and studied in [7, 8], namely the ring $\mathbf{G}$ of G-values (which contains $\overline{\mathbb{Q}}$, π, Catalan's constant G, $\zeta(3)$, multiple zeta values, Beta values $B(a,b)$ with $a, b \in \mathbb{Q}$, powers of Gamma values $\Gamma(a/b)^b$ with $a, b \in \mathbb{N}$) and the ring $\mathbf{E}$ of E-values (which contains e^a and Bessel's $J_0(a)$, $a \in \overline{\mathbb{Q}}$). $P(\overline{\mathbb{Q}})$ also contains elements which are conjecturally neither in $\mathbf{G}$ nor in $\mathbf{E}$, like Euler's constant γ and more generally $\gamma + \log(x)$ ($x \in \overline{\mathbb{Q}}$), $\Gamma(a/b)$ with $a, b \in \mathbb{N}$, and Gompertz constant $\delta := \int_0^\infty e^{-x}/(x+1)dx$; see [4, 18, 19] and [9, p. 424]. It is not known if all periods in Konsevich-Zagier's sense are in $P(\overline{\mathbb{Q}})$.

Because of Apéry's example, important efforts have been devoted to prove that certain classical numbers, generically denoted ξ here, are PA-numbers with underlying sequences in $\overline{\mathbb{Q}}^{\mathbb{N}}$ of order 2 and coefficients in $\mathbb{Q}[n]$; this then proves ξ to be in $P_2(\mathbb{Q})$. Besides an obvious arithmetic motivation, another one is that this yields in general a continued fraction representation for ξ with *ultimately polynomial elements* , ie we have

$$\xi = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cdots}} = b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \cdots \tag{1.1}$$

where for any $n \geq N_0$, $a_n = A(n)$, $b_n = B(n)$ for some $A(x), B(x) \in \overline{\mathbb{Q}}[x]$ and some integer N_0; the a_n's and b_n's are called the elements of the continued fraction [16, p. 5]. Following [5], we say that ξ is represented by a *polynomial continued fraction.* Recall that the sequence of convergents $(p_n/q_n)_n$ of the continued fraction on the right-hand side of (1.1) is $\frac{p_n}{q_n} := b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \cdots + \frac{a_n|}{|b_n}$ with $p_0 = b_0$, $q_0 = 1$ and (by convention) $p_{-1} = 1$, $q_{-1} = 0$, and where the sequences $(p_n)_{n\geq -1}$ and $(q_n)_{n\geq -1}$ both satisfy the linear recurrence $u_n = b_n u_{n-1} + a_n u_{n-2}$, $n \geq 1$. Amongst numbers already known to be in $P_2(\mathbb{Q})$, we have algebraic numbers, $\log(2)$, π, $\zeta(n)$ ($n \geq 2$), G, e, δ, $\pi\coth(\pi)$, $\Gamma(1/3)^3/(\pi\sqrt{3})$; see [1, 17, 20, 23], [9, pp. 15, 23, 46, 57, 426] and [16, pp. 266–268]. It is not known if γ, $e\pi$ or $e + \pi$ is in $P_2(\mathbb{Q})$.

Our main result is an addition to this list:

Theorem 1. *For every $r \in \mathbb{R}$ such that $r^2 \in \mathbb{Q}$, the number $e^{r\pi}$ is in $P_2(\mathbb{Q})$.*

More specifically in the case $r = 1$, the number e^π is representable by the two (inequiv-

alent) polynomial continued fractions:

$$e^{\pi} = 1 + \frac{6|}{|1} - \frac{560|}{|800} - \frac{C(2)|}{|D(2)} - \frac{C(3)|}{|D(3)} - \cdots - \frac{C(n)|}{|D(n)} - \cdots \tag{1.2}$$

$$= -3 + \frac{200|}{|\ 4} + \frac{3744|}{|1064} - \frac{E(2)|}{|F(2)} - \frac{E(3)|}{|F(3)} - \cdots - \frac{E(n)|}{|F(n)} - \cdots, \tag{1.3}$$

where $A(x) := 2(x+1)^2 + 8,\ B(x) := 2x+3$ *and*

$$\begin{aligned}
C(x) &:= 4A(2x-2)A(2x-1)B(2x-4)B(2x),\\
D(x) &:= 2B(2x-2)A(2x) + 4B(2x-2)B(2x-1)B(2x) + 2A(2x-1)B(2x),\\
E(x) &:= A(2x-1)A(2x)B(2x-3)B(2x+1),\\
F(x) &:= B(2x-1)A(2x+1) + 2B(2x-1)B(2x)B(2x+1) + A(2x)B(2x+1).
\end{aligned}$$

By inequivalent, we mean that the continued fractions (1.2) and (1.3) don't have the same sequences of convergents, denoted by $(\widehat{p}_n/\widehat{q}_n)_n$ and $(\widetilde{p}_n/\widetilde{q}_n)_n$ respectively; note that $\widehat{p}_n, \widehat{q}_n, \widetilde{p}_n$ and $\widetilde{q}_n$ are integers for all $n \geq 0$ because $C(n), D(n), E(n), F(n)$ are integers for any integer n. We shall also prove that

$$\lim_{n\to+\infty} \left| e^{\pi} - \frac{\widehat{p}_n}{\widehat{q}_n} \right|^{1/n} = \lim_{n\to+\infty} \left| e^{\pi} - \frac{\widetilde{p}_n}{\widetilde{q}_n} \right|^{1/n} = \frac{1}{4}(2-\sqrt{2})^4 \tag{1.4}$$

and that

$$\lim_{n\to+\infty} |\widehat{q}_n/n!^3|^{1/n} = 32(2+\sqrt{2})^2 \quad \text{and} \quad \lim_{n\to+\infty} |\widetilde{q}_n/n!^3|^{1/n} = 16(2+\sqrt{2})^2. \tag{1.5}$$

The justifications of these limits are given in §4. The growth of $\widehat{q}_n$ and $\widetilde{q}_n$ is unfortunately too fast to get a new proof of the irrationality of e^{π} from these continued fractions, or to prove that e^{π} is not a Liouville number, a long standing problem for this number.

We shall deduce Eqs. (1.2) and (1.3) from another continued fraction expansion that will be proved first using Proposition 1 stated in §2:

$$e^{\pi} = 1 + \frac{4\ |}{|-1} + \frac{A(0)\ |}{|2B(0)} + \frac{A(1)|}{|B(1)} + \cdots + \frac{A(2n)\ |}{|2B(2n)} + \frac{A(2n+1)|}{|B(2n+1)} + \cdots \tag{1.6}$$

where $A(x)$ and $B(x)$ are as in Theorem 1. This is formally not a polynomial continued fraction and it does not immediately imply that $e^{\pi} \in P_2(\mathbb{Q})$. More generally, an adaptation of the proof of (1.6) shows that, for every $r \in \mathbb{R}$ such that $r^2 \in \mathbb{Q}$,

$$e^{r\pi} = 1 + \frac{4r\ \ |}{|1-2r} + \frac{A_r(0)|}{|2B(0)} + \frac{A_r(1)|}{|B(1)} + \cdots + \frac{A_r(2n)|}{|2B(2n)} + \frac{A_r(2n+1)|}{|B(2n+1)} + \cdots, \tag{1.7}$$

where $A_r(x) := 2(x+1)^2 + 8r^2 \in \mathbb{Q}[x]$ and $B(x) := 2x+3$ (with a simple modification to the continued fraction if $r = 1/2$). Two polynomial continued fractions for $e^{r\pi}$ can then be obtained in the same way as (1.2) and (1.3) follow from (1.6). Anyone of these two continued fractions proves the first assertion of Theorem 1, ie that $e^{r\pi}$ is in $P_2(\mathbb{Q})$ for every $r \in \mathbb{R}$ such that $r^2 \in \mathbb{Q}$.

