FACTORS OF E-OPERATORS WITH AN η-APPARENT SINGULARITY AT ZERO

TANGUY RIVOAL

Abstract. In 1929, Siegel defined E-functions as power series in $\overline{\mathbb{Q}}[[z]]$, with Taylor coefficients satisfying certain growth conditions, and solutions of linear differential equations with coefficients in $\overline{\mathbb{Q}}(z)$. The Siegel-Shidlovskii Theorem (1956) generalized to E-functions the Diophantine properties of the exponential function. In 2000, André proved that the finite singularities of a differential operator in $\overline{\mathbb{Q}}(z)[\frac{d}{dz}] \setminus \{0\}$ of minimal order for some non-zero E-function are apparent, except possibly 0 which is always regular singular. We pursue the classification of such operators and consider those for which 0 is η-apparent, in the sense that there exists $\eta \in \mathbb{C}$ such that L has a local basis of solutions at 0 in $z^\eta\mathbb{C}[[z]]$. We prove that they have a $\mathbb{C}$-basis of solutions of the form $Q_j(z)z^\eta e^{\beta_j z}$, where $\eta \in \mathbb{Q}$, the $\beta_j \in \overline{\mathbb{Q}}$ are pairwise distinct and the $Q_j(z) \in \overline{\mathbb{Q}}[z]\setminus\{0\}$. This generalizes a previous result by Roques and the author concerning E-operators with an apparent singularity or no singularity at the origin, of which certain consequences are also given here.

1. Introduction

An E-function is a power series $f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n \in \overline{\mathbb{Q}}[[z]]$ (where $\overline{\mathbb{Q}}$ is embedded into $\mathbb{C}$) such that:

(i) $f(z)$ satisfies a non-zero linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$;

(ii) there exists $C > 0$ such that for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we have $|\sigma(a_n)| \leq C^{n+1}$;

(iii) there exist $D > 0$ and a sequence of positive integers d_n such that $d_n \leq D^{n+1}$ and $d_n a_m$ is an algebraic integer for all $m \leq n$.

From (i), we deduce that the a_n's all live in some number field, so that there are only finitely many Galoisian automorphisms to consider in (ii). A G-function at $z = 0$ is a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[[z]]$ such that $\sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$ is an E-function. The exponential $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ and Bessel function $J_0(z) = \sum_{n=0}^{\infty} \frac{(-z^2/4)^n}{n!^2}$ are E-functions, while $-\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$ and algebraic functions over $\overline{\mathbb{Q}}(z)$ holomorphic at $z = 0$ are G-functions. Both classes of functions have been first introduced by Siegel in 1929 to generalize the results of the Diophantine nature of the exponential and logarithmic functions of Hermite, Lindemann and Weierstrass.

Throughout the paper, by "a solution y of a differential operator L", it must be understood "a solution y of the differential equation $Ly = 0$"; a minimal non-zero operator in $\mathbb{C}(z)[\frac{d}{dz}]$ for some given function is unique up to factor of $\mathbb{C}(z) \setminus \{0\}$ and we make

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the slight abuse of terminology of saying "the minimal operator". Results of André [3], Chudnovsky [8] and Katz [12] all together imply that if $L \in \overline{\mathbb{Q}}(z)[\frac{d}{dz}] \setminus \{0\}$ is the minimal differential operator annihilating a non-zero G-function, then it is of a special type called G-operator. In particular, L is Fuchsian with rational exponents and all its solutions at any point $\alpha \in \overline{\mathbb{Q}} \cup \{\infty\}$ are (essentially) G-functions of the variable $z-\alpha$ or $1/z$ if $\alpha = \infty$; see [3, p. 717, §3].

The Fourier-Laplace transform $\widehat{L} \in \overline{\mathbb{Q}}[z, \frac{d}{dz}]$ of an operator $L \in \overline{\mathbb{Q}}[z, \frac{d}{dz}]$ is the image of L by the automorphism of the Weyl algebra $\overline{\mathbb{Q}}[z, \frac{d}{dz}]$ defined by $z \mapsto -\frac{d}{dz}$ and $\frac{d}{dz} \mapsto z$. André [4] defined an E-operator as a differential operator in $L \in \overline{\mathbb{Q}}[z, \frac{d}{dz}]$ such that $\widehat{L}$ is a G-operator. Transfering by Fourier-Laplace transform the properties of G-operators, André initiated the study of the structure of E-operators. For instance, the regularity of a G-operator at ∞ implies that 0 is the only possible finite singularity of an E-operator, and André showed that it is a regular of singular regular with rational exponents.

Any E-function is solution of an E-operator so that the finite non-zero singularities of the minimal non-zero operator satisfied by a given E-function are apparent. We recall that an apparent singularity α of $M \in \mathbb{C}(z)[\frac{d}{dz}]$ is a singularity of M at which there exists a local basis of solutions in $\mathbb{C}[[z-\alpha]]$, which are automatically all holomorphic at $z=\alpha$ (see [1, Appendix]); in other words an apparent singularity of M behaves like an ordinary point of M in terms of a local basis of solutions. We say that α is an *η-apparent singularity* for a parameter $\eta \in \mathbb{C}$ (for want of a better terminology) if M has a basis of solutions at α all of the form $(z-\alpha)^\eta \mathbb{C}[[z-\alpha]]$, and again the involved power series are necessarily all holomorphic at $z=\alpha$ (by the above cited result applied to $z^{-\eta} M z^\eta \in \mathbb{C}(z)[\frac{d}{dz}]$). Note that a 0-apparent singularity is either an apparent singularity or an ordinary point of M.

In [16], the following result was proved for E-operators for which 0 is an ordinary point or an apparent singularity.

Theorem 1 (Rivoal-Roques, 2017)**.** *Consider an E-operator $R \in \overline{\mathbb{Q}}[z, \frac{d}{dz}] \setminus \{0\}$ of order μ having an apparent singularity or no singularity at* 0*. Then R has a $\mathbb{C}$-basis of solutions each of the form*

$$P_1(z)e^{\beta_1 z} + \cdots + P_\ell(z)e^{\beta_\ell z} \tag{1.1}$$

for some integer $\ell \leq \mu$, $\beta_1, \ldots, \beta_\ell \in \overline{\mathbb{Q}}$, and $P_1(z), \ldots, P_\ell(z) \in \overline{\mathbb{Q}}[z]$ not all identically zero.

In this paper, we generalize Theorem 1 to right-factors of E-operators with an η-apparent singularity at 0. Note that η must be a rational number by the rationality of the local exponents at 0 of E-operators.

Theorem 2. *Consider an operator $L \in \overline{\mathbb{Q}}(z)[\frac{d}{dz}] \setminus \{0\}$ of order μ, which is a right factor of an E-operator. Let us assume that L has an η-apparent singularity at $z=0$ for a parameter $\eta \in \mathbb{Q}$.*

(*i*) *Then L has a $\mathbb{C}$-basis of solutions given by*

$$Q_1(z)z^\eta e^{\beta_1 z}, \ldots, Q_\mu(z)z^\eta e^{\beta_\mu z} \tag{1.2}$$

where $\beta_1, \ldots, \beta_\mu \in \overline{\mathbb{Q}}$ and $Q_1(z), \ldots, Q_\mu(z) \in \overline{\mathbb{Q}}[z] \setminus \{0\}$.

