EXTREMALITY PROPERTIES OF SOME DIOPHANTINE SERIES

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Abstract. We study the convergence properties of the series $\Psi_s(\alpha) := \sum_{n\ge1} \frac{||n^2\alpha||}{n^{s+1}||n\alpha||}$ with respect to the values of the real numbers α and s, where $||x||$ is the distance of x to $\mathbb{Z}$. For example when $s \in (0,1]$, the convergence of $\Psi_s(\alpha)$ strongly depends on the diophantine nature of α, mainly its irrationality exponent. We also conjecture that $\Psi_s(\alpha)$ is minimal at $\sqrt{5}$ for $s \in (0,1]$ and we present evidences in favor of that conjecture. For $s=1$, we formulate a more precise conjecture about the value of the abscissa u_k where the F_k-partial sum of $\Psi_1(\alpha)$ is minimal, F_k being the k-th Fibonacci number. A similar study it made for the partial sums of the series $\widetilde{\Psi}_1(\alpha) := \sum_{n\ge1}(-1)^n \frac{||n^2\alpha||}{n^2||n\alpha||}$ that we conjecture to be minimal at $\sqrt{2}/2$.

1. Introduction

The main goal of this paper is to study the following Dirichlet series, which is one of the "diophantine series" mentioned in the title (others appear in Sections 3, 7 and 8):

$$\Psi_s(\alpha) := \sum_{n=1}^{\infty} \frac{||n^2\alpha||}{n^{s+1}||n\alpha||}$$

for $\alpha, s \in \mathbb{R}$. Here, $||x||$ stands for the distance of x to $\mathbb{Z}$, i.e., $||x|| := |x - \lfloor x \rceil|$ with $\lfloor x \rceil$ the nearest integer to x (with $\lfloor 1/2 \rceil = 0$ say, even though this arbitrary choice has no influence on the value $||1/2||$). For future use, $\{x\}$ denotes the fractional part of x. For any integer $n \ge 1$, the function $D_n(\alpha) := \frac{||n\alpha||}{||\alpha||}$ is non-negative and continuous on $(0,1)$ with right limit at $\alpha = 0$ and left limit at $\alpha = 1$ both equal to n; it is also clearly 1-periodic on $\mathbb{R} \setminus \mathbb{Z}$. Furthermore, in [11, Lemma 2], it is shown that

$$D_n(\alpha) \le \frac{n}{1 + 2\lfloor n||\alpha|| \rfloor} \le n$$

for any $\alpha \in \mathbb{R}$. Therefore, for any integer $n \ge 1$, the function $D_n(n\alpha) = \frac{||n^2\alpha||}{||n\alpha||}$ is non-negative and continuous on $\mathbb{R}$, bounded by n, with the value n at rational numbers of the form j/n, $j \in \mathbb{Z}$. It follows that the partial sum

$$\Psi_{s,N}(\alpha) := \sum_{n=1}^{N} \frac{D_n(n\alpha)}{n^{s+1}}$$

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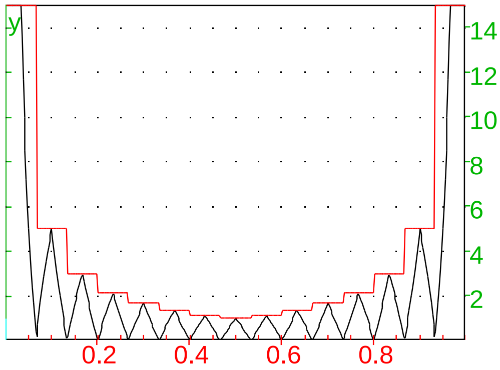

FIGURE 1. D_{15} and its upper bound

of $\Psi_s(\alpha)$ is a continuous function of α on $\mathbb{R}$. If $\alpha = a/b$ with $(a, b) = 1$, then the value of the summand is $1/n^s$ when n is divisible by b. Moreover, for any $\alpha \in \mathbb{R}$,

$$0 \le \Psi_{s,N}(\alpha) \le \sum_{n=1}^{N} \frac{1}{n^s} =: H_N(s).$$

The convergence/divergence of $\Psi_s(\alpha)$ strongly depends on the diophantine properties of α and before stating our results and conjectures, we recall some standard notation. For any irrational number α, let $(p_n/q_n)_{n\ge 0}$ denote the sequence of the convergents to α and let $(a_n)_{n\ge 0}$ denote the sequence of partial quotients, defined by $q_{n+1} = a_{n+1}q_n + q_{n-1}$. An irrational number α is said to have *a* finite irrationality exponent $\mu(\alpha) \ge 2$ if there exists a constant $c(\alpha) > 0$ such that

$$\left|\alpha - \frac{p}{q}\right| \ge \frac{1}{c(\alpha) q^{\mu(\alpha)}} \tag{1.1}$$

for all integers p, q with $q \ge 1$. We denote by $m(\alpha)$ *the* irrationality exponent of α, defined as the infinimum of all possible $\mu(\alpha)$, regardless of the value of $c(\alpha)$. By definition, Liouville numbers are precisely those real numbers which don't have a finite irrationality exponent; they are not only irrational but also transcendental.

When $s \in (0, 1)$, let us consider the sets $\mathscr{A}_s$ of irrational numbers α such that

$$\sum_{n=1}^{\infty} \frac{q_{n+1}^{1-s}}{q_n} < \infty \tag{1.2}$$

and, when $s = 1$, let us define $\mathscr{A}_s$ as the set of irrational numbers α such that

$$\sum_{n=1}^{\infty} \frac{\log\big(\max(q_{n+1}/q_n, q_n)\big)}{q_n} < \infty. \tag{1.3}$$

The following lemma was proved in [11]. We recall it for completeness.

Lemma 1 ([11], Lemma 1)**.** *(i) The set $\mathscr{A}_1$ contains all irrational numbers with a finite irrationality exponent. Some Liouville numbers belong to $\mathscr{A}_1$, some do not.*

(*ii*) *For any* $s \in (0,1)$*, the set* $\mathscr{A}_s$ *contains all irrational numbers with* $m(\alpha) < \frac{2-s}{1-s}$ *but none whose irrationality exponent* $m(\alpha)$ *is* $> \frac{2-s}{1-s}$*. Some irrational numbers with* $m(\alpha) = \frac{2-s}{1-s}$ *belong to* $\mathscr{A}_s$*, some do not.*

(*iii*) *For any* $s \in (0,1]$*, the set* $\mathscr{A}_s$ *has full measure.*

We can now state our result concerning the convergence/divergence of $\Psi_s(\alpha)$.

Theorem 1. (*i*) *For any* $s \in (0,1]$ *and any rational number* a/b *with* $(a,b)=1$*, we have*

$$\lim_{N\to+\infty} \frac{1}{H_N(s)}\, \Psi_{s,N}\Big(\frac{a}{b}\Big) = \frac{1}{b}.$$

Thus $\Psi_s(\frac{a}{b}) = +\infty$*.*

(*ii*) *For any* $s \in (0,1)$ *and any irrational number* α*, there exist two constants* $c_s, d_s > 0$ *(that also depend on* α*) such that*

$$c_s \sum_{k=1}^{m-1} \frac{q_{k+1}^{(1-s)/2}}{q_k^{(1+s)/2}} \le \Psi_{s,N}(\alpha) \le d_s \sum_{k=1}^{m} \frac{q_{k+1}^{1-s}}{q_k} \tag{1.4}$$

for any N *such that* $q_m \le N < q_{m+1}$*.*

For $s=1$*, there exist two constants* $c_1, d_1 > 0$ *(that depend on* α*) such that*

$$c_1 \sum_{k=1}^{m-1} \frac{\log(q_{k+1}/q_k)}{q_k} \le \Psi_{1,N}(\alpha) \le d_1 \sum_{k=1}^{m} \frac{\log\big(\max(q_{k+1}/q_k, q_k)\big)}{q_k} \tag{1.5}$$

for any N *such that* $q_m \le N < q_{m+1}$*.*

(*iii*) *For any* $s \in (0,1]$ *and any* $\alpha \in \mathscr{A}_s$*, the series* $\Psi_s(\alpha)$ *is convergent.*

(*iv*) *For any* $s \in (0,1)$*, the series* $\Psi_s(\alpha)$ *converges, respectively diverges, for any irrational number* α *such that* $m(\alpha) < \frac{2-s}{1-s}$*, respectively* $m(\alpha) > \frac{2}{1-s}$*.*

For $s=1$*, the series* $\Psi_1(\alpha)$ *converges for any irrational number* α *such that* $m(\alpha)$ *is finite. On the other hand, there exists a dense set of Liouville numbers* ξ *such that for any* $\varepsilon > 0$*,*

$$\limsup_{N\to+\infty} \frac{\Psi_{1,N}(\xi)}{\log(N)^{1-\varepsilon}} = +\infty. \tag{1.6}$$

(*v*) *When* $s \le 0$*, the series* $\Psi_s(\alpha)$ *diverges for all* $\alpha \in \mathbb{R}$ *while if* $s \ge 1$*, it converges for all* $\alpha \in \mathbb{R}$*.*

Remark. Diophantine series similar to those in (1.5) appear in [8, 12] in related contexts.