More generally, Proposition 1 in §2 implies that, for every $\alpha \in \overline{\mathbb{Q}} \setminus \{0, 1\}, \beta \in \overline{\mathbb{Q}} \setminus \mathbb{Q}$, the transcendental Gel'fond-Schneider number α^β (with $|\arg(\alpha)| < \pi$) is in $P_2(\mathbb{Q}(\alpha, \beta^2))$. Observe that if $\alpha \in \mathbb{Q}$ and $\beta^2 \in \mathbb{Q}$, then the elements of the continued fractions (deduced by specialisation of Proposition 1) for these numbers are rational numbers, except possibly the second and third ones. This applies for instance to $2^{\sqrt{2}}$ which is thus proved to be in $P_2(\mathbb{Q})$: taking $\alpha = 2$ and $\beta = \sqrt{2}$ in (2.2), we get

$$2^{\sqrt{2}} = 1 - \frac{2\sqrt{2}\,|}{|\sqrt{2}-3} - \frac{-1|}{|\,9} - \frac{2|}{|15} - \frac{7|}{|21} - \cdots - \frac{n^2+2n-1|}{|\,3(2n+3)} - \cdots .$$

It would be interesting from a number theoretical point of view to obtain a polynomial continued fraction for $2^{\sqrt{2}}$ with all its elements in $\mathbb{Q}$ and ultimately in $\mathbb{Q}[n]$; none of those listed in [16, pp. 269–270] seems to provide one. The case of e^π is in fact particularly remarkable. Taking $\alpha = e^{i\pi/2}$ and $\beta = -2i$, or $\alpha = e^{-i\pi/2}$ and $\beta = 2i$ (both choices lead in the end to the fractions in Theorem 1), Proposition 1 provides for e^π a polynomial continued fraction with elements ultimately in $\mathbb{Q}(i)[n]$. This *a priori* only proves that e^π is in $P_2(\mathbb{Q}(i))$ and surprisingly a more careful analysis leads to Theorem 1. This choice of parameters is very special in the sense that, taking $\alpha = e^{i\pi/k}$ and $\beta = -ki$ for some integer k such that $|k| \geq 3$, we do not get a continued fraction for e^π with all its elements in $\mathbb{Q}$.

2 Proof of Theorem 1

We first state and prove a useful lemma that make the connection between linear recurrences of order 2 and continued fractions.

Lemma 1. *Let $(p_n)_{n\geq -1} \in \mathbb{C}^\mathbb{N}$ and $(q_n)_{n\geq -1} \in \mathbb{C}^\mathbb{N}$ be two solutions of a recurrence*

$$u_n = B_n u_{n-1} + A_n u_{n-2}, \quad n \geq 1 \tag{2.1}$$

with initial conditions $p_{-1} = a \neq 0$, $p_0 = b$, $q_{-1} = c$, $q_0 = d$ such that $ad \neq bc$. Assume that $\xi := \lim_{n\to+\infty} p_n/q_n$ exists and is finite.

If $c\xi \neq a$, then

$$\frac{A_1|}{|B_1} + \frac{A_2|}{|B_2} + \frac{A_3|}{|B_3} + \cdots = \frac{(ad-bc)\xi}{a(a-c\xi)} - \frac{b}{a}.$$

A continued fraction representation for ξ is readily obtained under the above assumptions:

$$\xi = \frac{1\,|}{|c/a} + \frac{(ad-bc)/a^2|}{|\quad b/a} + \frac{A_1|}{|B_1} + \frac{A_2|}{|B_2} + \frac{A_3|}{|B_3} + \cdots .$$

Proof. The sequences $\hat{p}_n := \frac{1}{a}p_n$ and $\hat{q}_n = \frac{1}{ad-bc}(aq_n - cp_n)$ are solutions of (2.1) with initial conditions

$$\widehat{p}_{-1} = 1, \quad \widehat{p}_0 = \frac{b}{a}, \quad \widehat{q}_{-1} = 0, \quad \widehat{q}_0 = 1.$$

Moreover,

$$\lim_{n\to+\infty} \frac{\widehat{p}_n}{\widehat{q}_n} = \frac{(ad-bc)\xi}{a(a-c\xi)}.$$

The theory of continued fractions [16, p. 6] then ensures that

$$\frac{(ad-bc)\xi}{a(a-c\xi)} = \frac{b}{a} + \frac{A_1|}{|B_1} + \frac{A_2|}{|B_2} + \frac{A_3|}{|B_3} + \cdots .$$

This completes the proof. □

To prove Eq. (1.6) stated in the Introduction and then Theorem 1, we shall need the following proposition. For $\alpha \in \mathbb{C}^*$, we set $\log(\alpha) := \ln|\alpha| + i\arg(\alpha)$ where $-\pi < \arg(\alpha) < \pi$ and for any $\beta \in \mathbb{C}$, $\alpha^\beta := \exp(\beta \log(\alpha))$.

Proposition 1. *Let $\alpha \in \mathbb{C}^*$ be such that $|\arg(\alpha)| < \pi$ and $\beta \in \mathbb{C} \setminus \mathbb{Z}$. Set $U(x) := (x+1)^2 - \beta^2$ and $V(x) := (\alpha+1)(2x+3)$. Then*

$$\alpha^\beta = 1 + \frac{2\beta(\alpha-1)|}{|\alpha+\beta+1-\alpha\beta} - \frac{(\alpha-1)^2U(0)|}{|V(0)} - \frac{(\alpha-1)^2U(1)|}{|V(1)} - \frac{(\alpha-1)^2U(2)|}{|V(2)} - \cdots - \frac{(\alpha-1)^2U(n)|}{|V(n)} - \cdots . \tag{2.2}$$

This continued fraction is not new, see for instance [12, p. 105] or [16, p. 269, Eq. (2.3.2)] (the latter without proof). Even though it is not formally necessary for us to give a proof of it, we shall give one different of that of [12] (based on an analysis of solutions of Riccati equation that goes back to Euler). The reason to do that is that our method directly provides explicit closed form formulas for the convergents of (2.2) as well as rates of convergence, not given in [12]. The latter are needed to justify (1.4) and (1.5) above. Moreover, the method can also be adapted to produce other continued fractions for α^β; see §5 for details. We postpone the proof of Proposition 1 to §3.