(ii) Let us also assume that L is of minimal order in $\mathbb{C}(z)[\frac{d}{dz}]$ for one of its non-zero solution. Then the β_j's in (1.2) *are pairwise distinct.*

The paper is organized as follows. We make some remarks on Theorem 2 in §2. The proof of Theorem 2 is decomposed into several steps in §3. In §3.1 and §3.2, we first prove two general propositions that are used in §4.2 to prove Theorem 2. This proof is based on a result of Halphen and is different from that of Theorem 1. As this may provide further ideas to go beyond it, we present in §4 another proof of Theorem 2(ii) based on the Fourier-Laplace transform method used in [16]. We recall in §5 a result of [16] obtained there as a by-product of the proof of Theorem 1 and present two applications of it in §5.1 and §5.2 respectively. The first application concerns the solution of a functional equation involving two E-functions, while the second is a Diophantine statement concerning the values at rational points of E-functions with rational Taylor coefficients.

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2. Some remarks about Theorem 2

(a) If L is an E-operator with an apparent singularity or no singularity at $z = 0$, then both Theorems 1 and 2 can be applied (the latter with $\eta = 0$) and (1.2) is a stronger conclusion than (1.1). In this situation, it is in fact possible to deduce (1.2) from (1.1) using Proposition 1 in §3.2. See the details at the end of §3.2, where we also explain how to deduce Theorem 2(i) from Theorem 1 when L is also assumed to be an E-operator.

(b) In (i), it is not possible to claim that the β's are pairwise distinct without another hypothesis, like in (ii). Indeed, the operator $L := z(\frac{d}{dz})^2 - 2z(\frac{d}{dz}) + z$ admits e^z and ze^z as a $\mathbb{C}$-basis of solutions, so that 0 is an apparent singularity (which was of course obvious at first sight), and L is an E-operator because $\widehat{L} = -(z-1)^2\frac{d}{dz} - 2(z-1)$ is a G-operator (it is minimal for the G-function $1/(z-1)^2$). Note that L is in fact of minimal order in $\mathbb{C}(z)[\frac{d}{dz}]$ for none of its solutions.

(c) If we assume more specifically in (ii) that L is of minimal order for some non-zero E-function, then L is automatically a right-factor of an E-operator by [4, p. 720, Theorem 4.2]. If an E-operator R can be factorized as $R = ML$ with both L, M in $\overline{\mathbb{Q}}[z, \frac{d}{dz}]$, then L is also an E-operator because the analoguous property holds for G-operators. On the other hand, L might not necessarily be an E-operator if factorization is performed in $\overline{\mathbb{Q}}(z)[\frac{d}{dz}]$. For instance, $\frac{d}{dz} - \frac{z}{z-1}$ (with $(z-1)e^z$ for solution) is clearly not an E-operator but is a right-factor of the E-operator considered in (b): $z(\frac{d}{dz})^2 - 2z(\frac{d}{dz}) + z = (z\frac{d}{dz} - \frac{2z-z^2}{z-1})(\frac{d}{dz} - \frac{z}{z-1})$.

(d) The operator $R = 6z^2(\frac{d}{dz})^2 + z(12z-1)\frac{d}{dz} + 6z^2 - z + 1$ is an E-operator because $\widehat{R} = 6(z-1)^2(\frac{d}{dz})^2 + 23(z-1)\frac{d}{dz} + 12$ is a G-operator (it is minimal for the G-function $(1-z)^{1/2} + (1-z)^{1/3}$). Hence, Theorem 2 can be applied (trivially) to the two factors of

R given by $L := 2z\frac{d}{dz} - 2z - 1$ with $\eta = 1/2$ and $L := 3z\frac{d}{dz} - 3z - 1$ with $\eta = 1/3$, which are minimal for $z^{1/2}e^z$ and $z^{1/3}e^z$ respectively.

(e) As example (d) shows, an E-operator may have a right-factor with an η-apparent singularity at 0 without having itself an η-apparent singularity at 0. More generally, let R_1 be an E-operator with an η-apparent singularity at 0, and let R_2 be an E-operator with a true singularity at 0 of a different nature than R_1. Then there exists an E-operator R which is a common left multiple of both R_1 and R_2 (left Ore property, [4, p. 720]) and thus without an η-apparent singularity at 0.

(f) Any E-function is a solution of some E-operator but the example considered in (d) shows that an E-operator need not necessarily have a non-zero E-function for solution. On the other hand, by the *Théorème de pureté* in [4, p. 706], an E-operator R necessarily have a solution of the form $z^\eta f(z)$ where $\eta \in \mathbb{Q}$ and $f(z)$ is an E-function. Denoting by μ the order of R, $f(z)$ is a solution of the E-operator $z^{\mu-\eta}Rz^\eta$ (see Proposition 2 in §3.3).

(g) It would be interesting to extend further the classification of (factors of) E-operators of order μ with a singularity at 0 of a prescribed form. For instance, when the singularity at 0 is not apparent and the μ exponents at 0 are positive integers, a $\mathbb{C}$-basis of solutions can be made of μ functions of the form $\sum_{k=0}^{\ell} \log(z)^k f_{k,\ell}(z)$, $0 \le \ell \le \mu-1$, where the $f_{k,\ell}(z)$ are E-functions. If a classification is possible in that case, Bessel functions like $J_0(z)$ will appear in the result (see Eq. (5.3) in §5.2). The methods of the present paper do not seem to be easily transposable though. If $\mu = 2$, it might be possible to build upon Gorelov's results [10].

3. Proof of Theorem 2

In this section, we first state two results that will be important in the proof of the theorem, which is then proved in §3.3.

3.1. **A result of Halphen.** The proof of Theorem 1 in [16] was based in particular on the fact that Fourier-Laplace transform of an E-operator is Fuchsian (being a G-operator). The following result of Halphen can be used instead, and it leads to a stronger conclusion. It is proved in [11, pp. 372–375] by induction on the order of the operator using *ad hoc* changes of functions and computations with local exponents.

Theorem 3 (Halphen)**.** *Let* $M = \sum_{j=0}^{\mu} p_j(z)(\frac{d}{dz})^j \in \mathbb{C}[z][\frac{d}{dz}]$ *be such that*

(*i*) $\deg(p_j) \le \deg(p_\mu)$ *for all* $j \in \{1, \dots, \mu\}$*;*
(*ii*) *The finite singularities of* M *are regular;*
(*iii*) *The solutions of* M *are uniform on* $\mathbb{C}$*.*

Then, M *has a* $\mathbb{C}$*-basis of solutions of the form* $R_j(z)e^{\beta_j z}$ *with* $R_j(z) \in \mathbb{C}(z) \setminus \{0\}$ *and* $\beta_j \in \mathbb{C}$ *not all necessarily distinct.*

A solution of M is said to be "uniform on $\mathbb{C}$" if it has no monodromy after analytic continuation along a closed loop around any point of $\mathbb{C}$.

We recall that $M \in \overline{\mathbb{Q}}[z, \frac{d}{dz}]$ is an E-operator if, by definition, its Fourier-Laplace transform $\widehat{M} \in \overline{\mathbb{Q}}[z, \frac{d}{dz}]$ is a G-operator. André proved that for any E-operator $M \in \overline{\mathbb{Q}}[z, \frac{d}{dz}]$, there also exists a G-operator $L \in \overline{\mathbb{Q}}[z, \frac{d}{dz}]$ such that $\widehat{L} = M$. (In general, $L \neq \widehat{M}$ because the Fourier-Laplace transform is not an involution, but is of order 4.)