The first part of item (*iv*) is just a consequence of (1.4). We formulate it because it shows the link between convergence/divergence of $\Psi_s(\alpha)$ and the irrationality exponent $m(\alpha)$. It would be very interesting to obtain the exact threshold.

Moreover, (1.6) is essentially optimal because $|\Psi_{1,N}(\alpha)| \ll \log(N)$ for any α. In fact, the proof yields more: for any function $\varepsilon_N = o(1)$, we can find a dense set of Liouville numbers ξ such that (1.6) holds with ε_N instead of ε.

Theorem 1 is proved in Section 3. We also show the highly discontinuous behavior of Ψ_s in Section 4. In Section 5, we obtain an upper bound for the speed of convergence of the partial sums of $\Psi_s(\alpha)$: without surprise, this bound is not uniform and strongly depends on the diophantine properties of α.

A real surprise comes from the following conjecture, which we motivate in Section 6.

Conjecture 1. *For any $s \in (0,1]$, the function Ψ_s is minimal at the points of $\sqrt{5}+\mathbb{Z}$ and $-\sqrt{5}+\mathbb{Z}$, where it takes the same value.*

We remark that $m(\pm\sqrt{5}+k) = 2$ for any $k \in \mathbb{Z}$, hence that $\Psi_s(\pm\sqrt{5}+k)$ is convergent for any $s > 0$. In Section 6, we will also present evidences ([1]) for the following "finite version" of Conjecture 1 in the case $s = 1$.

Conjecture 2. (i) *For any integer $k \geq 4$, the partial sum Ψ_{1,F_k} is minimal on $[0,1]$ at the points*

$$u_k := \frac{F_{k-1}F_{k-2}}{F_k^2} \quad \text{and} \quad 1 - u_k.$$

Here, $(F_k)_{k\geq 0}$ is the Fibonacci sequence defined by $F_0 = 0, F_1 = 1$ and $F_{k+2} = F_{k+1} + F_k$.

(ii) *We have*

$$\lim_{k\to+\infty} \Psi_{1,F_k}(u_k) = \Psi_1(\sqrt{5}-2).$$

For $s \in (0,1)$, Conjecture 2 seems to hold sometimes, but it also fails sometimes. Note that $u_k \to \sqrt{5}-2$ at geometric rate, but we don't see how to deduce (ii) from this fact. The expression "finite version" is justified by the fact that Conjecture 2 implies the case $s = 1$ of Conjecture 1. Indeed, by 1-periodicity and symmetry of Ψ_1 with respect to the vertical axis $\alpha = 1/2$, it is enough to prove minimality at $\sqrt{5}-2$. (i) implies that, for any $\alpha \in [0,1]$, $\Psi_{1,F_k}(\alpha) \geq \Psi_{1,F_k}(u_k)$. Hence, by (ii),

$$\lim_{k\to+\infty} \Psi_{1,F_k}(\alpha) \geq \Psi_1(\sqrt{5}-2). \tag{1.7}$$

If α belongs to the domain of convergence of Ψ_1, (1.7) implies that $\Psi_1(\alpha) \geq \Psi_1(\sqrt{5}-2)$ whereas if α belongs to the domain of divergence of Ψ_1, the value of $\Psi_1(\alpha)$ is $+\infty$ and we still have $\Psi_1(\alpha) \geq \Psi_1(\sqrt{5}-2)$.

In Section 7, we will shortly consider the case of the series

$$\widetilde{\Psi}_1(\alpha) := \sum_{n=1}^{\infty} (-1)^n \frac{||n^2\alpha||}{n^2||n\alpha||}$$

which seems to present a minimum at any point of $\pm\frac{\sqrt{2}}{2}+\mathbb{Z}$; see Conjecture 3 for a more precise statement in the spirit of Conjecture 2. It is often the case that quadratic numbers are extremal for various diophantine statistics: $\sqrt{2}$ is minimal for the star discrepancy

[1]These are numerical datas/graphs computed/ploted with Maple, XCAS and GP-PARI. For a same graph ploted with the three programs, zooms on interesting parts all revealed the pattern shown in Conjecture 2.

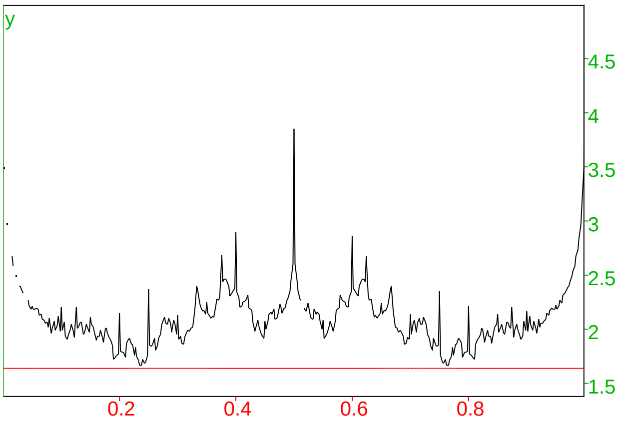

FIGURE 2. Graphs of $\Psi_{1,200}$ and the constant $\Psi_{1,200}(\sqrt{5}-2)$ on $[0,1]$

of $\{n\alpha\}$-sequences (Dupain-Sós [4]), $\frac{\sqrt{5}-1}{2}$ is conjecturally minimal for the discrepancy of $\{n\alpha\}$-sequences (see [1]), $\frac{\sqrt{5}-1}{2}$ is minimal for the circular dispersion of Niederreiter [9] and its variation of Jager-de Jong [5].

2. Motivations behind Ψ_s

Even though the series Ψ_s is an interesting object in itself, it does not come from nowhere. Indeed, in order to study how far the finite sequence $(\{k\alpha\})_{1\le k\le n}$ is from a subset of $\{\frac{0}{n}, \frac{1}{n}, \dots, \frac{n-1}{n}\}$, the author introduced in [11] the function

$$F_n(\alpha) := \sum_{k=1}^{n} \left| k\alpha - \frac{\lfloor kn\alpha \rceil}{n} \right| = \frac{1}{n} \sum_{k=1}^{n} ||kn\alpha||.$$

The function F_n is 1-periodic and symmetric with respect to the vertical axis $\alpha = \frac{1}{2}$. The study of the fluctuations of $F_n(\alpha)$ around $1/4$ led in particular to consider the Dirichlet series

$$\mathscr{G}_s(\alpha) := \sum_{n=1}^{\infty} \frac{F_n(\alpha) - 1/4}{n^s}$$

for $s \in \mathbb{R}$ and to determine for which α and s the equality ([2])

$$\mathscr{G}_s(\alpha) = -\frac{2}{\pi^2} \sum_{k=0}^{\infty} \frac{\Phi_s\big((2k+1)\alpha\big)}{(2k+1)^2} \tag{2.1}$$

holds, where

$$\Phi_s(\alpha) := \sum_{n=1}^{\infty} \frac{1}{n^{1+s}} \sum_{k=1}^{n} \cos(2\pi kn\alpha). \tag{2.2}$$

[2]This is formally obtained by means of the Fourier expansion $||\alpha|| = \frac{1}{4} - \frac{2}{\pi^2} \sum_{k=0}^{\infty} \frac{\cos(2(2k+1)\pi\alpha)}{(2k+1)^2}$. Finding when (2.1) holds is a problem similar to finding when Davenport's identities hold (see [3, 8] for some examples).

For $s > 1$, both $\mathscr{G}_s(\alpha)$ and $\Phi_s(\alpha)$ clearly converge absolutely for any $\alpha \in \mathbb{R}$, and (2.1) holds. It is a little less easy to prove that both diverge for any $\alpha \in \mathbb{R}$ when $s \leq 0$. Again, the situation is much more interesting when $s \in (0,1]$. The following theorem is a survey of some of the results proved in [11].