Proof of Theorem 1. In Proposition 1, we take $\alpha = e^{i\pi/2}$ and $\beta = -2i$, so that $\alpha^\beta = e^\pi$. We have $A(x) = 2U(x)$ and $B(x) = V(x)/(i+1)$ where $U(x) = (x+1)^2 + 2$ and $V(x) = (i+1)(2x+3)$ are as in Proposition 1, that yields

$$e^\pi = 1 + \frac{4(i+1)|}{|-(i+1)} + \frac{iA(0)|}{|(i+1)B(0)} + \frac{iA(1)|}{|(i+1)B(1)} + \frac{iA(2)|}{|(i+1)B(2)} + \cdots + \frac{iA(n)|}{|(i+1)B(n)} + \cdots . \tag{2.3}$$

This polynomial continued fraction has elements ultimately in $\mathbb{Q}(i)[n]$ and we seek one with elements ultimately in $\mathbb{Q}[n]$. The first step is to prove (1.6) in the Introduction, i.e. that

$$e^{\pi} = 1 + \frac{4|}{|-1} + \frac{A(0)|}{|2B(0)} + \frac{A(1)|}{|B(1)} + \frac{A(2)|}{|2B(2)} + \cdots + \frac{A(2n-1)|}{|B(2n-1)} + \frac{A(2n)|}{|2B(2n)} + \cdots . \tag{2.4}$$

To do that, we shall prove that (2.3) and (2.4) are *equivalent* continued fractions in the sense that they have the same sequence of convergents (this is not obvious at first glance), so that they both converge to e^{π} because the right-hand side of (2.3) does. By [14, p. 235, Theorem 2.1], two continued fractions $b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \cdots$ and $d_0 + \frac{c_1|}{|d_1} + \frac{c_2|}{|d_2} + \cdots$ are equivalent if and only if there exists a sequence $(\rho_n)_{n\geq 0}$ such that $\rho_0 = 1$, $\rho_n \neq 0$ for all $n \geq 0$ and $c_n = \rho_n \rho_{n-1} a_n$ $(n \geq 1)$, $d_n = \rho_n b_n$ $(n \geq 0)$. Here, considering that $b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \cdots$ is the right-hand side of (2.4) and $d_0 + \frac{c_1|}{|d_1} + \frac{c_2|}{|d_2} + \cdots$ is the right-hand side of (2.3), we see that taking $\rho_0 = 1$, $\rho_{2n+1} = i + 1$ and $\rho_{2n+2} = (i+1)/2$ for $n \geq 0$, we have $c_n = \rho_n \rho_{n-1} a_n$ $(n \geq 1)$ and $d_n = \rho_n b_n$ $(n \geq 0)$. Similarily, to prove the continued fraction (1.7) for $e^{r\pi}$ stated in the Introduction for any $r \in \mathbb{R}$ such that $r^2 \in \mathbb{Q}$, we take $\alpha = e^{i\pi/2}$ and $\beta = -2ri$ in Proposition 1, so that $\alpha^{\beta} = e^{r\pi}$ can be represented by the continued fraction (2.2) with $U(x) = (x+1)^2 + 4r^2$ and $V(x) = (i+1)(2x+3)$, which is then proved to be equivalent to (1.7) by the same method as above.

Observe the alternance of the factors 1 and 2 in front of $B(n)$ in (2.4), which is formally not a polynomial continued fraction. Consider now the sequence $(p_n/q_n)_n$ of its convergents, where the sequences $(p_n)_n$ and $(q_n)_n$ are solutions of the same system of two linear recurrences:

$$\begin{cases} u_{2n} = \mu_{2n} u_{2n-1} + \lambda_{2n} u_{2n-2}, \\ u_{2n+1} = \delta_{2n+1} u_{2n} + \lambda_{2n+1} u_{2n-1} \end{cases} \tag{2.5}$$

and where, for any n large enough, $\lambda_n = A(n-2)$, $\mu_n = 2B(n-2)$, $\delta_n = B(n-1)$. A quick computation then shows that $(p_{2n+1})_n$ and $(q_{2n+1})_n$ both satisfy

$$\delta_{2n+1} u_{2n+3} = \big((\delta_{2n+3}\mu_{2n+2} + \lambda_{2n+3})\delta_{2n+1} + \delta_{2n+3}\lambda_{2n+2}\big) u_{2n+1} - \delta_{2n+3}\lambda_{2n+2}\lambda_{2n+1} u_{2n-1}. \tag{2.6}$$

This is a linear recurrence of order 2 with coefficients in $\mathbb{Q}[n]$. Hence, not only (2.6) implies that $e^{\pi} = \lim_n \frac{p_{2n+1}}{q_{2n+1}}$ is in $P_2(\mathbb{Q})$, but Lemma 1 applied to (2.6) shows that e^{π} can be represented by a polynomial continued fraction with elements in $\mathbb{Q}[n]$. We can similarly obtain the linear recurrence of order 2 satisfied by $(p_{2n})_n$ and $(q_{2n})_n$:

$$\mu_{2n} u_{2n+2} = \big((\delta_{2n+1}\mu_{2n+2} + \lambda_{2n+2})\mu_{2n} + \mu_{2n+2}\lambda_{2n+1}\big) u_{2n} - \mu_{2n+2}\lambda_{2n+1}\lambda_{2n} u_{2n-2} \tag{2.7}$$

and make the same deductions.

Let us now more precisely show how to obtain (1.2) and (1.3) from (2.4). The process consists of 'taking the even and odd parts' of a continued fraction. The two resulting continued fractions are given by general formulas given in [16], which we repeat below. For the even part, we have ([16, p. 87, Eq. (2.4.5)])

$$\begin{aligned}&\frac{a_1|}{|b_1}+\frac{a_2|}{|b_2}+\frac{a_3|}{|b_3}+\cdots\\&=\frac{b_2a_1|}{|b_2b_1+a_2}-\frac{a_2a_3b_4|}{|b_2(a_4+b_3b_4)+a_3b_4}-\frac{a_4a_5b_6b_2|}{|b_4(a_6+b_5b_6)+a_5b_6}-\frac{a_6a_7b_8b_4|}{|b_6(a_8+b_7b_8)+a_7b_8}-\cdots.\end{aligned}$$

Applying this formula to (2.4), we obtain (1.2) after some simplications, which is obviously a polynomial continued fraction. It is important to note that even though the sequence of convergents $(\widehat{p}_n/\widehat{q}_n)_{n\geq 0}$ of (1.2) coincides with $(p_{2n}/q_{2n})_{n\geq 0}$, we do not have $\widehat{p}_n = p_{2n}$ and $\widehat{q}_n = q_{2n}$ for all $n \geq 0$ because (1.2) is not the *canonical* even part of (1.6).

The formula for the odd part ([16, p. 87, Eq. (2.4.5)]) is

$$\begin{aligned}&\frac{a_1|}{|b_1}+\frac{a_2|}{|b_2}+\frac{a_3|}{|b_3}+\cdots\\&\qquad=\frac{a_1}{b_1}-\frac{a_1a_2b_3/b_1|}{|b_1(a_3+b_2b_3)+a_2b_3}-\frac{a_3a_4b_5b_1|}{|b_3(a_5+b_4b_5)+a_4b_5}-\frac{a_5a_6b_7b_3|}{|b_5(a_7+b_6b_7)+a_6b_7}-\cdots,\end{aligned}$$

from which we compute the odd part of (2.4), given by the polynomial continued fraction (1.3). Again, we mention that even though the sequence of convergents $(\widetilde{p}_n/\widetilde{q}_n)_{n\geq 0}$ of (1.3) coincides with $(p_{2n+1}/q_{2n+1})_{n\geq 0}$, we do not we have $\widetilde{p}_n = p_{2n+1}$ and $\widetilde{q}_n = q_{2n+1}$ for all $n \geq 0$ because (1.2) is not the *canonical* odd part of (1.6).