The G-operator $L = \sum_{j=0}^{\nu} q_j(z)(\frac{d}{dz})^j$ is Fuchsian, so that $\deg(q_j) \leq \deg(q_\nu) + j - \nu \leq \deg(q_\nu)$ for all j (see [14, Chap V, p. 77, Eq. (14)]). This implies that the E-operator

$$M = \sum_{j=0}^{\nu} q_j\Big(-\frac{d}{dz}\Big)z^j = \sum_{j=0}^{\mu} p_j(z)\Big(\frac{d}{dz}\Big)^j$$

is such that $\deg(p_j) \leq \deg(p_\mu)$ for all j and that $p_\mu(z) = cz^\nu$ for some $c \in \overline{\mathbb{Q}}^*$, so that $z = 0$ is the only possible finite singularity of M. Moreover, using the regularity of L at the singularities $z = 0$ and $z = \infty$, André [4, p. 726] proved that 0 is the only slope of the Newton polygon of M at $z = 0$, which is thus an ordinary point or a regular singularity of M.

It follows that every E-operator satisfies Conditions (i) and (ii) in Halphen's Theorem and this explains its relevance in the study of E-operators.

3.2. **Minimal differential equation of exponential polynomials.** The following proposition is probably well-known and we give a proof of it for the reader's convenience.

Proposition 1. *Let $n \geq 1$ and $\beta_1, \ldots, \beta_n$ be pairwise distinct complex numbers, and $Q_1(z)$, ..., $Q_n(z) \in \mathbb{C}[z] \setminus \{0\}$. The minimal non-zero operator in $\mathbb{C}(z)[\frac{d}{dz}]$ satisfied by $\sum_{j=1}^{n} Q_j(z)e^{\beta_j z}$ is the operator of order n with $Q_1(z)e^{\beta_1 z}, \ldots, Q_n(z)e^{\beta_n z}$ as elements of a $\mathbb{C}$-basis of solutions.*

Proof of Proposition 1. Below, the various $\mathcal{O}(1/z)$ are with respect to $z \to \infty$. We set $f(z) := \sum_{j=1}^{n} Q_j(z)e^{\beta_j z}$ and for $k = 1, \ldots, n$, $g_k(z) := Q_k(z)e^{\beta_k z}$. For every k, $g_k(z)$ is solution of $N_k y(z) = 0$ where $N_k := \frac{d}{dz} - a_k(z) \in \mathbb{C}(z)[\frac{d}{dz}]$ and

$$a_k(z) := \beta_k + Q_k'(z)/Q_k(z).$$

If $n = 1$, the assumption that $\beta_1, \ldots, \beta_n$ are pairwise distinct complex numbers is empty, and there is in fact nothing to prove.

Let us assume that $n \geq 2$ from now on. The functions $g_1(z), \ldots, g_n(z)$ are solutions of a same differential equation $Ny(z) = 0$ given by $N = \mathrm{LCLM}(N_1, \ldots, N_n) \in \mathbb{C}(z)[\frac{d}{dz}]$ of order $\leq n$ *a priori.* ([1]) This order is equal to n because the g_j's form a $\mathbb{C}$-basis of N (they are even $\mathbb{C}(z)$-linearly independent because the β's are pairwise distinct; see [11, p. 136]). The function $f(z)$ is also a solution of $Ny(z) = 0$. Hence, the minimal non-zero differential equation $My(z) = 0$ satisfied by $f(z)$ is a right-factor of N. We shall now prove that the order of M is equal to n, so that $M = N$ up to a factor of $\mathbb{C}(z) \setminus \{0\}$.

[1]The Least Common Left Multiple (LCLM) of k differential operators $M_1, \ldots, M_k$ of respective degree $\mu_1, \ldots, \mu_k$ is defined in [15, p. 40]. Its order is $\leq \sum_{j=1}^{k} \mu_j$.

We first assume that none of the β's is equal to 0. It is then checked by induction on ℓ that, for any integers $k,\ell \geq 1$, $g_k^{(\ell)}(z) = a_{\ell,k}(z)g_k(z)$ for some $a_{\ell,k}(z) \in \mathbb{C}(z)$ such that

$$a_{\ell,k}(z) = \beta_k^\ell + \mathcal{O}\big(1/z\big). \tag{3.1}$$

For $\ell = 0$, we set $a_{0,k}(z) := 1$ for all $k \geq 1$. Then, for any integer $\ell \geq 0$, we have

$$f^{(\ell)}(z) = \sum_{k=1}^{n} a_{\ell,k}(z)g_k(z).$$

We claim that $\Delta(z) := \det(a_{\ell,k}(z))_{\substack{0\leq \ell\leq n-1\\ 1\leq k\leq n}} \in \mathbb{C}(z)$ is not identically zero. Indeed, one readily checks that

$$\Delta(z) = \det(\beta_k^\ell)_{\substack{0\leq \ell\leq n-1\\ 1\leq k\leq n}} + \mathcal{O}\big(1/z\big) \tag{3.2}$$

and the Vandermonde determinant on the right-hand side of (3.2) is non-zero because the β_k's are pairwise distinct. Therefore, the functions $f(z), f'(z), \ldots, f^{(n-1)}(z)$ are linearly independent over $\mathbb{C}(z)$ because the functions $g_1(z), g_2(z), \ldots, g_n(z)$ are. In other words, the function $f(z)$ does not satisfy any non-zero linear differential equation of order $< n$ with coefficients in $\mathbb{C}(z)$.

It remains to deal with the case when exactly one of the β's is zero, say $\beta_1 = 0$. We define the $a_{\ell,k}(z)$ and $\Delta(z)$ as above. For $Q_1(z) \in \mathbb{C}[z] \setminus \{0\}$ of degree $d \geq 0$, we have

$$a_{1,1}(z) = d/z + \mathcal{O}\big(1/z^2\big), \quad a_{\ell,1}(z) = \mathcal{O}\big(1/z^2\big) \quad (\ell \geq 2).$$

For every $k \geq 2$ and $\ell \geq 1$, the estimate (3.1) still holds, and we recall that $a_{0,k}(z) = 1$ for all $k \geq 1$. It follows that

$$\Delta(z) = \beta_2 \cdots \beta_n \det(\beta_k^\ell)_{\substack{0\leq \ell\leq n-2\\ 2\leq k\leq n}} + \mathcal{O}\big(1/z\big).$$

Hence $\Delta(z)$ is not identically 0 because $\beta_2, \ldots, \beta_n$ are all non-zero and pairwise distinct, and we conclude again that $f(z)$ does not satisfy any non-zero differential equation of order $< n$ with coefficients in $\mathbb{C}(z)$. $\square$

3.3. **Proof of Theorem 2.** Let $R \in \overline{\mathbb{Q}}[z, \frac{d}{dz}]\setminus\{0\}$ be an E-operator of which $L \in \overline{\mathbb{Q}}(z)[\frac{d}{dz}]\setminus \{0\}$ is a right-factor: we have $R = KL$ for some $K \in \overline{\mathbb{Q}}(z)[\frac{d}{dz}] \setminus \{0\}$. Since 0 is the only possible finite singularity of R and since it is a regular one, the finite singularities of L are regular (and even apparent if non-zero).