Theorem ([11], Theorems 1 and 2)**.** (i) *For any rational number α and any $s \in (0,1]$, the series $\mathscr{G}_s(\alpha)$ and $\Phi_s(\alpha)$ diverge to $-\infty$ and $+\infty$ respectively.*

(ii) *For any $s \in (0,1]$ and any $\alpha \in \mathscr{A}_s$, the series $\Phi_s(\alpha)$ converges to a finite limit.*

(iii) *For any $s \in (0,1)$ and any $\alpha \in \mathscr{A}_s$, the series $\mathscr{G}_s(\alpha)$ converges and identity* (2.1) *holds. This is also the case when $s = 1$ and $m(\alpha)$ is finite.*

(iv) *For any $s \in (0,1)$ and any irrational number α such that $m(\alpha) > \frac{6-4s}{1-s}$, the series $\mathscr{G}_s(\alpha)$ and $\Phi_s(\alpha)$ both diverge, to $-\infty$ and $+\infty$ respectively. When $s = 1$, there exists a dense set of Liouville numbers α such that the same conclusion holds.*

(v) *When $s > 1$, $\mathscr{G}_s(\alpha)$ and $\Phi_s(\alpha)$ converge for all α. When $s \leq 0$, both diverge for all α.*

The series $\Psi_s(\alpha)$ appears as follows. We have

$$\Phi_s(\alpha) = \sum_{n=1}^{\infty} \frac{\cos(\pi n(n+1)\alpha)\sin(\pi n^2\alpha)}{n^{s+1}\sin(\pi n\alpha)}$$

and

$$\left|\frac{\cos(\pi n(n+1)\alpha)\sin(\pi n^2\alpha)}{\sin(\pi n\alpha)}\right| \leq \frac{\sin(\pi||n^2\alpha||)}{\sin(\pi||n\alpha||)} \leq \frac{\pi}{2}\frac{||n^2\alpha||}{||n\alpha||}$$

because $2x \leq \sin(\pi x) \leq \pi x$ for $x \in [0, \pi/2]$. Hence,

$$|\Phi_s(\alpha)| \leq \sum_{n=1}^{\infty}\left|\frac{\cos(\pi n(n+1)\alpha)\sin(\pi n^2\alpha)}{n^{s+1}\sin(\pi n\alpha)}\right| \leq \frac{\pi}{2}\Psi_s(\alpha). \tag{2.3}$$

As we have seen earlier, $\Psi_s(\alpha)$ converges at least for any $\alpha \in \mathscr{A}_s$, which explains part of the above theorem.

Like Ψ_s, the functions $\mathscr{G}_s$ and $-\Phi_s$ also have surprising extremal properties, namely for any fixed $s \in (0,1]$, they seem to attain their respective maxima over $[0,1]$ at $\frac{\sqrt{5}-1}{2}$ and $1 - \frac{\sqrt{5}-1}{2}$. See Figures 3 and 4 in the case $s = 1$. The shift in the apparent position of extremal values ($\frac{\sqrt{5}-1}{2} \to \sqrt{5}$) in (2.3) is curious.

3. Proof of Theorem 1

(i) We write $n = kb + r$ with $k \geq 0$ and $1 \leq r \leq b$, so that

$$\Psi_{s,N}(a/b) = \sum_{r=1}^{b-1}\frac{||r^2a/b||}{||ra/b||}\sum_{k=0}^{\lfloor (N-r)/b\rfloor}\frac{1}{(kb+r)^{s+1}} + \frac{1}{b^s}\sum_{k=1}^{\lfloor N/b\rfloor}\frac{1}{k^s}.$$

(Since $(a,b) = 1$, ra/b is an integer if and only if $r = b$.)

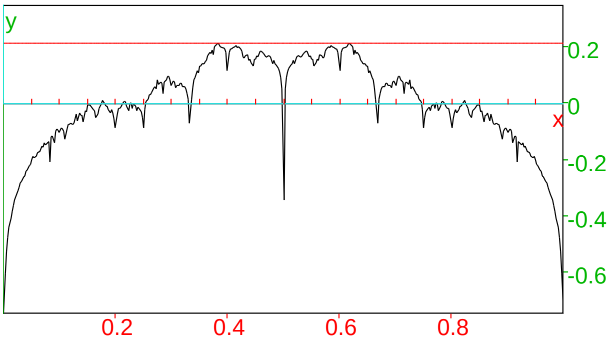

FIGURE 3. Graphs of $\mathscr{G}_{1,200}$ and the constant $\mathscr{G}_{1,200}\left(\frac{\sqrt{5}-1}{2}\right)$ on $[0,1]$

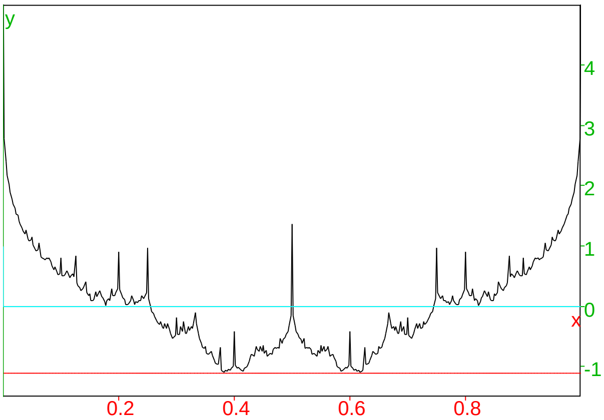

FIGURE 4. Graphs of $\Phi_{1,200}$ and the constant $\Phi_{1,200}\left(\frac{\sqrt{5}-1}{2}\right)$ on $[0,1]$

Since $s>0$, the term

$$\sum_{r=1}^{b-1} \frac{||r^2 a/b||}{||ra/b||} \sum_{k=0}^{\lfloor (N-r)/b \rfloor} \frac{1}{(kb+r)^{s+1}}$$

converges to a finite limit when $N \to +\infty$. On the other hand,

$$\frac{1}{b^s} \sum_{k=1}^{\lfloor N/b \rfloor} \frac{1}{k^s} \sim \frac{1}{b} H_N(s)$$

when $N \to +\infty$, which proves the result.

(ii) We don't repeat the proof of the right inequalities in (1.4) and (1.5), which have been proved in [11]. Let us prove the left inequalities. Obviously, we have

$$\Psi_{s,N}(\alpha) \geq \sum_{n=0}^{m-1} \sum_{\substack{k=q_n \\ q_n|k}}^{q_{n+1}-1} \frac{||k^2\alpha||}{k^{s+1}||k\alpha||} = \sum_{n=0}^{m-1} \frac{1}{q_n^{s+1}} \sum_{\ell=1}^{\lfloor (q_{n+1}-1)/q_n \rfloor} \frac{||\ell^2 q_n^2 \alpha||}{\ell^{s+1}||\ell q_n \alpha||}$$

where m is such that $q_m \le N < q_{m+1}$. We recall that

$$\frac{1}{q_n + q_{n+1}} \le |q_n\alpha - p_n| \le \frac{1}{q_{n+1}}.$$

Hence

$$||\ell q_n \alpha|| \le |\ell q_n\alpha - \ell p_n| \le \frac{\ell}{q_{n+1}}.$$

We also have

$$\frac{\ell^2 q_n}{q_n + q_{n+1}} \le |(\ell q_n)^2\alpha - \ell^2 q_n p_n| \le \frac{\ell^2 q_n}{q_{n+1}}. \tag{3.1}$$

Provided that $\ell \le Q := \sqrt{\frac{q_{n+1}}{2q_n}}$, we deduce from (3.1) that $|(\ell q_n)^2\alpha - \ell^2 q_n p_n| = ||\ell^2 q_n^2 \alpha||$ and that

$$||\ell^2 q_n^2 \alpha|| \ge \frac{\ell^2 q_n}{q_n + q_{n+1}}.$$

It follows from all this that

$$\Psi_{s,N}(\alpha) \ge \sum_{n=0}^{m-1} \frac{1}{q_n^{s+1}} \sum_{\ell=1}^{\lfloor Q\rfloor} \frac{||\ell^2 q_n^2\alpha||}{\ell^{s+1}||\ell q_n \alpha||} \ge \sum_{n=0}^{m-1} \frac{q_{n+1}}{(q_{n+1}+q_n)q_n^s} \sum_{\ell=1}^{\lfloor Q\rfloor} \frac{1}{\ell^s}.$$