Finally, let us prove that the continued fractions (1.2) and (1.3) are not equivalent. Indeed, if they were equivalent, by the characterization of equivalence used above, the quantity $\rho_n := D(n)/F(n)$ (well defined for all integer $n \geq 2$ because F vanishes at no integer point) would satisfy

$$\frac{C(n)}{E(n)} = \rho_n\rho_{n-1} = \frac{D(n)D(n-1)}{F(n)F(n-1)}$$

for every integer $n \geq 3$, hence for all $n \in \mathbb{C}$ (but finitely many points) because this is an equality involving rational functions of n. But

$$\frac{C(n)}{E(n)} = \frac{(5+4n+4n^2)(5+4n)(4n-3)}{4(4n^2-4n+5)(4n+3)(4n-5)}$$

and

$$\frac{D(n)D(n-1)}{F(n)F(n-1)} = \frac{(43n+54n^2+24n^3+12)(24n^3+7n-18n^2-1)}{4(24n^3+18n^2+7n+1)(24n^3-54n^2+43n-12)}$$

are not equal because 5/4 is a pole of the former and not of the latter. This completes the proof of Theorem 1. □

Remark. We can produce many more inequivalent polynomial continued fractions for e^π by taking the odd and even parts of (1.2) and (1.3), and so on and so forth.

3 Proof of Proposition 1

We first need to specify the branch of logarithm. It will be necessary below to have $|\sqrt{\alpha}+1| \neq |\sqrt{\alpha}-1|$, ie that $\alpha \notin \mathbb{R}^-$. Thus we consider $\mathbb{R}^-$ as a cut and we set $\log(\alpha) := \ln|\alpha| + i\arg(\alpha)$, where $-\pi + 2k_0\pi < \arg(\alpha) < \pi + 2k_0\pi$ for some $k_0 \in \mathbb{Z}$. Moreover, we shall first prove Proposition 1 under the following supplementary technical assumption (which will then be lifted):

$$\left|e^{|\log(\alpha)|/2} - 1\right| < \max|\sqrt{\alpha} \pm 1|, \tag{3.1}$$

where for simplicity, we set $\max|\sqrt{\alpha} \pm 1| := \max(|\sqrt{\alpha}+1|, |\sqrt{\alpha}-1|)$, and similarly for min. A necessary condition for Eq. (3.1) to hold is that $k_0 = 0$. Hence, from now on, we assume that $\log(\alpha)$ denotes its principal branch. Accordingly, for any $z \in \mathbb{C}$, we set $\alpha^z := \exp(z\log(\alpha))$.

Note that this weaker version of Proposition 1 is enough to prove Theorem 1, with $\alpha = i$ and $\beta = -2i$. Moreover, (3.1) is always satisfied if $\alpha > 1$ because $\left|e^{|\log(\alpha)|/2} - 1\right| = |\sqrt{\alpha} - 1|$ in this case.

3.1 Construction of the convergents

For integers $m, n \geq 0$, let

$$I(m,n) := \frac{m!n!}{2i\pi} \int_{\mathcal{C}} \frac{\alpha^z dz}{\left(\prod_{j=0}^m (z-j)\right)\left(\prod_{j=0}^n (z-j-\beta)\right)}$$

where $\mathcal{C}$ is any simple positively oriented loop surrounding the poles of the integrand. If $\alpha = 1$, then $I(n,m) = 0$ for all $m, n \geq 0$; this case is not interesting even though some lemmas below are valid for $\alpha = 1$. The integral $I(m,n)$ is a variation of a family of integrals appearing in interpolation theory, especially for functions of exponential type, where the interpolating sets are here $\mathbb{N}$ and $\mathbb{N} + \beta$. When $\alpha = e^{i\pi/2}$, $\beta = -2i$ or $\alpha = 2$, $\beta = \sqrt{2}$, suitable generalisations of $I(m,n)$ enabled Gel'fond [10] and Kuzmin [13] to prove the transcendence of e^π and $2^{\sqrt{2}}$ respectively ([1]) by interpolating $e^{\pi z}$ and 2^z on $\mathbb{Z}[i]$ and $\mathbb{Z}[\sqrt{2}]$ respectively. Taking $n = 0$ and allowing multiplicities for the poles at $0, 1, \ldots, m$, the case $\alpha = e$ leads to a proof of the transcendence of e.

Lemma 2. *Let $\alpha \in \mathbb{C}^*$ such that $|\arg(\alpha)| < \pi$ and $\beta \in \mathbb{C} \setminus \mathbb{Z}$. For all integers $m, n \geq 0$, we have*

$$I(m,n) = Q(m,n)\alpha^\beta - P(m,n),$$

where

$$Q(m,n) := (-1)^n \sum_{k=0}^n \frac{(-1)^k \binom{n}{k} m!}{\prod_{j=0}^m (k-j+\beta)} \alpha^k$$

and

$$P(m,n) := (-1)^{m+1} \sum_{k=0}^m \frac{(-1)^k \binom{m}{k} n!}{\prod_{j=0}^n (k-j-\beta)} \alpha^k.$$

[1] The interpolating integral is implicit in Kuzmin's paper.

are polynomials in α of respective degree n and m.

Proof. This is an immediate application of the residue theorem applied to $I(m,n)$ because all the poles of the integrand are simple. $\square$

Note that $(P(m,n))_{m,n\geq 0}$ and $(Q(m,n))_{m,n\geq 0}$ are in fact well-defined for any $\alpha \in \mathbb{C}$ and any $\beta \in \mathbb{C}\setminus\mathbb{Z}$. Note also that

$$\int_{\mathcal{C}} \frac{z^k dz}{\big(\prod_{j=0}^m (z-j)\big)\big(\prod_{j=0}^n (z-j-\beta)\big)} = 0$$

for any integer $k \in \{0,\dots,m+n\}$ (because for such an integer k, the integrand is a rational fraction of z of degree ≤ -2); hence the expansion $\alpha^z = \sum_{k=0}^\infty \frac{\log(\alpha)^k}{k!} z^k$ shows that $I(m,n)$ is the remainder term of the $[m/n]$ Padé approximant of α^β at $\alpha=1$. See [2] for an alternative approach (and different goals) to the computation of the $[n/n]$ Padé approximants, with more complicated expressions and no recurrence given as in the next lemma.

We shall now be interested in the case $m=n$ only and for $S \in \{I,P,Q\}$, we set $S_n := S(n,n)$.