Let us denote by $N(D)$ the Newton polygon of a differential operator $D \in \mathbb{C}(z)[\frac{d}{dz}]$. Theorem 4.3 (*iv*) of [4] ensures that the slopes of $N(R)$ at ∞ are in $\{0, 1\}$. Hence, the slopes of L at ∞ are also in $\{0, 1\}$ because $N(R) = N(K) + N(L)$ (where the $+$ sign refers to the Minkowsky sum of subsets of $\mathbb{R}^2$; see [15, p. 92, Lemma 3.45]).

R, and thus L as well, admits a $\mathbb{C}$-basis of solutions in the Nilsson-Gevrey arithmetic class $\mathrm{NGA}\{0\}_{-1}$ by the *Théorème de pureté* in [4, p. 706]. We recall that an element of $\mathrm{NGA}\{0\}_{-1}$ is a function of the form

$$\sum_{\alpha,k,\ell} c_{\alpha,k,\ell} z^\alpha \log(z)^k f_{\alpha,k,\ell}(z)$$

where α and (k,ℓ) run through a finite subset of $\overline{\mathbb{Q}}$ and $\mathbb{N}^2$ respectively, $c_{\alpha,k,\ell} \in \mathbb{C}$ and the $f_{\alpha,k,\ell}(z)$ are E-functions. But by assumption, L has an η-apparent singularity at $z=0$, ie it also admits a $\mathbb{C}$-basis at 0 made of functions in $z^\eta\mathbb{C}[[z]]$. Hence, L admits a $\mathbb{C}$-basis of solutions $z^\eta f_1(z), \ldots, z^\eta f_\mu(z)$ where each $f_j(z)$ is a $\mathbb{C}$-linear combination of E-functions, hence is an entire function.

Write $z^{-\eta}Lz^\eta = \sum_{j=0}^{\mu} \frac{c_j(z)}{b(z)}(\frac{d}{dz})^j \in \overline{\mathbb{Q}}(z)[\frac{d}{dz}]$ where $b(z) \in \overline{\mathbb{Q}}[z]$ and for each j, $c_j(z) \in \overline{\mathbb{Q}}[z]$. Consider now the operator $M \in \overline{\mathbb{Q}}[z, \frac{d}{dz}]$ of order μ defined by $M := b(z)z^{-\eta}Lz^\eta$. The slopes of M at ∞ are the same as those of L. Hence they are in $\{0,1\}$ so that $\deg(c_j) \leq \deg(c_\mu)$ (see for instance [7, Eq. (15)]). The non-zero finite singularities of M are apparent, and moreover M has a 0-apparent singularity at $z=0$, with a $\mathbb{C}$-basis of solutions given by $f_1(z), \ldots, f_\mu(z)$, which are all entire functions. Hence, M satisfies all the assumptions of Halphen's Theorem, from which it follows that M has a $\mathbb{C}$-basis of solutions given by

$$P_1(z)e^{\beta_1 z}, \ldots, P_\mu(z)e^{\beta_\mu z}$$

where $\beta_1, \ldots, \beta_\mu \in \mathbb{C}$ (not necessarily distinct) and $P_1(z), \ldots, P_\mu(z) \in \mathbb{C}(z) \setminus \{0\}$. The numbers $\beta_1, \ldots, \beta_\mu$ are in $\overline{\mathbb{Q}}$ because M has algebraic coefficients.

For each j, $P_j(z)$ is a solution of the operator $M_j := e^{-\beta_j z}Me^{\beta_j z} \in \overline{\mathbb{Q}}[z, \frac{d}{dz}]$. Therefore, by Proposition 7 in [9, § 5], for each j, there exist n_j functions $P_{k,j}(z)$ algebraic over $\overline{\mathbb{Q}}(z)$ ($k = 1, \ldots, n_j$) which are all solutions of M_j and $c_1, \ldots, c_{n_j} \in \mathbb{C}$ such that $P_j(z) = \sum_{k=1}^{n_j} c_k P_{k,j}(z)$. Hence, the functions $P_{k,j}(z)e^{\beta_j z}$, $j = 1, \ldots, \mu$, $k = 1, \ldots, n_j$, form a generating set of the solutions of M, from which we extract a $\mathbb{C}$-basis of solutions of M that we denote by

$$Q_1(z)e^{\beta_1 z}, \ldots, Q_\mu(z)e^{\beta_\mu z} \tag{3.3}$$

where $\beta_j \in \overline{\mathbb{Q}}$ (recall that the β_j's are not necessarily distinct). Since the solutions of M are entire functions, the various algebraic functions $Q_j(z)$ are in $\overline{\mathbb{Q}}[z]$.

Since $L = b(z)^{-1}z^\eta M z^{-\eta}$, it follows that L has a $\mathbb{C}$-basis of solutions given by

$$Q_1(z)z^\eta e^{\beta_1 z}, \ldots, Q_\mu(z)z^\eta e^{\beta_\mu z}$$

where $\beta_1, \ldots, \beta_\mu \in \overline{\mathbb{Q}}$ and $Q_1(z), \ldots, Q_\mu(z) \in \overline{\mathbb{Q}}[z] \setminus \{0\}$. This proves (i).

Let us now prove (ii). Let $z^\eta f(z) \neq 0$ be a solution of L for which L is of minimal order in $\mathbb{C}(z)[\frac{d}{dz}]$. Then, M is of minimal order μ for $f(z)$ in $\mathbb{C}(z)[\frac{d}{dz}]$ and by (3.3), there exists a non-empty set $J \subset \{1, \ldots, \mu\}$ such that

$$f(z) = \sum_{j \in J} \widetilde{Q}_j(z)e^{\beta_j z}$$

where, for $j \in J$, the β_j's are pairwise distinct and $\widetilde{Q}_j(z) \in \mathbb{C}[z] \setminus \{0\}$. By Proposition 1, the minimal equation in $\mathbb{C}(z)[\frac{d}{dz}]$ for $f(z)$ is of order $\#J$ with a $\mathbb{C}$-basis given by $(\widetilde{Q}_j(z)e^{\beta_j z})_{j\in J}$. Hence necessarily, $\#J = \mu$ and the β_j's are all pairwise distinct.

Remark. Though this was not used, the operator $z^{-\eta}Lz^\eta$ introduced in the above proof is a right-factor of an E-operator; this is a consequence of Proposition 2 below, which is of independent interest (see below) and apparently not yet recorded in the literature.