We remark now that $\lfloor Q\rfloor = 0$ iff $a_{n+1} = 1$, and then $\sum_{\ell=1}^{\lfloor Q\rfloor} \frac{1}{\ell^s} = 0$. Let us first discard this case and consider only those $n \ge 0$ such that $a_{n+1} \ge 2$. (Note that $q_{n+1} = a_{n+1}q_n + q_{n-1}$ implies that $q_{n+1}/(2q_n) > 1$.) Then

$$\sum_{\ell=1}^{\lfloor Q\rfloor} \frac{1}{\ell^s} \ge \begin{cases} e_1 \log(q_{n+1}/q_n) > 0 & \text{if} \quad s = 1 \\ e_s (q_{n+1}/q_n)^{(1-s)/2} & \text{if} \quad 0 < s < 1, \end{cases}$$

for some constants $e_s > 0$ that depend on s and α. Hence if $s = 1$,

$$\Psi_{s,N}(\alpha) \ge e_1 \sum_{\substack{n=0 \\ a_{n+1}\ge 2}}^{m-1} \frac{q_{n+1}}{(q_{n+1}+q_n)q_n} \log(q_{n+1}/q_n) \ge \frac{e_1}{2} \sum_{\substack{n=0 \\ a_{n+1}\ge 2}}^{m-1} \frac{\log(q_{n+1}/q_n)}{q_n}$$

while if $s \in (0,1)$,

$$\Psi_{s,N}(\alpha) \ge e_s \sum_{\substack{n_0 \\ a_{n+1}\ge 2}}^{m-1} \frac{q_{n+1}}{(q_{n+1}+q_n)q_n^s} \cdot \frac{q_{n+1}^{(1-s)/2}}{q_n^{(1-s)/2}} \ge \frac{e_s}{2} \sum_{\substack{n_0 \\ a_{n+1}\ge 2}}^{m-1} \frac{q_{n+1}^{(1-s)/2}}{q_n^{(1-s)/2}}.$$

It remains to deal with the case $a_{n+1} = 1$, which implies that q_{n+1}/q_n is bounded by 2. Hence the series

$$\sum_{\substack{n=0 \\ a_{n+1}=1}}^{\infty} \frac{\log(q_{n+1}/q_n)}{q_n} \quad \text{and} \quad \sum_{\substack{n=0 \\ a_{n+1}=1}}^{\infty} \frac{q_{n+1}^{(1-s)/2}}{q_n^{(1+s)/2}}$$

are convergent because the sequence $(q_n)_{n\geq 0}$ grows at least geometrically. This implies that

$$\sum_{\substack{n=0\\ a_{n+1}\geq 2}}^{m-1} \frac{\log(q_{n+1}/q_n)}{q_n} \geq f_1 \sum_{n=0}^{m-1} \frac{\log(q_{n+1}/q_n)}{q_n}$$

and

$$\sum_{\substack{n=0\\ a_{n+1}\geq 2}}^{m-1} \frac{q_{n+1}^{(1-s)/2}}{q_n^{(1+s)/2}} \geq f_s \sum_{n=0}^{m-1} \frac{q_{n+1}^{(1-s)/2}}{q_n^{(1+s)/2}},$$

for some constants f_s that depend on s and α. This completes the proof of (1.4) and (1.5).

(iii) This was proved in [11] as a consequence of the right inequalities in (1.4) and (1.5).

(iv) By the definition of $m(\alpha)$, for any $\mu > m(\alpha)$, we have

$$\frac{1}{q_n^{\mu}} \leq \left|\alpha - \frac{p_n}{q_n}\right| \leq \frac{1}{q_n q_{n+1}}$$

for $n \geq n_\mu$. Hence $q_{n+1} \leq q_n^{\mu-1}$ and

$$\sum_{k=n_\mu}^{m} \frac{q_{k+1}^{1-s}}{q_k} \leq \sum_{k=n_\mu}^{m} \frac{1}{q_k^{1-(\mu-1)(1-s)}}.$$

If $\mu < \frac{2-s}{1-s}$, then $1-(\mu-1)(1-s) > 0$ and the series

$$\sum_{k=0}^{\infty} \frac{1}{q_k^{1-(\mu-1)(1-s)}}$$

is convergent. Hence by the right inequality in (1.4), the series $\Psi_s(\alpha)$ is convergent for any irrational number α such that $m(\alpha) < \frac{2-s}{1-s}$.

On the other hand, if $m(\alpha) > \mu$ for some μ, then we must have $q_{k+1} > q_k^{\mu-1}$ for infinitely many k (denoted by $(k_n)_n$ below), otherwise we would have $m(\alpha) \leq \mu$ because of the inequalities

$$\left|\alpha - \frac{p_n}{q_n}\right| \geq \frac{1}{q_n(q_{n+1}+q_n)} \gg \frac{1}{q_n^{\mu}}$$

for all $n \gg 1$. Therefore,

$$\sum_{k=0}^{m-1} \frac{q_{k+1}^{(1-s)/2}}{q_k^{(1+s)/2}} \geq \sum_{0\leq k_n \leq m-1} q_k^{\frac{(1-s)(\mu-1)-(1+s)}{2}}. \tag{3.2}$$

If $\mu \geq \frac{2}{1-s}$, we have $(1-s)(\mu-1)-(1+s) \geq 0$ and the series on the right hand side of (3.2) diverges. Then, by the left inequality in (1.4), the series $\Psi_s(\alpha)$ is divergent.

If $s=1$ and $m(\alpha) < +\infty$, then from $q_{n+1} \leq q_n^{\mu-1}$ for some $\mu > m(\alpha)$, we deduce that

$$\sum_{k=0}^{\infty} \frac{\log\big(\max(q_{k+1}/q_k, q_k)\big)}{q_k} \ll \sum_{k=0}^{\infty} \frac{\log(q_k)}{q_k} < +\infty,$$

which proves the first claim by the right inequality in (1.5).

The left inequality in (1.5) shows that

$$\frac{\log(q_k/q_{k-1})}{q_{k-1}} \leq \Psi_{1,q_k}(\alpha)$$

for any α. We consider now any number ξ such that $q_{k-1} = o\big(\log(q_k)\big)$ as $k \to +\infty$ (which implies that ξ is a Liouville number), so that $\log(q_k)^{1-o(1)} \leq \Psi_{1,q_k}(\xi)$ and thus for any $\varepsilon > 0$,

$$\limsup_{N\to+\infty} \frac{\Psi_{1,N}(\xi)}{\log(N)^{1-\varepsilon}} = +\infty.$$

Since we can assume that the condition $q_{k-1} = o\big(\log(q_k)\big)$ holds for k large enough, we can construct a dense set of Liouville numbers with the claimed property by chosing freely the first partial quotients of ξ.

(v) The proof of item (i) above works for $s \leq 0$ and shows that $\Psi_s(\alpha)$ diverges for any rational number α when $s \leq 0$. Let us now consider the case where α is irrational. For any $\varepsilon > 0$, there exist infinitely many n such that

$$|q_n\alpha - p_n| \leq \frac{1}{(L(\alpha) - \varepsilon)q_n}$$

where $L(\alpha)$ is the Lagrange constant of α (defined as $\liminf_q \frac{1}{q||q\alpha||}$). It is well-known that for any irrational number α, we have $L(\alpha) \geq \sqrt{5}$ (see [2]). Therefore, for ε small enough, we have $|q_n^2\alpha - q_np_n| < \frac{1}{2}$ for infinitely many n. It follows that, for infinitely many n,

$$||q_n^2\alpha|| = q_n|q_n\alpha - p_n| = q_n||q_n\alpha||$$

or, written differently,

$$\frac{||q_n^2\alpha||}{q_n||q_n\alpha||} = 1.$$

Hence, the series $\Psi_s(\alpha)$ cannot converge when $s \leq 0$. Finally, since $0 \leq \frac{||n^2\alpha||}{||n\alpha||} \leq n$, we have $0 \leq \Psi_s(\alpha) \leq \zeta(s) < +\infty$ when $s > 1$.

This conclude the proof of the theorem.

4. Discontinuity of Ψ_s

We now deduce from Theorem 1 a result concerning the analytic behavior of Ψ_s. We set

$$\mathscr{D}_s = \{\alpha \in \mathbb{R} : \Psi_s(\alpha) \text{ is convergent}\}.$$

We know that $\mathscr{D}_s = \emptyset$ for $s \leq 0$, $\mathscr{A}_s \subset \mathscr{D}_s$ for any $s \in (0,1]$ and $\mathscr{D}_s = \mathbb{R}$ for $s > 1$. In particular, $\mathscr{D}_s$ has full measure when $s > 0$.