Lemma 3. *Let $\alpha \in \mathbb{C}^*$ such that $|\arg(\alpha)| < \pi$ and $\beta \in \mathbb{C}\setminus\mathbb{Z}$. The three sequences $(P_n)_{n\geq 0}$, $(Q_n)_{n\geq 0}$ and $(I_n)_{n\geq 0}$ are solutions of the linear recurrence of order 2:*

$$\begin{aligned}((n+2)^2-\beta^2)u_{n+2} \\ -(\alpha+1)(n+2)(2n+3)u_{n+1} + (\alpha-1)^2(n+1)(n+2)u_n = 0. \qquad (3.2)\end{aligned}$$

The roots of the characteristic equation associated to this recurrence are $(\sqrt{\alpha}\pm 1)^2$. They are distinct and such that $|\sqrt{\alpha}+1| \neq |\sqrt{\alpha}-1|$.

Note that

$$P_0 = \frac{1}{\beta}, \quad P_1 = -\frac{\alpha\beta+\alpha-\beta+1}{\beta(\beta^2-1)}, \quad Q_0 = \frac{1}{\beta}, \quad Q_1 = \frac{\alpha\beta-\alpha-\beta-1}{\beta(\beta^2-1)}.$$

Proof. We apply Zeilberger's algorithm (as implemented in Maple) to the two sequences $(P_n)_{n\geq 0}$ and $(Q_n)_{n\geq 0}$. They are both found to be satisfy (3.2) (under even less restrictive assumptions on α and β). By linearity, the same holds for the sequence $(I_n)_{n\geq 0}$.

Let us give the details for $P_n = \sum_{k=0}^n T(n,k)$ where

$$T(n,k) := (-1)^{n+1} \frac{\binom{n}{k} n!}{\prod_{j=0}^n (k-j-\beta)} (-\alpha)^k.$$

Note that $P_n = \sum_{k=0}^{n+\ell} T(n,k)$ for any integer $\ell \geq 0$ because $\binom{n}{k} = 0$ for any integer $k \geq n+1$. Zeilberger's algorithm (as implemented in Maple 16) shows that

$$\begin{aligned}((n+2)^2-\beta^2)T(n+2,k) - (\alpha+1)(n+2)(2n+3)T(n+1,k) \\ + (\alpha-1)^2(n+1)(n+2)T(n,k) = G(n,k+1) - G(n,k) \qquad (3.3)\end{aligned}$$

where the certificate

$$G(n,k) = \big(4n^2 - \alpha n^2 - \alpha\beta n + 13n + 2kn\alpha - 4kn - 4n\alpha + 2n\beta - k\beta + 10 + k^2 \\ - 4\alpha - 2\alpha\beta - 6k + 3\beta + 4k\alpha + k\alpha\beta - k^2\alpha\big)\frac{(-\alpha)^k\Gamma(1-k+\beta)n!(n+2)!}{\Gamma(n+3-k+\beta)\Gamma(k)\Gamma(n+3-k)}.$$

Since $G(n,0) = G(n,n+3) = 0$ (because of the factor $\Gamma(k)\Gamma(n+3-k)$ at the denominator), summing both sides of (3.3) for k from 0 to $n+2$ proves that $(P_n)_{n\geq 0}$ is solution of (3.2). The sequence $(Q_n)_{n\geq 0}$ is also one of its solution because Q_n is obtained from $-P_n$ by changing β to $-\beta$, and the polynomial coefficients of (3.2) depend on β only through β^2.

The characteristic equation of the recurrence (3.2) is $x^2-2(\alpha+1)x+(\alpha-1)^2 = 0$, with solutions given by $(\sqrt{\alpha}\pm 1)^2$, obviously distinct because $\alpha \neq 0$. Finally, $|\sqrt{\alpha}+1| = |\sqrt{\alpha}-1|$ implies that $\alpha \in \mathbb{R}^-$, which is excluded. □

Lemma 4. *Let us assume that $\alpha \in \mathbb{C}\setminus\{1\}$ and $\beta \in \mathbb{C}\setminus\mathbb{Z}$. Then, the sequences $(P_n)_{n\geq 0}$ and $(Q_n)_{n\geq 0}$ are $\mathbb{C}$-linearly independent for $n \geq 0$.*

Proof. We define the Casoratian $C_n := P_nQ_{n+1} - P_{n+1}Q_n$. It is readily checked that C_n is the solution of the recurrence

$$(\beta+n+2)(\beta-n-2)C_{n+1} = -(\alpha-1)^2(n+1)(n+2)C_n$$

with initial condition $C_0 = \frac{2(\alpha-1)}{\beta(\beta^2-1)}$. It follows that

$$C_n = (-1)^n\frac{2(\alpha-1)^{2n+1}n!(n+1)!}{(\beta)_{n+2}(\beta-n-1)_{n+1}}$$

and this implies that $C_n \neq 0$ for every $n \geq 0$. We then use the fact that the sequences $(P_n)_{n\geq 0}$ and $(Q_n)_{n\geq 0}$ are $\mathbb{C}$-linearly independent for $n \geq 0$ if and only if their Casoratian does not vanish for every $n \geq 0$; see [6, p. 71, Theorem 2.15]. □

3.2 Convergence of the sequence of convergents

In this section, we determine the behavior of the sequences P_n, Q_n and I_n as $n \to +\infty$.

Lemma 5. *Let $\alpha \in \mathbb{C}^*$ such that $|\arg(\alpha)| < \pi$ and $\beta \in \mathbb{C}\setminus\mathbb{Z}$. We have*

$$\limsup_{n\to+\infty} |I_n|^{1/n} \leq \big(e^{|\log(\alpha)|/2} - 1\big)^2. \tag{3.4}$$

Observe that both sides of (3.4) vanish when $\alpha = 1$. Moreover, as we shall see in Lemma 6 below, Eq. (3.4) is sharp when $\alpha \geq 1$.

Proof. In the integral for I_n, we choose $\mathcal{C}$ as the circle of center 0 and radius $R := vn$ for some $v > 1$ to be specified later, with n large enough to ensure that the poles of the integrand are all inside the open disk delimited by $\mathcal{C}$. For $z \in \mathcal{C}$, we have

$$\prod_{j=0}^{n} |z-j| \geq c_{v,n} \frac{\Gamma(vn)}{\Gamma((v-1)n)} \quad \text{and} \quad \prod_{j=0}^{n} |z-j-\beta| \geq d_{v,\beta,n} \frac{\Gamma(vn)}{\Gamma((v-1)n)},$$

where $c_{v,n}^{1/n}$ and $d_{v,\beta,n}^{1/n}$ both tend to 1 as $n \to +\infty$. Moreover, for $z \in \mathcal{C}$, we have $|\alpha^z| \leq e^{|\log(\alpha)|vn}$. Hence, using Stirling's formula $\Gamma(x) = x^{x-1/2}e^{-x}\sqrt{2\pi}(1+O(1/x))$ (valid for $|\arg(x)| < \pi$), we see that

$$\limsup_{n\to+\infty} |I_n|^{1/n} \leq e^{|\log(\alpha)|v}(v-1)^{2(v-1)}v^{-2v}.$$

It remains to minimize the right-hand side with respect to $v > 1$, which is immediate: it is minimal for $v = 1/(1-e^{-|\log(\alpha)|/2}) > 1$, which leads to (3.4). $\square$