Proposition 2. *Let $\eta \in \mathbb{Q}$ and $S \in \overline{\mathbb{Q}}[z, \frac{d}{dz}] \setminus \{0\}$ be an E-operator of order μ. Then $z^{\mu-\eta} S z^\eta \in \overline{\mathbb{Q}}[z, \frac{d}{dz}]$ is also an E-operator.*

Proof. The argument is inspired by that of [13, Corollary 3.3.3] in a related but different context. Note that $z^{-\eta} S z^\eta \in z^{-\delta}\overline{\mathbb{Q}}[z, \frac{d}{dz}]$ for some $\delta \in \{0, 1, \dots, \mu\}$, which justifies that $M := z^{\mu-\eta} S z^\eta \in \overline{\mathbb{Q}}[z, \frac{d}{dz}]$. For later use, we write $M = \sum_{k=0}^{\mu} b_k(z)(\frac{d}{dz})^k$ where $b_k(z) \in \overline{\mathbb{Q}}[z]$ and $b_\mu(z) = cz^\kappa$ for $\kappa \in \mathbb{N}$ and $c \in \overline{\mathbb{Q}}^*$. (The latter property holds because it does for the E-operator S.) The operators S and M both have a $\mathbb{C}$-basis of solutions in $\mathrm{NGA}\{0\}_{-1}$. Let $f_1, \dots, f_\mu$ be such a $\mathbb{C}$-basis of M. By [4, Theorem 6.1], each f_j is a solution of some E-operator $R_j \in \overline{\mathbb{Q}}[z, \frac{d}{dz}] \setminus \{0\}$ and moreover by the left Ore property (which holds for E-operators, see [4, p. 720]) there exists an E-operator $R = \sum_{p=0}^{\rho} r_j(z)(\frac{d}{dz})^j \in \overline{\mathbb{Q}}[z, \frac{d}{dz}] \setminus \{0\}$ of which each R_j is a right-factor. M is then a right-factor of R, ie there exists $N \in \overline{\mathbb{Q}}(z)[\frac{d}{dz}]$ of order ω such that $R = NM$, and $\rho = \omega + \mu$. Writing $N = \sum_{j=0}^{\omega} a_j(z)(\frac{d}{dz})^j$, we claim that each $a_j(z) \in \overline{\mathbb{Q}}(z)$ has at most a pole at $z = 0$. Indeed, we have

$$NM = \sum_{k=0}^{\mu} \sum_{\ell=0}^{\omega} \left(\sum_{j=\ell}^{\omega} \binom{j}{\ell} a_j(z) b_k(z)^{(j-\ell)} \right) \left(\frac{d}{dz} \right)^{k+\ell}$$

so that by equating on both sides of $R = NM$ the coefficients of $\left(\frac{d}{dz}\right)^n$ for each $n \in \{\mu, \mu+1 \dots, \rho\}$, we obtain

$$r_n(z) = b_\mu(z) a_{n-\mu}(z) + \sum_{j=1}^{\rho-n} c_j(z) a_{n-\mu+j}(z),$$

for some $c_j(z) \in \overline{\mathbb{Q}}[z]$ for $j \in \{1, 2, \dots, \rho - n\}$. Since $r_n(z) \in \overline{\mathbb{Q}}[z]$ and $b_\mu(z) = cz^\kappa$, it follows by induction on $n = \rho, \rho-1, \dots, \mu$ that $a_{n-\mu}(z) \in \overline{\mathbb{Q}}(z)$ has at most a pole at $z = 0$. The claim follows.

We can thus write $N = z^{-\nu} K$ with $\nu \in \mathbb{Z}_{\geq 0}$ and $K \in \overline{\mathbb{Q}}[z, \frac{d}{dz}]$. Since the factorization $z^\nu R = KM$ holds in $\overline{\mathbb{Q}}[z, \frac{d}{dz}]$, M is an E-operator by [4, p. 720] because $z^\nu R$ is an E-operator. Indeed, $\widehat{z^\nu R} = \widehat{z^\nu}\widehat{R} = (-\frac{d}{dz})^\nu \widehat{R}$ is a G-operator as a product of two G-operators. $\square$

We conclude this section with the proof of the remark made in (a) of §2, i.e. that (1.1) implies (1.2) with $\eta = 0$. Indeed with the notations of Theorem 1, consider a solution $f(z) \neq 0$ of the E-operator R of the form (1.1). Let $M_f \in \mathbb{C}(z)[\frac{d}{dz}] \setminus \{0\}$ be minimal for $f(z)$, so that M_f is a right factor of R. By Proposition 1, M_f as a $\mathbb{C}$-basis of solutions of the form (1.2) with $\eta = 0$. Considering all solutions of R, we deduce that R has a generating set of solutions of the form (1.2) with $\eta = 0$, hence a $\mathbb{C}$-basis of the same form. Moreover, using Proposition 2, it follows from the above considerations that when we also assume that L is also an E-operator in Theorem 2, then (1.2) is an immediate consequence of Theorem 1 (of which the E-operator $z^{\mu-\eta} L z^\eta$ satisfies the assumptions).

4. Another proof of Theorem 2(ii)

In this section, we present another proof of Case (ii) of Theorem 2, which could be useful in other contexts. It is based on the Fourier-Laplace method used in [16] to prove Theorem 1, and it avoids Halphen's Theorem. On the other hand, it is not clear how Case (i) of Theorem 2 could be obtained by this method. Since this section can be read (essentially) independently of the rest of the paper, we repeat certain arguments used in §3.

We first state a useful proposition, that will be used in the proofs of Theorem 2(ii) in §4.2 below and of Theorem 6 in §5.2.

4.1. **Taylor coefficients of solutions of differential operators.** The following result is a generalization of [16, Proposition 1]. Given a power series $g(z) = \sum_{n=0}^{\infty} b_n (z-\beta)^n \in \overline{\mathbb{Q}}[[z-\beta]]$ with $\beta \in \overline{\mathbb{Q}}$, we set $g^\sigma(z) = \sum_{n=0}^{\infty} \sigma(b_n)(z-\sigma(\beta))^n$ for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Proposition 3. *Consider a differential operator $L \in \overline{\mathbb{Q}}(z)[\frac{d}{dz}]$. Let*

$$F(z-\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{n!}(z-\alpha)^n \in \overline{\mathbb{Q}}[[z-\alpha]]$$

be a local solution of L at $\alpha \in \overline{\mathbb{Q}}$ which is either an ordinary point or an apparent singularity of L.

(i) *Let d_n denote the smallest positive integer such that $d_n a_0, d_n a_1, \ldots, d_n a_n$ are algebraic integers. There exists a positive integer C_α such that, for all $n \geq 0$, d_n divides C_α^{n+1}.*

(ii) *If $F^\sigma(z)$ is an entire function for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and if the slopes of the Newton polygon of L at ∞ are in $\{0,1\}$, then $F(z)$ is an E-function.*

Proof of Proposition 3. Proposition 1 in [16] deals with the case where α is an ordinary point, and we now consider the case where α is an apparent singularity of L.

It is then known that there exist $\widetilde{L}, M \in \overline{\mathbb{Q}}(z)[\frac{d}{dz}] \setminus \{0\}$ such that $\widetilde{L} = ML$ and α is not a singularity of $\widetilde{L}$; see [1, Appendix] for instance. Hence, [16, Proposition 1(i)] applies to the ordinary solution $F(z-\alpha)$ of $\widetilde{L}$ and (i) follows.

Concerning (ii), we first note that for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, $F^\sigma(z-\sigma(\alpha))$ is a local solution of L^σ (obtained from L by the action of σ on the coefficients of L) at the point $\sigma(\alpha)$, which is apparent for L^σ. The proof of (ii) uses (i) but the value of C_α is irrelevant; we thus proceed exactly as for the corresponding statement in [16, Proposition 1], *mutatis mutandis.* □

4.2. **Proof of Case** (ii)**.** $L \in \overline{\mathbb{Q}}(z)[\frac{d}{dz}] \setminus \{0\}$ is of minimal order for some non-zero solution, which we denote by $f(z)$. L is a right-factor of an E-operator $R \in \overline{\mathbb{Q}}[z, \frac{d}{dz}] \setminus \{0\}$ of which $f(z)$ is also a solution. The slopes of L at ∞ are also in $\{0,1\}$ (see the beginning of §3.3).