Theorem 2. *For any $s \in (0,1]$ and any $u, v \in \mathbb{R}$, the function Ψ_s has no upper bound in $[u,v] \cap \mathscr{A}_s$. In particular, the function Ψ_s restricted to $\mathscr{D}_s$ is nowhere continuous.*

Proof. An interval $[u,v]$ determines the first $m+1$ partial quotients $(a_n)_{0\le n\le m}$ of any of its elements, where m depends on u and v. The partial quotients $(a_n)_{n>m}$ can be chosen freely, in particular a_{m+1}. When $s\in(0,1)$, the left inequality of (1.4) shows that, for any $\alpha\in[u,v]$,

$$\Psi_{s,q_{m+1}}(\alpha)\ge c_s\frac{q_{m+1}^{(1-s)/2}}{q_m^{(1+s)/2}}.$$

Since $q_{m+1}=a_{m+1}q_m+q_{m-1}$, we can choose a_{m+1} large enough so that $\Psi_{s,q_{m+1}}(\alpha)\ge A$ for any given $A>0$. The other partial quotients $(a_n)_{n>m+1}$ can then be chosen such that $\alpha\in\mathscr{A}_s$. If $s=1$, the left inequality of (1.5) shows that

$$\Psi_{1,q_{m+1}}(\alpha)\ge c_1\frac{\log(q_{m+1}/q_m)}{q_m}.$$

and we conclude similarly. □

5. Computation of Ψ_s

We present in this section (cf Proposition 1 below) bounds that ensure that we obtain an approximation of $\Psi_s(\alpha)$ to a prescribed accuracy by computing $\Psi_{s,N}(\alpha)$ for N large enough or even $\Psi_{s,N}(p/q)$ where p/q is a good rational approximation of α. We need two lemmas.

Lemma 2. *For any $\alpha,\beta\in\mathbb{R}$ and any integer $n\ge1$, we have*

$$|D_n(\alpha)-D_n(\beta)|\le 4n^2|\alpha-\beta|.$$

Proof. There are five cases to consider.

1) Assume that $\alpha,\beta\in[\frac{j}{n},\frac{j+1/2}{n}]$ with $0\le j\le\frac{n-1}{2}$, and also in $[\frac12-\frac1n,\frac12-\frac1{2n}]$ if n is even. Then

$$\Delta(\alpha,\beta):=D_n(\alpha)-D_n(\beta)=\frac{n\alpha-j}{\alpha}-\frac{n\beta-j}{\beta}=j\Big(\frac1\beta-\frac1\alpha\Big)=\frac{j}{\alpha\beta}(\alpha-\beta).$$

If $j=0$, then $\Delta(\alpha,\beta)=0$.
If $j\ge1$, we have $\alpha\beta\ge(j/n)^2$, so that

$$|\Delta(\alpha,\beta)|\le\frac{n^2}{j}|\alpha-\beta|\le n^2|\alpha-\beta|.$$

2) Assume that $\alpha,\beta\in[\frac{j+1/2}{n},\frac{j+1}{n}]$ with $0\le j\le\frac{n-2}{2}$, and also in $[\frac12-\frac1{2n},\frac12]$ if n is odd. Then

$$\Delta(\alpha,\beta)=\frac{j+1-n\alpha}{\alpha}-\frac{j+1-n\beta}{\beta}=\frac{j+1}{\alpha\beta}(\beta-\alpha).$$

It follows that

$$|\Delta(\alpha,\beta)|\le\frac{j+1}{(j+1/2)^2}n^2|\alpha-\beta|\le4n^2|\alpha-\beta|.$$

3) Assume that $\alpha, \beta \in [\frac{j}{n}, \frac{j+1/2}{n}]$ with $\frac{n}{2} \leq j \leq n-1$, and also in $[\frac{1}{2}, \frac{1}{2}+\frac{1}{2n}]$ if n is odd. Then

$$\Delta(\alpha,\beta) = \frac{n\alpha - j}{1-\alpha} - \frac{n\beta - j}{1-\beta} = \frac{j-n}{(1-\alpha)(1-\beta)}(\beta-\alpha).$$

Since $(1-\alpha)(1-\beta) \geq ((n-j-1/2)/n)^2$, we get again that

$$|\Delta(\alpha,\beta)| \leq \frac{n^2(n-j)}{(n-j-1/2)^2}|\alpha-\beta| \leq 4n^2|\alpha-\beta|.$$

4) Assume that $\alpha, \beta \in [\frac{j+1/2}{n}, \frac{j+1}{n}]$ with $\frac{n-1}{2} \leq j \leq n-1$, and also in $[\frac{1}{2}+\frac{1}{2n}, \frac{1}{2}+\frac{1}{n}]$ if n is even. Then

$$\Delta(\alpha,\beta) = \frac{j+1-n\alpha}{1-\alpha} - \frac{j+1-n\beta}{1-\beta} = \frac{j+1-n}{(1-\alpha)(1-\beta)}(\alpha-\beta).$$

If $j = n-1$, then $\Delta(\alpha,\beta) = 0$.
If $j < n-1$, then

$$|\Delta(\alpha,\beta)| \leq \frac{n^2}{(n-j-1)}|\alpha-\beta| \leq n^2|\alpha-\beta|.$$

5) So far, we have proved that for any $\alpha, \beta \in [\frac{j}{n}, \frac{j+1}{n}]$ for some $j \in \{0, \ldots, n-1\}$, we have $|\Delta(\alpha,\beta)| \leq 4n^2|\alpha-\beta|$.

In the general case where $\alpha \leq \beta$ are anywhere in $[0,1]$, we consider the sequence $x_0 = \alpha < x_1 = \frac{j+1}{n} < x_2 = \frac{j+2}{n} < \ldots < x_k = \frac{j+k}{n} < x_{k+1} = \beta$, where $\alpha \in [\frac{j}{n}, \frac{j+1}{n}]$ and $\beta \in [\frac{j+k}{n}, \frac{j+k+1}{n}]$. Then

$$|\Delta(\alpha,\beta)| = \Big|\sum_{\ell=0}^{k} \Delta(x_\ell, x_{\ell+1})\Big| \leq \sum_{\ell=0}^{k} |\Delta(x_\ell, x_{\ell+1})| \leq 4n^2 \sum_{\ell=0}^{k} |x_\ell - x_{\ell+1}| = 4n^2|\alpha-\beta|.$$

This conclude the proof of the lemma. □

The following lemma was proved in [11]. Here, $\mu(\alpha)$ and $c(\alpha)$ are any positive real numbers satisfying (1.1).

Lemma 3 ([11], Proposition 1)**.** *Let us fix an integer $m \geq 6$. (i) For any $\alpha \in \mathscr{A}_s$ (for some $s \in (0,1)$) and with $\mu(\alpha) < \frac{2-s}{1-s}$, we have*

$$\sum_{n=q_m+1}^{\infty} \frac{||n^2\alpha||}{n^{s+1}||n\alpha||} \leq \frac{2\big(1+\zeta(s+1)\big)}{(1-s)q_m^{1-(\mu(\alpha)-1)(1-s)}} \Big(3(1+c(\alpha)^{1-s})\log(q_m) + \frac{c(\alpha)^{1-s}}{1-\sqrt{2}^{(\mu(\alpha)-1)(1-s)-1}}\Big) =: R_{s,m}.$$

(ii) For any $\alpha \in \mathscr{A}_1$ *with* $m(\alpha) < +\infty$*, we have*

$$\sum_{n=q_m+1}^{\infty} \frac{||n^2\alpha||}{n^2||n\alpha||}$$
$$\leq 2(1+\zeta(2))\Big(3(1+\log c(\alpha))\frac{\log(q_m)}{q_m} + 5(\mu(\alpha)-1)\frac{\log(q_m)^2}{q_m}\Big) =: R_{1,m}.$$

We can now state a result that enables us to compute approximations of $\Psi_s(\alpha)$.

Proposition 1. *In the conditions of Lemma 3, for any* $s \in (0,1]$*, any real number* β *and any integer* $N \geq q_m$*, we have*

$$\big|\Psi_s(\alpha) - \Psi_{s,N}(\beta)\big| \leq R_{s,m} + 4q_m^{3-s}\big|\alpha - \beta\big|. \tag{5.1}$$

Proof. For $N \geq q_m$, we have

$$\begin{aligned}\big|\Psi_s(\alpha) - \Psi_{s,N}(\beta)\big| &\leq \big|\Psi_s(\alpha) - \Psi_{s,q_m}(\beta)\big| \\ &\leq \big|\Psi_s(\alpha) - \Psi_{s,q_m}(\alpha)\big| + \big|\Psi_{s,q_m}(\alpha) - \Psi_{s,q_m}(\beta)\big|.\end{aligned}$$

The term $\big|\Psi_s(\alpha) - \Psi_{s,q_m}(\alpha)\big|$ is bounded by $R_{s,m}$ by Lemma 3. Moreover, using Lemma 2 with $n\alpha$ instead of α and $n\beta$ instead of β, we get

$$\begin{aligned}\big|\Psi_{s,q_m}(\alpha) - \Psi_{s,q_m}(\beta)\big| &\leq \sum_{n=1}^{q_m} \frac{1}{n^{1+s}}\bigg|\frac{||n^2\alpha||}{||n\alpha||} - \frac{||n^2\beta||}{||n\beta||}\bigg| \\ &\leq 4\big|\alpha-\beta\big| \cdot \sum_{n=1}^{q_m} \frac{n^3}{n^{1+s}} \\ &\leq 4q_m^{3-s}\big|\alpha-\beta\big|.\end{aligned}$$

The proposition follows. □

In order to use Proposition 1 for a given α, we have to choose a suitable β and to find upper bounds for $\mu(\alpha)$ and $c(\alpha)$.