Lemma 6. *Let $\alpha \in \mathbb{C} \in \setminus\{0,1\}$ such that $|\arg(\alpha)| < \pi$ and $\beta \in \mathbb{C} \setminus \mathbb{Z}$. Assume that* (3.1) *holds. Then*

$$\lim_{n\to+\infty} |P_n|^{1/n} = \lim_{n\to+\infty} |Q_n|^{1/n} = \max |\sqrt{\alpha} \pm 1|^2$$

and

$$\lim_{n\to+\infty} |I_n|^{1/n} = \min |\sqrt{\alpha} \pm 1|^2.$$

Proof. Classical results of Adams, Birkhoff and Trjitzinsky (see [6, p. 377, Theorem 8.36] or [22]) ensure the existence of two independent solutions $(U_n)_{n\geq 0}$ and $(V_n)_{n\geq 0}$ of (3.2) such that

$$\lim_{n\to+\infty} U_n^{1/n} = (\sqrt{\alpha}+1)^2 \quad \text{and} \quad \lim_{n\to+\infty} V_n^{1/n} = (\sqrt{\alpha}-1)^2.$$

(The assumption that $\alpha \neq 1$ is used here). Any other solution $(W_n)_{n\geq 0}$ of (3.2) is a $\mathbb{C}$-linear combination of $(U_n)_{n\geq 0}$ and $(V_n)_{n\geq 0}$ and is such that

$$\lim_{n\to+\infty} W_n^{1/n} = (\sqrt{\alpha}+1)^2 \quad \text{or} \quad \lim_{n\to+\infty} W_n^{1/n} = (\sqrt{\alpha}-1)^2.$$

because $|\sqrt{\alpha}+1| \neq |\sqrt{\alpha}-1|$ by the assumption $|\arg(\alpha)| < \pi$ (ie, there is no oscillating behavior, which typically happens when the modulus of characteristic roots are equal). By Lemma 4, the sequences $(P_n)_{n\geq 0}$ and $(Q_n)_{n\geq 0}$ are $\mathbb{C}$-independent (and thus generate the space of solutions of (3.2)), so that I_n is not 0 for any large enough n. Hence, Assumption (3.1) and Lemma 5 together imply that

$$\lim_{n\to+\infty} |I_n|^{1/n} = \min |\sqrt{\alpha} \pm 1|^2.$$

We now assume that $\max |\sqrt{\alpha} \pm 1| = |\sqrt{\alpha}+1|$; the other possibility would be dealt with in a similar way. There exist $a, b, c, d \in \mathbb{C}$ such that for all $n \geq 0$

$$P_n = aU_n + bV_n \quad \text{and} \quad Q_n = cU_n + dV_n.$$

We cannot have $a = 0$ and $c = 0$ because by Lemma 4, the sequences $(P_n)_{n\geq 0}$ and $(Q_n)_{n\geq 0}$ are $\mathbb{C}$-independent. Assume that $a \neq 0$, so that $|P_n|$ behave like $|\sqrt{\alpha}+1|^{2n}$. Then, $c \neq 0$ as well because $|I_n| = |Q_n\alpha^\beta - P_n|$ behaves like $|\sqrt{\alpha}-1|^{2n}$. Similarly, $c \neq 0$ forces $a \neq 0$. Hence we always $ac \neq 0$, so that

$$\lim_{n\to+\infty} |P_n|^{1/n} = \lim_{n\to+\infty} |Q_n|^{1/n} = \lim_{n\to+\infty} |U_n|^{1/n} = |\sqrt{\alpha}+1|^2$$

as expected. $\square$

3.3 Completion of the proof of Proposition 1

We first assume that (3.1) holds, and we recall that $|\arg(\alpha)| < \pi$ is a necessary condition for that. Consider the sequences $(P_n)_{n\geq 0}$ and $(Q_n)_{n\geq 0}$ as in Lemma 3 with the same initial conditions. Under the assumptions of Lemma 6, we have

$$\lim_{n\to+\infty} \left(\alpha^\beta - \frac{P_n}{Q_n}\right) = \lim_{n\to+\infty} \frac{I_n}{Q_n} = 0,$$

and even more precisely

$$\lim_{n\to+\infty} \left|\alpha^\beta - \frac{P_n}{Q_n}\right|^{1/n} = \frac{\min|\sqrt{\alpha}\pm 1|^2}{\max|\sqrt{\alpha}\pm 1|^2} < 1.$$

Set $p_n := P_{n+1}$ and $q_n := Q_{n+1}$ for all $n \geq -1$. We have

$$p_{-1} = \frac{1}{\beta}, p_0 = -\frac{\alpha\beta+\alpha-\beta+1}{\beta(\beta^2-1)}, q_{-1} = \frac{1}{\beta}, q_0 = \frac{\alpha\beta-\alpha-\beta-1}{\beta(\beta^2-1)}.$$

To apply Lemma 1 with $\xi := \alpha^\beta$, we first need to ensure that $\alpha^\beta \neq p_{-1}/q_{-1} = 1$ and that

$$0 \neq p_{-1}q_0 - p_0q_{-1} = \frac{2(\alpha-1)}{\beta(\beta^2-1)}$$

ie that $\alpha \neq 1$. With our assumptions on α, β and the definition of α^β for $|\arg(\alpha)| < \pi$, the condition $\alpha^\beta \neq 1$ is equivalent to $\alpha \neq 1$. Then, by Lemma 1, we have that

$$\frac{2\beta(\alpha-1)}{\beta^2-1} \cdot \frac{1}{\alpha^{-\beta}-1} = \frac{\beta-\alpha-\alpha\beta-1}{\beta^2-1} + \frac{W(1)/U(1)|}{|Z(1)/U(1)} + \frac{W(2)/U(2)|}{|Z(2)/U(2)} + \cdots \tag{3.5}$$

where $U(x) := (x+1)^2-\beta^2$, $W(x) := -(\alpha-1)^2x(x+1)$, and $Z(x) := (\alpha+1)(x+1)(2x+1)$. We also need $V(x) := (\alpha+1)(2x+3)$ because after some straighforward simplications, we obtain

$$\alpha^{-\beta} = 1 + \frac{2\beta(\alpha-1)|}{|\beta-\alpha-\alpha\beta-1} + \frac{(\alpha-1)^2A(0)|}{|B(0)} - \frac{(\alpha-1)^2A(1)|}{|B(1)} - \frac{(\alpha-1)^2A(2)|}{|B(2)} - \cdots - \frac{(\alpha-1)^2A(n)|}{|B(n)} - \cdots. \tag{3.6}$$

This is the statement of the Proposition 1 with $-\beta$ instead of β. Observe now that the assumptions of Proposition 1 (even under (3.1)) are the same for β and $-\beta$, and that $A(x)$ and $B(x)$ depend on β only through β^2. We thus obtain (2.2) by simply changing β to $-\beta$ in (3.6). We can get rid of the assumption $\alpha \neq 1$ because (3.6) holds for $\alpha = 1$ and $\beta \in \mathbb{C} \setminus \mathbb{Z}$.

It remain to get rid of the technical assumption (3.1). For this, we remark that by [12, pp. 102–106], the continued fraction on the right-hand side of (2.2) defines an analytic function in $\mathbb{C} \setminus \mathbb{R}^-$ (because its elements satisfy a general property implying that). Now, Assumption (3.1) holds for $\alpha > 1$. Hence, Identity (2.2) holds for $\alpha > 1$ and then for $|\arg(\alpha)| < \pi$ by analytic continuation of both sides.