We shall first prove the result in the case $\eta = 0$, to which the general case will then be reduced. By minimality of L for $f(z)$, L admits a $\mathbb{C}$-basis of solutions in the Nilsson-Gevrey arithmetic class $\mathrm{NGA}\{0\}_{-1}$ by the *Théorème de pureté* in [4, p. 706]. But by the assumption $\eta = 0$, L also admits a $\mathbb{C}$-basis at 0 made of functions in $\mathbb{C}[[z]]$, hence in $\overline{\mathbb{Q}}[[z]]$

because L has coefficients in $\overline{\mathbb{Q}}$. Hence, L admits a $\mathbb{C}$-basis of solutions $f_1, \dots, f_\mu$ which consists of E-functions. Then, for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the E-functions $f_1^\sigma, \dots, f_\mu^\sigma$ form a $\mathbb{C}$-basis of solutions of $L^\sigma \in \overline{\mathbb{Q}}(z)[\frac{d}{dz}]$. Indeed, if $f_1^\sigma, \dots, f_\mu^\sigma$ were $\mathbb{C}$-linearly dependent, they would also be $\overline{\mathbb{Q}}$-linearly dependent, and using σ^{-1} on the induced $\overline{\mathbb{Q}}$-relations of Taylor coefficients, this would imply that $f_1, \dots, f_\mu$ are $\overline{\mathbb{Q}}$-linearly dependent. In particular, for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, any solution of L^σ is an entire function.

Moreover, since an E-operator has at most 0 has finite singularity, the (putative) finite non-zero singularities of L are apparent. Let us fix $\alpha \in \overline{\mathbb{Q}}^*$. We can apply Proposition 3 to such an α and we deduce from the above remarks that L admits at $z = \alpha$ a basis of solutions of the form $F_1(z-\alpha), \dots, F_\mu(z-\alpha)$, where each $F_j(z)$ is an E-function. We fix j and set $F(z) := F_j(z)$. Observe that $R\big(F(z-\alpha)\big) = 0$ because L is a right-factor of R. Since R is an E-operator, we are now exactly in the same situation as in the proof of [16, Theorem 1] and by the arguments based on the Fourier-Laplace transform given there (and that we don't repeat), we obtain that

$$F(z) = \sum_\kappa Q_\kappa(z) e^{\kappa z} \tag{4.1}$$

where κ runs through a finite set of algebraic numbers and the $Q_\kappa(z) \in \overline{\mathbb{Q}}[z] \setminus \{0\}$. Of course, the κ's can be assumed to all be pairwise distinct in (4.1).

As $F(z-\alpha)$ represents any element of a $\mathbb{C}$-basis of local solutions of $Ly(z) = 0$ at $z = \alpha$, it follows that L has a $\mathbb{C}$-basis of solutions of the form (4.1) and that the non-zero solution $f(z)$ can be written

$$f(z) = \sum_{j=1}^{\nu} Q_j(z) e^{\beta_j z} \tag{4.2}$$

where $\nu \geq 1$, the $\beta_j \in \overline{\mathbb{Q}}$ can be assumed to all be pairwise distinct and the $Q_j(z) \in \mathbb{C}[z] \setminus \{0\}$.

The conclusion can now be obtained. Indeed, since L of order μ is of minimal order for $f(z)$ in $\mathbb{C}(z)[\frac{d}{dz}]$, Proposition 1 applied to (4.2) implies that $\nu = \mu$ and that L has a $\mathbb{C}$-basis made of $Q_1(z)e^{\beta_1 z}, \dots, Q_\mu(z)e^{\beta_\mu z}$. Moreover, since L has algebraic coefficients, we can even take $Q_j(z) \in \overline{\mathbb{Q}}[z] \setminus \{0\}$ for all j. This completes the proof of the case $\eta = 0$ of Theorem 2(ii).

Let us now prove the general case. By André's *Théorème de pureté* again, L has $\mathbb{C}$-basis made of $z^\eta f_1(z), \dots, z^\eta f_\mu(z)$ where each $f_j(z)$ is a non-zero E-function. By assumption, L is of minimal order in $\mathbb{C}(z)[\frac{d}{dz}]$ for some function of the form $z^\eta f(z)$ where $f(z) = \sum_{j=1}^\mu d_j f_j(z) \neq 0$ for some $d_j \in \mathbb{C}$.

Let $M := z^{-\eta} L z^\eta \in \overline{\mathbb{Q}}(z)[\frac{d}{dz}] \setminus \{0\}$. Then

— $Mf(z) = Mf_1(z) = \cdots = Mf_\mu(z) = 0$;
— M is of minimal order for $f(z)$;
— M has an apparent singularity at 0 because $f_1(z), \dots, f_\mu(z)$ make up a local $\mathbb{C}$-basis of M at $z = 0$.

Fix any $j \in \{1, \dots, \mu\}$ and set $M_j \in \overline{\mathbb{Q}}(z)[\frac{d}{dz}] \setminus \{0\}$ minimal for $f_j(z)$ in $\mathbb{C}(z)[\frac{d}{dz}]$, of order $\kappa_j \geq 1$. We observe that we can apply the already proven case $\eta = 0$ of Theorem 2(ii) to M_j because:

— it is a non-zero right-factor of the E-operator of which $f_j \neq 0$ is a solution;

— it has (at most) an apparent singularity at 0 because it is a right-factor of M, with the same property;

— it is minimal for f_j in in $\mathbb{C}(z)[\frac{d}{dz}]$.

Therefore, M_j has a $\mathbb{C}$-basis of solutions of the form $Q_{j,1}(z)e^{\beta_{j,1}z}, \dots, Q_{j,\kappa_j}(z)e^{\beta_{j,\mu_j}z}$ where $Q_{j,k}(z) \in \overline{\mathbb{Q}}[z] \setminus \{0\}$ and $\beta_{j,k} \in \overline{\mathbb{Q}}$ (pairwise distinct). Consequently, the operator $\mathcal{M} := \mathrm{LCLM}(M_1, \dots, M_\mu) \in \overline{\mathbb{Q}}(z)[\frac{d}{dz}] \setminus \{0\}$, which is of order $\delta \leq \sum_{j=1}^{\mu} \kappa_j$, has a $\mathbb{C}$-basis of solutions made from some of the functions $Q_{j,k}(z)e^{\beta_{j,k}z}$, $k = 1, \dots, \kappa_j$, $j = 1, \dots, \mu$. A $\mathbb{C}$-basis of $\mathcal{M}$ is thus of the form $P_j(z)e^{\alpha_j z}$, $j = 1, \dots, \delta$, with $P_j(z) \in \overline{\mathbb{Q}}[z] \setminus \{0\}$ and $\alpha_j \in \overline{\mathbb{Q}}$ (not necessarily distinct).