Concerning the former task, simple choices are $\beta = \alpha$ or, if one preferes to compute with rational numbers, $\beta = p_k/q_k$ where p_k/q_k is another convergent to α. In this case, we get

$$q_m^{3-s}\left|\alpha - \frac{p_k}{q_k}\right| \leq \frac{q_m^{3-s}}{q_{k+1}}$$

and one must take q_{k+1} large enough.

Concerning the problem of finding $\mu(\alpha)$ and $c(\alpha)$, there is unfortunately no general recipe: see the examples of e, π, π^2 and real algebraic numbers in [11, Proposition 4]. In particular, one form of the well-known Liouville's inequality reads as follows: for any real algebraic irrational number of degree d, with minimal polynomial $\sum_{j=0}^{d} s_j X^j \in \mathbb{Z}[X]$, we can take $\mu(\alpha) = d$ and $c(\alpha) = (|\alpha|+1)^{d-1}\sum_{j=1}^{d} j|s_j|$.

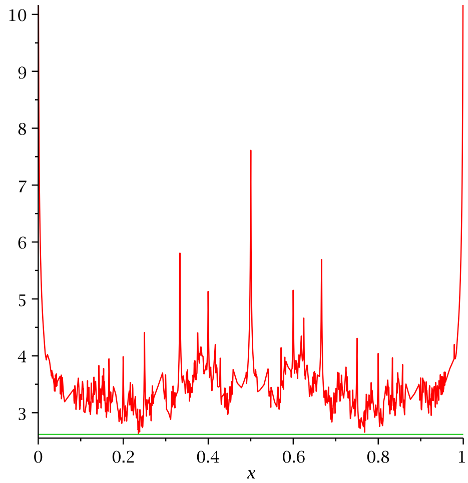

FIGURE 5. Graphs of $\Psi_{1/2,50}$ and the constant $\Psi_{1/2,50}(\sqrt{5}-2)$ in $[0,1]$

The only numbers which really interest us here are $\sqrt{5}+k$, with $k \in \mathbb{Z}$. They all have $m(\sqrt{5}+k) = 2$ and the constant $c(\sqrt{5}+k) = (4+2|k|)(1+|\sqrt{5}+k|)$ is minimal for $k = -2$. The 19th convergent of $\sqrt{5}-2$ is

$$\frac{p_{18}}{q_{18}} = \frac{31622993}{133957148}.$$

In the table below, we show approximations of $\Psi_s(\sqrt{5}-2)$ for various values of s, computed using GP-Pari. We use Proposition 1 with $\alpha = \beta = \sqrt{5}-2$ and $N = q_m = 133957148$. The digits between parenthesis are not certified to be correct with that value of q_m.

s	1/2	2/3	3/4	4/5	1
$\Psi_s(\sqrt{5}-2)$	3.6(04342)	2.500(415)	2.189(498)	2.0451(34)	1.6580(68)

6. Evidences for Conjectures 1 and 2

In this section, we arrive to what seems to be the most surprising property of $\Psi_s(\alpha)$, which was explicited as Conjecture 1. We present in this section various graphs which give evidences that, for any $s \in (0,1]$, Ψ_s is minimal at the points of $\sqrt{5}+\mathbb{Z}$ and $-\sqrt{5}+\mathbb{Z}$ (where it takes the same value): see Figures 5-6 in the case $s = 1/2$ and Figures 7-8 in the case $s = 1/5$. In the case $s = 1$, we present four graphs (Figure 9 to 12) in support of Conjecture 2(i). They are zooms centered at $u_{11} = \frac{F_9F_{10}}{F_{11}^2} = \frac{1870}{89^2}$ of the graph of $\Psi_{1,F_{11}}$. A similar verification was done for $u_2, \dots, u_{26}$; in particular it seems that Ψ_{1,F_k} is not differentiable at u_k.

We now make a few remarks about Conjecture 2(ii). Using the classical expression of Fibonacci numbers $F_k = \frac{1}{\sqrt{5}}\left(\varphi^k - (1-\varphi)^k\right)$, where $\varphi := (\sqrt{5}+1)/2$, one easily finds that $\left|u_k - \varphi^{-3}\right| \ll \varphi^{-2k}$. Note that $\varphi^{-3} = \sqrt{5}-2$. Unfortunately, the convergence is not fast enough to imply Conjecture 2(ii) by means of Lemma 2, like in the proof of Proposition 1.

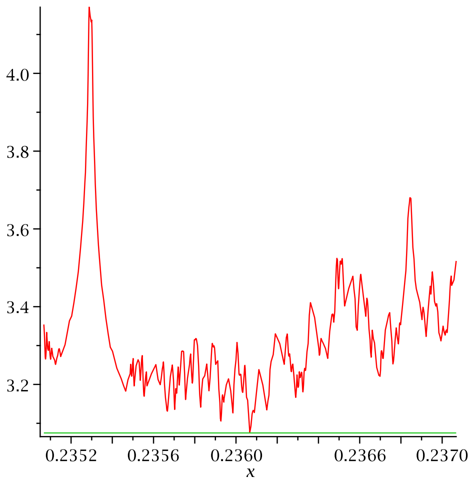

FIGURE 6. Graphs of $\Psi_{1/2,300}$ and the constant $\Psi_{1/2,300}(\sqrt{5}-2)$ in $[\sqrt{5}-2-10^{-3}, \sqrt{5}-2+10^{-3}]$

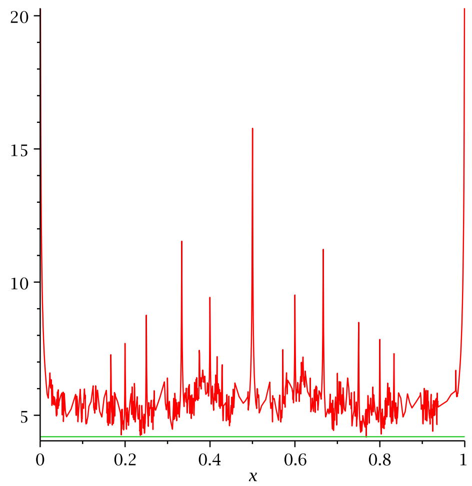

FIGURE 7. Graphs of $\Psi_{1/5,50}$ and the constant $\Psi_{1/5,50}(\sqrt{5}-2)$ in $[0,1]$

However, it seems that the value of the derivative of $E_n(\alpha) := \frac{||n^2\alpha||}{||n\alpha||}$ at $\alpha = \sqrt{5}-2$ is very often of the order of n^2 and not just bounded by $4n^3$ (by Lemma 2). If it were possible to quantify precisely this fact, then (ii) might follow. Note that one cannot expect to replace n^3 by n^2 for all n because it seems that, for any k,

$$\max_{n=1,\dots,F_k} |E_n'(\sqrt{5}-2)| = |E_{F_k}'(\sqrt{5}-2)| \gg F_k^3.$$

More generally, we tried to find the minima of the partial sum $\Psi_{1,N}$ for $N=1$ to 145: the data are summarized in the table below where $\alpha_N \in [0,1/2]$ is such that

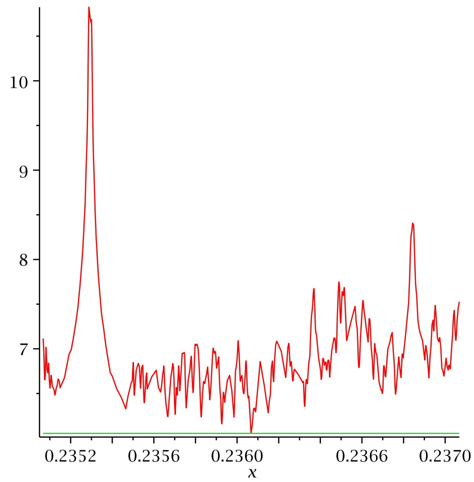

FIGURE 8. Graphs of $\Psi_{1/5,300}$ and the constant $\Psi_{1/5,300}(\sqrt{5}-2)$ in $[\sqrt{5}-2-10^{-3}, \sqrt{5}-2+10^{-3}]$

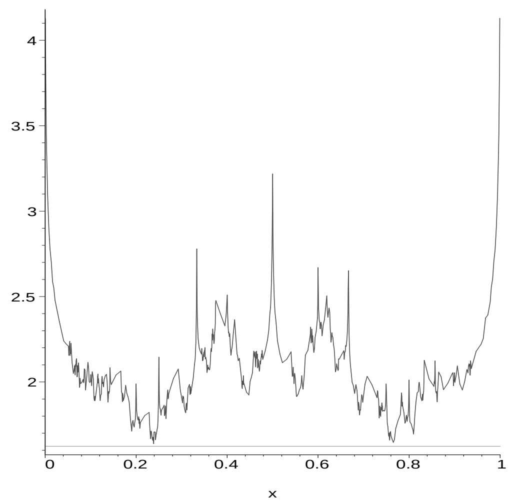

FIGURE 9. Graphs of $\Psi_{1,F_{11}}$ and the constant $\Psi_{1,F_{11},1}(u_{11})$ in $[0,1]$

$\Psi_{1,N}(\alpha_N)$ is apparently minimal. These conjectural values have been obtained by zooming on the part of the graphs where the minimum seemed to be attained. ([3]) Except for $N = 15, 17, 46, 50, 64, 65, 67, 73$, $\Psi_{1,N}$ does not seem to be differentiable at α_N. At these eight exceptional values, $\Psi_{1,N}$ seems to have a vanishing derivative at α_N; we are able to get only numerical approximations for these α_N that we don't mention (they are getting closer and closer to $\sqrt{5}-2$ as expected). It is also interesting to see that, when we are able to identify it, α_N is a rational number whose denominator is a square.