By construction the sequence $(P_n/Q_n)_{n\geq 0}$ coincides with the sequence of convergents of (2.2).

4 Proof of Eqs. (1.4) and (1.5)

In this section, we set $\alpha = i$ and $\beta = -2i$ in Proposition 1.

We first justify (1.4) in the Introduction. By construction, for all $n \geq 0$, we have $P_n/Q_n = p_n/q_n$ where $(p_n/q_n)_{n\geq 0}$ is the sequence of convergents of (1.6), and $\widehat{p}_n/\widehat{q}_n = p_{2n}/q_{2n}$ and $\widetilde{p}_n/\widetilde{q}_n = p_{2n+1}/q_{2n+1}$. It follows from Lemma 6 that

$$\lim_{n\to+\infty} \left| e^{\pi} - \frac{P_n}{Q_n} \right|^{1/n} = \frac{|e^{i\pi/4} - 1|^2}{|e^{i\pi/4} + 1|^2} = \frac{1}{2}(2 - \sqrt{2})^2 \tag{4.1}$$

from which (1.4) follows.

Let us now justify (1.5). For $n \geq 3$, we have

$$\begin{cases} \widetilde{p}_n = F(n-1)\widetilde{p}_{n-1} + E(n-1)\widetilde{p}_{n-2} \\ \widetilde{q}_n = F(n-1)\widetilde{q}_{n-1} + E(n-1)\widetilde{q}_{n-2} \end{cases} \tag{4.2}$$

where $E(x)$ and $F(x)$ are defined in Theorem 1. The sequences $(\widetilde{p}_n)_n$ and $(\widetilde{q}_n)_n$ are independent for $n \geq 2$ as we can see by an argument similar to the proof of Lemma 3: the Casoratian $c_n := \widetilde{p}_n\widetilde{q}_{n-1} - \widetilde{p}_{n-1}\widetilde{q}_n$ satisfies $c_{n+1} = E(n)c_n$ for $n \geq 2$, and $c_2 = -748800 \neq 0$, $E(n) \neq 0$ for all $n \geq 2$. We now observe that $\widetilde{p}_n/n!^3$ and $\widetilde{q}_n/n!^3$ are solutions of the recurrence deduced from (4.2): $u_n = \alpha(n)u_{n-1} + \beta(n)u_{n-2}$, where

$$\alpha(x) := \frac{F(x-1)}{x^3} \in \mathbb{Q}(x) \quad \text{and} \quad \alpha(x) := \frac{E(x-1)}{x^3(x-1)^3} \in \mathbb{Q}(x)$$

are such that

$$\lim_{x\to+\infty} \alpha(x) = 192 \quad \text{and} \quad \lim_{x\to+\infty} \beta(x) = -1024.$$

The resulting characteristic polynomial is $x^2 - 192x + 1024$, with roots $96 \pm 64\sqrt{2} = 16(2 \pm \sqrt{2})^2$. Using again Adams, Birkhoff and Trjitzinsky's results, we deduce that the limits of

$$\left|\frac{\widetilde{p}_n}{n!^3}\right|^{1/n}, \quad \left|\frac{\widetilde{q}_n}{n!^3}\right|^{1/n} \quad \text{and} \quad \left|\frac{\widetilde{q}_n e^{\pi} - \widetilde{p}_n}{n!^3}\right|^{1/n}$$

all exist and are equal to either $16(2-\sqrt{2})^2$ or $16(2+\sqrt{2})^2$. Since

$$\frac{\widetilde{q}_n e^{\pi} - \widetilde{p}_n}{n!^3} = \frac{\widetilde{q}_n}{n!^3}\Big(e^{\pi} - \frac{P_{2n}}{Q_{2n}}\Big),$$

we deduce from (4.1) that

$$\lim_{n\to+\infty}\left|\frac{\widetilde{q}_n e^{\pi} - \widetilde{p}_n}{n!^3}\right|^{1/n} \leq 4(2+\sqrt{2})^2(2-\sqrt{2})^4 = 16(2-\sqrt{2})^2$$

hence this limit is indeed equal to $16(2-\sqrt{2})^2$. Arguing as in the proof of Lemma 6, we conclude that the limits of $|\widetilde{p}_n/n!^3|^{1/n}$ and $|\widetilde{q}_n/n!^3|^{1/n}$ are both equal to $16(2+\sqrt{2})^2$.

We proceed similarly for $\widehat{p}_n$ and $\widehat{q}_n$. For $n \geq 3$, we have

$$\begin{cases} \widehat{p}_n = D(n-1)\widehat{p}_{n-1} + C(n-1)\widehat{p}_{n-2} \\ \widehat{q}_n = D(n-1)\widehat{q}_{n-1} + C(n-1)\widehat{q}_{n-2} \end{cases}$$

from which we deduce that $\widehat{p}_n/n!^3$ and $\widehat{q}_n/n!^3$ are solutions of $u_n = \delta(n)u_{n-1} + \kappa(n)u_{n-2}$ where

$$\delta(x) := \frac{D(x-1)}{x^3} \in \mathbb{Q}(x) \quad \text{and} \quad \kappa(x) := \frac{C(x-1)}{x^3(x-1)^3} \in \mathbb{Q}(x)$$

are such that

$$\lim_{x\to+\infty} \delta(x) = 384 \quad \text{and} \quad \lim_{x\to+\infty} \kappa(x) = -4096.$$

The resulting characteristic polynomial is $x^2 - 384x + 4096$, with roots $32(2\pm\sqrt{2})^2$. We obtain exactly as above that the limits of $|\widehat{p}_n/n!^3|^{1/n}$ and $|\widehat{q}_n/n!^3|^{1/n}$ exist and are equal to $32(2+\sqrt{2})^2$.

5 Variations on Proposition 1

The method used to prove Proposition 1 is flexible. We started with the integral $I(n,n)$ but we can also consider an integral of the form $I(kn,\ell n)$, where k,ℓ are positive integers. By Lemma 2, we have $I(kn,\ell n) = Q(kn,\ell n)\alpha^{\beta} - P(kn,\ell n)$. Our method then rests on the explicit computation of the linear recurrence satisfied by $P(kn,\ell n)$ and $Q(kn,\ell n)$. Provided k and ℓ are given specific values, Zeilberger's algorithm can again be used. For instance, for $k = 2$ and $\ell = 1$, we find that both satisfy the following order 2 linear recurrence:

$$
\begin{aligned}
&(\beta-3-2n)(\beta-4-2n)(n+2+\beta)(\alpha\beta-\beta+\alpha n+8n+5+\alpha)u_{n+2}\\
&\quad+(n+2)\big(86+260n+246\alpha-23\beta-27\alpha\beta n+69\alpha^2n+51\alpha^2n^2+612\alpha n^2+30\alpha^2\\
&\quad+64n^3+10\beta^2+702\alpha n-15\alpha\beta+232n^2-28n\beta+39\alpha^2\beta-8n^2\beta+8n\beta^2+57\alpha^2\beta n\\
&\quad+3\alpha\beta^3-3\alpha^2\beta^3+\alpha^3\beta^3-2\alpha^3-5\alpha^3n-4\alpha^3n^2-\alpha^3n^3-\alpha^3\beta+2\alpha^3\beta^2+6\alpha^2\beta^2\\
&\quad-18\alpha\beta^2+12\alpha^2n^3+168\alpha n^3-b^3+21\alpha^2n^2\beta-12\alpha n^2\beta+6\alpha^2n\beta^2-15\alpha n\beta^2-2\alpha^3n\beta\\
&-\alpha^3n^2\beta+\alpha^3\beta^2n\big)u_{n+1}+2(1-\alpha)^3(2n+1)(n+2)(n+1)(\alpha n+8n+\alpha\beta+2\alpha+13-\beta)u_n=0.
\end{aligned}
$$

It is then possible to perform the same study done in the previous sections and in the end we obtain a new polynomial continued fraction for α^β, though obviously quite cumbersome to write down explicitely.