Since $f(z)$ is a solution of $\mathcal{M}$ and M is of minimal order for $f(z)$, M is a right-factor of $\mathcal{M}$. Hence, a $\mathbb{C}$-basis of M is also of the form $A_j(z)e^{\gamma_j z}$, $j = 1, \dots, \mu$, with $A_j(z) \in \overline{\mathbb{Q}}[z] \setminus \{0\}$ and $\gamma_j \in \overline{\mathbb{Q}}$. Hence, there exists a non-empty set $J \subset \{1, \dots, \mu\}$ such that

$$f(z) = \sum_{j \in J} \widetilde{A}_j(z)e^{\gamma_j z}$$

where, for $j \in J$, the γ_j's are pairwise distinct and $\widetilde{A}_j(z) \in \mathbb{C}[z] \setminus \{0\}$. M being of minimal order for $f(z)$ in $\mathbb{C}(z)[\frac{d}{dz}] \setminus \{0\}$, Proposition 1 implies that $\#J = \mu$, so that the γ_j's are all pairwise distinct.

By definition of M, we have $L = z^\eta M z^{-\eta}$ and it follows that L has a $\mathbb{C}$-basis of the form stated in Theorem 2 (ii), given by $A_j(z)z^\eta e^{\gamma_j z}$, $j = 1, \dots, \mu$.

5. Two applications of a result in [16]

We now recall a result obtained in [16] as a by-product of the proof of Theorem 1; see also Eq. (4.1) above. It is not clear if it could also be obtained using Halphen's Theorem. It has two interesting applications we present in the next sections.

Theorem 4 (Rivoal-Roques, 2017)**.** *Let $L \in \overline{\mathbb{Q}}(z)[\frac{d}{dz}] \setminus \{0\}$ be an E-operator. Let us assume there exist a non-zero E-function $F(z)$ and $\alpha \in \overline{\mathbb{Q}}^*$ such that $F(z - \alpha)$ is a solution of the equation $Ly(z) = 0$. Then*

$$F(z) = \sum_{\kappa} Q_\kappa(z)e^{\kappa z}, \tag{5.1}$$

where κ runs through a finite set of algebraic numbers and the $Q_\kappa(z) \in \overline{\mathbb{Q}}[z] \setminus \{0\}$.

5.1. **A functional equation between two E-functions.** The exponential function satisfies $e^z = e^{-\alpha}e^{z+\alpha}$, which can be interpreted as a functional equation of the form $F(z) = \beta G(z + \alpha)$ between two E-functions F and G. Theorem 4 enables us to solve this functional equation when $\alpha \in \overline{\mathbb{Q}}^*$.

Theorem 5. *Let $F(z)$ and $G(z)$ be two non-zero E-functions. Let us assume there exist $\alpha \in \overline{\mathbb{Q}}^*$ and $\beta \in \mathbb{C}^*$ such that $F(z) = \beta G(z+\alpha)$ for all $z \in \mathbb{C}$. Then $F(z)$ and $G(z)$ are both of the form $Q(z)e^{\kappa z}$ for some $\kappa \in \overline{\mathbb{Q}}$ and $Q(z) \in \mathbb{Q}[z] \setminus \{0\}$.*

The converse is true: the E-functions $F(z) = Q(z)e^{\kappa z}$ and $G(z) = Q(z-\alpha)e^{\kappa z}$ satisfy the equation $F(z) = \beta G(z+\alpha)$ with $\beta = e^{-\alpha\kappa}$. It would be interesting to determine the non-zero E-functions F_j solutions of the functional equation $\sum_{k=1}^{d} \beta_j F_j(\delta_j z + \alpha_j) = 0$ with given $\alpha_j, \delta_j \in \overline{\mathbb{Q}}^*$ and $\beta_j \in \mathbb{C}^*$ for all $j = 1, \ldots, d$.

Proof of Theorem 5. Let $F(z), G(z)$ be two non-zero E-functions, $\alpha \in \overline{\mathbb{Q}}^*$ and $\beta \in \mathbb{C}^*$ be such that $F(z) = \beta G(z+\alpha)$ for all $z \in \mathbb{C}$.

Let $L \in \overline{\mathbb{Q}}(z)[\frac{d}{dz}] \setminus \{0\}$ be an E-operator for $F(z)$. Then $G(z+\alpha)$ is also a solution of L and by Theorem 4,

$$G(z) = \sum_{j=1}^{r} Q_j(z)e^{\kappa_j z}$$

for some pairwise distinct $\kappa_j \in \overline{\mathbb{Q}}$ and $Q_j(z) \in \overline{\mathbb{Q}}[z] \setminus \{0\}$. The situation being symmetric, we can consider an E-operator for $G(z)$ of which $F(z-\alpha)$ is also a solution, and Theorem 4 implies that

$$F(z) = \sum_{j=1}^{s} P_j(z)e^{\delta_j z}$$

for some pairwise distinct $\delta_j \in \overline{\mathbb{Q}}$ and $P_j(z) \in \overline{\mathbb{Q}}[z] \setminus \{0\}$.

The equality $F(z) = \beta G(z+\alpha)$ now reads

$$\sum_{j=1}^{s} P_j(z)e^{\delta_j z} - \beta \sum_{j=1}^{r} e^{\alpha\kappa_j} Q_j(z+\alpha)e^{\kappa_j z} = 0. \tag{5.2}$$

Now, for any sequence of pairwise distinct complex numbers $(\omega_j)_{j=1,\ldots,n}$, the functions $e^{\omega_j z}$, $j = 1, \ldots, n$, are $\mathbb{C}(z)$-linearly independent ([11, p. 136]). Therefore, Eq. (5.2) implies that $r = s$ and, up to relabelling, that $\delta_j = \kappa_j$ and $P_j(z) = \beta e^{\alpha\kappa_j} Q_j(z+\alpha)$ for all $j = 1, \ldots, r$.

Let us assume that $r \geq 2$. Then, at least one of the $\kappa_j' s$ is non-zero, say κ_1. In that case, $e^{\alpha\kappa_j} \notin \overline{\mathbb{Q}}$ because $\alpha\kappa_1 \in \overline{\mathbb{Q}}^*$. Therefore the equality $P_1(z) = \beta e^{\alpha\kappa_1} Q_1(z+\alpha)$ with $P_1(z), Q_1(z+\alpha) \in \overline{\mathbb{Q}}[z] \setminus \{0\}$ forces that $\beta = e^{-\alpha\kappa_1}\eta$ for some $\eta \in \overline{\mathbb{Q}}^*$. Hence, for all $j = 2, \ldots, r$, we have $P_j(z) = e^{\alpha(\kappa_j - \kappa_1)}\eta Q_j(z+\alpha)$ with $P_j(z), Q_j(z+\alpha) \in \overline{\mathbb{Q}}[z] \setminus \{0\}$. This is impossible because the κ_j's are pairwise distinct algebraic numbers and thus $e^{\alpha(\kappa_j-\kappa_1)} \notin \overline{\mathbb{Q}}$ when $j \in \{2, \ldots, r\}$. It follows that $r = 1$, which completes the proof. $\square$

5.2. **Irrationalité sans irrationalité.** The second application was suggested to the author by André. As explained below, it is not a new result and it is even weaker than what is currently known with the same hypothesis. But its proof is new, simple and fits into the "*transcendance sans transcendance*" point of view developped by André in [5]. The latter is based on the special properties of the differential equations satisfied by E-functions and avoids a direct use of standard arguments in Diophantine theory such as Siegel's lemma and Shidlovskii's lemma, etc. ([2])

Theorem 6. *Let $f(z) \in \mathbb{Q}[[z]]$ be an non-zero E-function and let $L \in \mathbb{Q}(z)[\frac{d}{dz}] \setminus \{0\}$ of order μ be the minimal operator for $f(z)$ (in $\mathbb{C}(z)[\frac{d}{dz}]$). Let us assume that $Ly(z) = 0$ has a non-entire solution.*

Then, for any $\alpha \in \mathbb{Q}^*$ which is not a singularity of L, the μ numbers $f(\alpha)$, $f'(\alpha)$, $\ldots, f^{(\mu-1)}(\alpha)$ generate a $\mathbb{Q}$-vector space of dimension at least 2. In particular at least one of these numbers is irrational.