[3]The most difficult part is to guess the exact value of α_N by successive zooms on the graph. Once it is guessed, one can center the subsequent zooms at that point to check if it is a good choice.

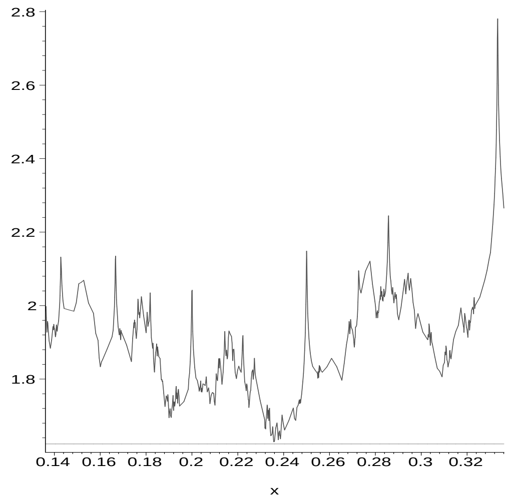

FIGURE 10. Graphs of $\Psi_{1,F_{11}}$ and the constant $\Psi_{1,F_{11}}(u_{11})$ in $[u_{11} - 0.1, u_{11} + 0.1]$

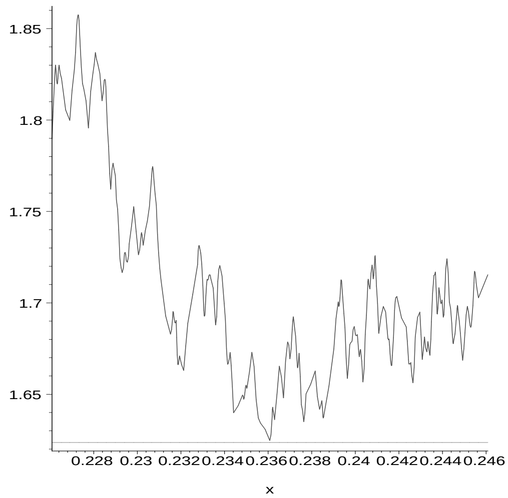

FIGURE 11. Graphs of $\Psi_{1,F_{11}}$ and the constant $\Psi_{1,F_{11}}(u_{11})$ in $[u_{11} - 10^{-2}, u_{11} + 10^{-2}]$

N	2	3	4	5	6	7	8	9	10	11	12	13	14	15
α_N	$\frac{1}{2^2}$	$\frac{2}{3^2}$	$\frac{2}{3^2}$	$\frac{6}{5^2}$	$\frac{6}{5^2}$	$\frac{6}{5^2}$	$\frac{15}{8^2}$	$\frac{19}{9^2}$	$\frac{15}{8^2}$	$\frac{15}{8^2}$	$\frac{19}{9^2}$	$\frac{40}{13^2}$	$\frac{40}{13^2}$	$\star$
16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$\frac{40}{13^2}$	$\star$	$\frac{61}{16^2}$	$\frac{61}{16^2}$	$\frac{61}{16^2}$	$\frac{104}{21^2}$	$\frac{104}{21^2}$	$\frac{104}{21^2}$	$\frac{104}{21^2}$	$\frac{104}{21^2}$	$\frac{53}{15^2}$	$\frac{104}{21^2}$	$\frac{104}{21^2}$	$\frac{53}{15^2}$	$\frac{53}{15^2}$
31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
$\frac{53}{15^2}$	$\frac{53}{15^2}$	$\frac{53}{15^2}$	$\frac{273}{34^2}$	$\frac{273}{34^2}$	$\frac{273}{34^2}$	$\frac{273}{34^2}$	$\frac{341}{38^2}$	$\frac{341}{38^2}$	$\frac{341}{38^2}$	$\frac{341}{38^2}$	$\frac{341}{38^2}$	$\frac{341}{38^2}$	$\frac{341}{38^2}$	$\frac{341}{38^2}$
46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
$\star$	$\frac{273}{34^2}$	$\frac{273}{34^2}$	$\frac{273}{34^2}$	$\star$	$\frac{341}{38^2}$	$\frac{341}{38^2}$	$\frac{341}{38^2}$	$\frac{341}{38^2}$	$\frac{714}{55^2}$	$\frac{714}{55^2}$	$\frac{714}{55^2}$	$\frac{714}{55^2}$	$\frac{714}{55^2}$	$\frac{714}{55^2}$
61	62	63	64	65	66	67	68	69	70	71	72	73	74	75
$\frac{714}{55^2}$	$\frac{714}{55^2}$	$\frac{714}{55^2}$	$\star$	$\star$	$\frac{714}{55^2}$	$\star$	$\frac{714}{55^2}$	$\frac{714}{55^2}$	$\frac{714}{55^2}$	$\frac{1190}{71^2}$	$\frac{323}{37^2}$	$\star$	$\frac{323}{37^2}$	$\frac{613}{51^2}$
76	77	78	79	80	81	82	83	84	85	86	87	88	89	$\cdots$
$\frac{613}{51^2}$	$\frac{613}{51^2}$	$\frac{613}{51^2}$	$\frac{613}{51^2}$	$\frac{613}{51^2}$	$\frac{613}{51^2}$	$\frac{613}{51^2}$	$\frac{613}{51^2}$	$\frac{613}{51^2}$	$\frac{1058}{67^2}$	$\frac{1663}{84^2}$	$\frac{1058}{67^2}$	$\frac{1663}{84^2}$	$\frac{1870}{89^2}$	$\cdots$
109	110	$\cdots$	113	114	$\cdots$	123	124	125	126	127	$\cdots$	143	144	145
$\frac{1870}{89^2}$	$\frac{967}{64^2}$	$\cdots$	$\frac{967}{64^2}$	$\frac{1870}{89^2}$	$\cdots$	$\frac{1870}{89^2}$	$\frac{967}{64^2}$	$\frac{967}{64^2}$	$\frac{967}{64^2}$	$\frac{3808}{127^2}$	$\cdots$	$\frac{3808}{127^2}$	$\frac{4895}{144^2}$	$\frac{4895}{144^2}$

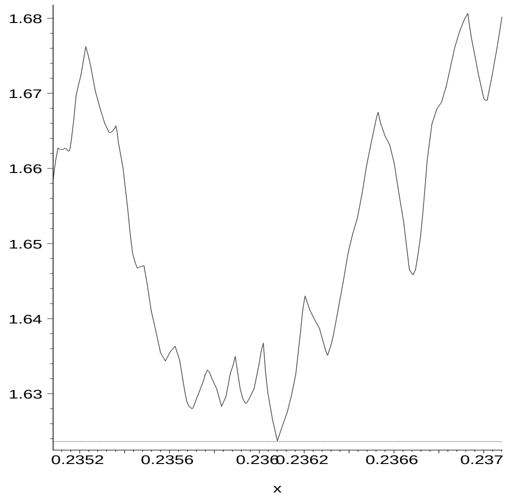

FIGURE 12. Graphs of $\Psi_{1,F_{11}}$ and the constant $\Psi_{1,F_{11}}(u_{11})$ in $[u_{11}-10^{-3}, u_{11}+10^{-3}]$

The dots indicate that, for example, from 89 to 109, the minimum seemingly occurs at the same point $\frac{1870}{89^2}$. In that table, is easy to recognize that when $N = F_k$, then F_k^2 is a denominator of $\alpha_{F_k} = u_k$. To get an expression of the numerator, we simply plugged the sequence of numerators of u_k into the On-line Encyclopedia of Integer Sequences [10] to see that it matches the sequence A001654 defined by $F_{k-1}F_{k-2}$. This led to Conjecture 2.