We conclude with the following simple considerations. Euler gave a formal transformation of a series into a continued fraction, the sequence of convergents of which coincide with the sequence of partial sums of the series (so that the convergence of the series is equivalent to that of the continued fraction); see [15, p. 458, Eq. (12)]. It is more easily stated as

$$\sum_{n=0}^{\infty}\Big(\prod_{j=0}^{n}a_j\Big)=\frac{a_0|}{|1}-\frac{a_1|}{|1+a_1}-\frac{a_2|}{|1+a_2}-\cdots-\frac{a_n|}{|1+a_n}-\cdots. \tag{5.1}$$

That transformation is at the origin of numerous polynomial continued fractions for classical numbers such as e and π (see for instance [5, 15]). It can be applied as well to the series $\sum_{n=0}^{\infty}\binom{\beta}{n}(\alpha-1)^n$ which converges (for any $\alpha\in\mathbb{C}$ such that $|\alpha-1|<1$ and any $\beta\in\mathbb{C}$) to α^β with $|\arg(\alpha)|<\pi$. In this case, the corresponding sequence $(a_n)_{n\geq 0}$ in (5.1) is $a_0=1$ and $a_n=(\alpha-1)(\beta-n+1)/n$ for $n\geq 1$. This leads to the continued fraction (this is [16, p. 269, Eq. (2.3.5)] that holds for $|\alpha-1|<1$ and any $\beta\in\mathbb{C}\setminus\mathbb{Z}$:

$$\alpha^\beta=\frac{1|}{|1}-\frac{(\alpha-1)\beta|}{|1+(\alpha-1)\beta}-\frac{(\alpha-1)(\beta-1)1|}{|2+(\alpha-1)(\beta-1)}-\cdots-\frac{(\alpha-1)(\beta-n)n|}{|n+1+(\alpha-1)(\beta-n)}-\cdots. \tag{5.2}$$

Applying (5.2) with $\alpha=1/2$ and $\beta=-\sqrt{2}$, we obtain a polynomial continued fraction for $2^{\sqrt{2}}$ with elements ultimately in $\mathbb{Q}(\sqrt{2})[n]$. We can also apply it with $\alpha=e^{i\pi\ell/k}$ for any rational number ℓ/k such that $0<|\ell/k|<1/3$ (so that $0<|\alpha-1|<1$) and $\beta=-ik/\ell$; this yields a polynomial continued fraction for e^π with elements ultimately in $\mathbb{Q}(i,e^{i\pi\ell/k})[n]$. These are weaker results than those obtained above using Proposition 1 but they are much simpler to obtain.

References

[1] R. Apéry, Interpolation de fractions continues et irrationalité de certaines constantes, *Bull. Sect. Sci. Mathématiques* III (1981), 37–53.

[2] K. Alladi and M. L. Robinson, Legendre polynomials and irrationality, *J. Reine Angew. Math.* **318** (1980), 137–155.

[3] G. Almkvist, D. van Straten and W. Zudilin, Apéry limits of differential equations of order 4 and 5, *Fields Inst. Commun. Ser.* **54** (2008), 105–123.

[4] A. I. Aptekarev (éditeur), *Rational approximants for Euler constant and recurrence relations*, Sovre- mennye Problemy Matematiki ('Current Problems in Mathematics'), vol. 9, MIAN (Steklov Institute), Moscow, 2007.

[5] D. Bowman and J. McLaughlin, Polynomial continued fractions, *Acta Arith.* **103**.4 (2002), 329–342.

[6] S. Elaydi, *An Introduction to Difference Equations*, Undergraduate Texts in Mathematics, Springer-Verlag New York, xxii+540 p., 2005.

[7] S. Fischler and T. Rivoal, On the values of G-functions, *Commentarii Math. Helv.* **89**.2 (2014), 313–341.

[8] S. Fischler and T. Rivoal, Arithmetic theory of E-operators, *Journal de l'École polytechnique - Mathématiques* **3** (2016), 31–65.

[9] S. Finch, *Mathematical constants*, Encyclopedia of Mathematics and its Applications **94**, Cambridge University Press, 2003.

[10] A. O. Gel'fond, Sur les nombres transcendants, *C. R. Acad. Sci. Paris* **189** (1929), 1224–1226.

[11] M. Hata, Rational approximations to π and some other numbers, *Acta Arith.* **63**.4 (1993), 335–349.

[12] A. N. Khovanskii, *The application of continued fractions and their generalizations to problems in Approximation theory*, P. Noordhoff, Groningen (1963).

[13] R. Kuzmin, On a new class of transcendental numbers, *Izv. Akad. Nauk SSSR* **7**.6 (1930), 585–597, in russian.

[14] L. J. Lange, *Continued fraction representations for functions related to the Gamma function*, Lect. Notes Pure Appl. Math. **154** (1994), 233–279.

[15] L. J. Lange, An elegant continued fraction for π, *Amer. Math. Monthly.* **106**.5 (1999), 456–458.

[16] L. Lorentzen and H. Waadeland, *Continued Fractions, Vol. 1 : Convergence Theory*, Atlantis Studies in Mathematics, 2008.

[17] Kh. Hessami Pilehrood and T. Hessami Pilehrood, An Apéry-like continued fraction for $\pi \coth \pi$, *J. Difference Equ. Appl.* **14**.12 (2008), 1279–1287.

[18] T. Rivoal, Rational approximations for values of derivatives of the Gamma function, *Trans. Amer. Math. Soc.* **361** (2009), 6115–6149.

[19] T. Rivoal, Approximations rationnelles des valeurs de la fonction Gamma aux rationnels, *J. Number Theory* **130**.4 (2010), 944–955.

[20] T. Rivoal, Values of the Beta function: from Ramanujan's continued fraction to Hermite-Padé approximants, *Proc. Steklov Inst. Math* **298**.1 (2017), 57–69.

[21] B. Salvy, Linear Differential Equations as a Data Structure, *Foundations of Computational Mathematics*, **19**.5 (2019), 1071–1112.

[22] J. Wimp and D. Zeilberger, Resurrecting the asymptotics of linear recurrences, *J. Math. Anal. Appl.* **111**.1 (1985), 162–176.

[23] W. Zudilin, An Apéry-like difference equation for Catalan's constant, *Electron. J. Comb.* **10**.1 (2003), Research paper R14, 10 pages.

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