Proof of Theorem 6. Let $\alpha \in \mathbb{Q}^*$ which is not a singularity of L. The numbers $f(\alpha)$, $f'(\alpha)$, $\ldots, f^{(\mu-1)}(\alpha)$ cannot all be equal to zero because α is an ordinary point of L and $f(z)$ is a non-zero solution of L. Hence, they generate a $\mathbb{Q}$-vector space $\mathcal{F}_\alpha$ of dimension ≥ 1.

Let us assume that $\mathcal{F}_\alpha$ has dimension 1, i.e that $\mathcal{F}_\alpha = \mathbb{Q}\beta$ where $\beta := f^{(j)}(\alpha) \neq 0$ for some $j \in \{0, \ldots, \mu - 1\}$. The equation $Lf(z) = 0$ also implies that $f^{(n)}(\alpha) \in \mathcal{F}_\alpha$ for all $n \geq 0$. The function

$$F(z) := f(z + \alpha)/\beta = \sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)/\beta}{n!} z^n \in \mathbb{Q}[[z]]$$

is thus a non-zero E-function by Proposition 3 because the slopes at ∞ of L are in $\{0, 1\}$ (see the beginning of §3.3) and for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, $F^\sigma = F$ is an entire function.

Let R be an E-operator for $f(z)$, of which L is a right-factor. Since we have also $RF(z - \alpha) = 0$, Theorem 4 applied to R implies that $F(z)$ is an exponential polynomial of the form (5.1). Hence, by Proposition 1, the non-zero minimal differential equation $M_\alpha \in \mathbb{Q}(z)[\frac{d}{dz}]$ satisfied by $F(z)$ admits a $\mathbb{C}$-basis of functions of the form $Q(z)e^{\beta z}$ with $Q(z) \in \overline{\mathbb{Q}}[z] \setminus \{0\}$ and $\beta \in \overline{\mathbb{Q}}$; in particular the solutions of $M_\alpha y(z) = 0$ are all entire functions.

Consider now $L_\alpha \in \mathbb{Q}[z, \frac{d}{dz}]$ obtained from L by changing z to $z + \alpha$: clearly, L_α is of minimal order for $F(z)$ because L is minimal for $f(z)$. Hence $M_\alpha = L_\alpha$ (up to a non-zero polynomial factor). But $Ly(z) = 0$ has a non-entire solution and this is also the case of $L_\alpha y(z) = 0$. This contradiction proves that $\mathcal{F}_\alpha$ has dimension ≥ 2. □

Remarks. (a) This proof is not easily generalizable to an E-function with arbitrary algebraic Taylor coefficients. The reason is the following: as we can no longer say that $F^\sigma = F$ for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we need an argument to ensure that F^σ is an entire function, in order to be able to use Proposition 3. Such an argument is lacking at present.

[2]André's approach is based on the lifting of certain properties of G-operators to E-operators. For instance (see [4, p. 726]), the regularity at 0 and ∞ of a G-operator implies that 0 and ∞ are the only possible singularities of an E-operator, that 0 is regular and that the slopes at ∞ are in $\{0, 1\}$ (these facts are all used in the present paper); the proof of the regularity at 0 and ∞ of a G-operator is quite simple. On the other hand, checking that a differential operator is an E-operator is difficult in practice because the only known way to do that is to check that its Fourier-Laplace transform is a G-operator; a celebrated result of Chudnovsky [8] provides a sufficient condition for this, the proof of which is based on the above mentioned Diophantine lemmas.

(b) The minimal non-zero operator for $J_0(z)$ is $L := z(\frac{d}{dz})^2 + \frac{d}{dz} + z$, which also has for solution the non-entire function

$$\log(z/2)J_0(z) - \sum_{n=0}^{\infty}(H_n - \gamma)\frac{(-z^2/4)^n}{n!^2}, \tag{5.3}$$

where $H_n := \sum_{k=1}^{n} 1/k$ and γ is Euler's constant. Hence, Theorem 6 applies: for any $\alpha \in \mathbb{Q}^*$, $J_0(\alpha)$ and $J_0'(\alpha)$ are $\mathbb{Q}$-linearly independent. Of course, this is much weaker than what was already known to Siegel, who proved in [18, 19] that $J_0(\alpha)$ and $J_0'(\alpha)$ are algebraically independent over $\overline{\mathbb{Q}}$ for any $\alpha \in \overline{\mathbb{Q}}^*$.

(c) Theorem 6 does not seem to follow from a direct application of the Siegel-Shidlovskii Theorem [17]. A new proof of this theorem was given by André in [5] using his theory of E-operators, which also led to new Diophantine results on E-functions, see [2, 6] for instance. Beukers' Corollary 1.4 in [6] implies Theorem 6: that corollary proves that the dimension is equal to μ, and in fact this was also a consequence of a result of Shidlovskii [17, p. 115, Lemma 17] (which is weaker than Beukers' result in general but coincides with it over $\mathbb{Q}$). If, keeping the other hypothesis on L, we assume that L has only entire solutions, then by Theorem 2, L has a basis of solutions of the form $P(z)e^{\beta z}$ with $P(z) \in \overline{\mathbb{Q}}[z]$ and $\beta \in \overline{\mathbb{Q}}$; hence any Diophantine consequence falls under the scope of the Lindemann-Weierstrass Theorem.

(d) Theorem 6 is optimal when $\mu = 2$, and the form of E-functions solutions of L is given in [10]. Its conclusion might even be false if $\alpha \in \mathbb{Q}^*$ is allowed to be a singularity of L. Consider for instance $f(z) := (z-1)^2 J_0(z)$, whose minimal differential equation is of order 2, singular at $z = 1$ and with a non-entire solution, while $f(1) = f'(1) = 0$ giving $\dim_{\mathbb{Q}} \mathcal{F}_1 = 0$. Similarly, the function $f(z) := (z-1)J_0(z)$ gives a "'singular" example for which $\dim_{\mathbb{Q}} \mathcal{F}_1 = 1$.

(e) Because the non-zero finite singularities of L are apparent, the argument can be adapted to prove the following: under the same assumptions as in Theorem 6, for any $\alpha \in \mathbb{Q}^*$ which is a singularity of L, the numbers $f(\alpha), f'(\alpha), \ldots, f^{(\delta-1)}(\alpha)$ generate a $\mathbb{Q}$-vector space of dimension at least 2, where δ is the order of any operator $M \in \mathbb{Q}[z, \frac{d}{dz}] \setminus \{0\}$ such that α is not a singularity of M and $Mf(z) = 0$, and δ is minimal for these two properties. Note that δ depend on the singularity α.

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Tanguy Rivoal, Institut Fourier, CNRS et Université Grenoble Alpes, CS 40700, 38058 Grenoble cedex 9, France
tanguy.rivoal (@) univ-grenoble-alpes.fr