We also computed approximations to 6 digits of some values of $\Psi_{1,F_k}(u_k)$ for $k = 4, 5, \ldots 26$. They tend to confirm Conjecture 2(ii), even though the convergence is slow.

k	4	5	6	7	8
$\Psi_{1,F_k}(u_k)$	1.0625	1.334325	1.414417	1.459825	1.545960
9	10	11	12	13	14
1.580966	1.599159	1.623628	1.634958	1.641142	1.647968
15	16	17	18	19	20
1.651493	1.653337	1.655235	1.656236	1.656780	1.657293
21	22	23	24	25	26
1.657570	1.657723	1.657860	1.657935	1.657977	1.658013

7. Minimal values of the series $\widetilde{\Psi}_1$

In this section, we present a few results concerning the function

$$\widetilde{\Psi}_1(\alpha) = \sum_{n=1}^{\infty} (-1)^n \frac{||n^2\alpha||}{n^2||n\alpha||}.$$

which is an alternating analogue of Ψ_1. There are a number of differences with the behavior of Ψ_1. In particular, a straightforward modification of the proof of part (i) of Theorem 1 shows that $\widetilde{\Psi}_1(\alpha)$ converges at any rational number $\alpha = a/b$ with b odd and $(a, b) = 1$, while it diverges when b is even and $(a,b)=1$. Of course, $\widetilde{\Psi}_1(\alpha)$ converges almost everywhere because it converges for any irrational number $\alpha \in \mathscr{A}_1$.

Like in the case of Ψ_1, we focused on the extremal properties of $\widetilde{\Psi}_1$ and were led to a precise conjecture regarding the partial sums

$$\widetilde{\Psi}_{1,N}(\alpha) := \sum_{n=1}^{N} (-1)^n \frac{||n^2\alpha||}{n^2||n\alpha||}.$$

Set S_k the k-th denominator of the convergents to $\frac{\sqrt{2}}{2}$; for $k \geq 1$, the sequence starts with $1, 3, 7, 17, 41$. Set $T_k := 2R_k + (-1)^k$ where R_k is defined by $R_0 = 0$, $R_1 = 1$ and $R_{k+2} = 6R_{k+1} - R_k$.

Conjecture 3. (i) *For any $k \geq 2$, the sum $\widetilde{\Psi}_{1,S_k}$ is minimal on $[0,1]$ at the points*

$$v_k := \frac{T_k}{2S_k^2} \quad \text{and} \quad 1 - v_k.$$

(ii) *We have*

$$\lim_{k\to+\infty} \widetilde{\Psi}_{1,S_k}(v_k) = \widetilde{\Psi}_1\Big(\frac{\sqrt{2}}{2}\Big).$$

(iii) *On its set of convergence, the series $\widetilde{\Psi}_1$ is minimal at the points of $\pm\frac{\sqrt{2}}{2} + \mathbb{Z}$, where it takes the same value.*

It is easy to see that $R_k \sim \frac{\sqrt{2}}{8}(1+\sqrt{2})^{2k}$ and that $S_k \sim \frac{1}{2}(1+\sqrt{2})^k$. Hence

$$\lim_{k\to+\infty} v_k = \frac{\sqrt{2}}{2}.$$

The first few values of the sequence v_k are $\frac{13}{2\cdot 3^2}$, $\frac{69}{2\cdot 7^2}$, $\frac{409}{2\cdot 17^2}$, $\frac{2377}{2\cdot 41^2}$. They were guessed by successive zooms of the part of the graph of $\widetilde{\Psi}_{1,S_k}$ where the minimum seems to be attained. Again, the numerators of the sequence v_k were found by using the OEIS [10]: the sequence T_k matches `A105058` and the sequence R_k matches `A001109` (which is directly linked to `A105058` in the OEIS). Of course, parts (i) and (ii) of Conjecture 3 implies part (iii).

8. A related diophantine function

In this section, we define another "diophantine function", namely the series

$$\mathcal{Q}_{s,t}(\alpha) := \sum_{n=0}^{\infty} \frac{\log\big(q_{n+1}(\alpha)/q_n(\alpha)\big)^t}{q_n(\alpha)^s}$$

for $\alpha \in \mathbb{R}$ and $s,t > 0$. The case $s=1$ and $t=1$ is motivated by the similarity of both sides of the inequalities (1.5) in Theorem 1(ii). ([4]) The similary is also visible when one compares Figure 4 and Figure 13: it would interesting to understand better the link between $\mathcal{Q}_{1,1}$ and Φ_1.

[4]For $s \in (0,1)$, the left and right hand sides of the inequalities (1.4) are not very close. The extremality properties of the series $\sum_{n=0}^{\infty} \frac{q_{n+1}(\alpha)^t}{q_n(\alpha)^s}$ are not striking at first sight.

It is easy to prove that $\mathcal{Q}_{s,t}(\alpha)$ converges for almost all irrational numbers α, in particular for all α such that $m(\alpha)$ is finite. The infinite series $\mathcal{Q}_{s,t}(\alpha)$ is not defined for rational numbers α because the sequence $(q_n)_n$ is then finite. But this can be solved as follows: we assume that the sequence of partial quotients of $\alpha \in \mathbb{Q}$ is of the form $(a_n)_{n=0,\dots,K}$ with $a_K \geq 2$, so that we can can set ([5])

$$\mathcal{Q}_{s,t}(\alpha) := \sum_{n=0}^{K-1} \frac{\log\big(q_{n+1}(\alpha)/q_n(\alpha)\big)^t}{q_n(\alpha)^s}$$

for $\alpha \in \mathbb{Q}$.

Conjecture 4. *Fix the real numbers $s, t > 0$. The series $\mathcal{Q}_{s,t}$ attains its minimum in $\mathbb{R}\setminus\mathbb{Q}$ at the points of $\frac{\sqrt{5}-1}{2} + \mathbb{Z}$ and $\frac{3-\sqrt{5}}{2} + \mathbb{Z}$.*

The values at the minima are equal because $\mathcal{Q}_{s,t}(\alpha)$ is 1-periodic and $\mathcal{Q}_{s,t}(1-\alpha) = \mathcal{Q}_{s,t}(\alpha)$. In fact, it seems that a finite version of Conjecture 4 holds. Set

$$\mathcal{Q}_{N,s,t}(\alpha) := \sum_{n=0}^{N} \frac{\log\big(q_{n+1}(\alpha)/q_n(\alpha)\big)^t}{q_n(\alpha)^s}$$

for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and

$$\mathcal{Q}_{N,s,t}(\alpha) := \sum_{n=0}^{\min(N,K-1)} \frac{\log\big(q_{n+1}(\alpha)/q_n(\alpha)\big)^t}{q_n(\alpha)^s}$$

for $\alpha \in \mathbb{Q}$. Although this is not completely clear on the various graphs (which are mere approximations of the reality), $\mathcal{Q}_{N,s,t}$ is essentially a piecewise constant function. It is continuous at any irrational number, around which it is locally constant. It is also continuous and locally constant around any rational number whose sequence of partial quotients terminates at a position $> N+1$. But it is discontinuous at any rational number whose sequence of partial quotients terminates at position $\leq N+1$.

Conjecture 5. *Fix any integer $N \geq 0$ and any real numbers $s \geq 0$, $t > 0$. We consider $\mathcal{Q}_{N,s,t}$ as being defined on $\mathbb{R} \setminus \mathbb{Q}$.*

(i) The series $\mathcal{Q}_{N,s,t}$ is constant and minimal on the interval consisting of irrational numbers whose partial quotients satisfy $a_0 = 0$, $a_1 = a_2 = \cdots = a_{N+1} = 1$.

(ii) The second minimal value of $\mathcal{Q}_{N,s,t}$ is attained on the interval consisting of irrational numbers whose partial quotients satisfy $a_0 = 0$, $a_1 = 2, a_2 = \cdots = a_{N+1} = 1$. It is also constant there.

If the irrational number α is in $(1/2, 1)$ then $q_n(1-\alpha) = q_{n+1}(\alpha)$ for all $n \geq 1$ (with $q_0(1-\alpha) = q_0(\alpha) = 1$), so that $\mathcal{Q}_{N,s,t}(1-\alpha) = Q_{N+1,s,t}(\alpha)$: hence part (ii) of the conjecture follows from (i). It is also clear that Conjecture 5(i) together with the identity $\mathcal{Q}_{s,t}(1-\alpha) = \mathcal{Q}_{s,t}(\alpha)$ imply Conjecture 4 when $s > 0$. The first part can be reformulated

[5] The alternative definition "$(\widetilde{a}_n)_{n=0,\dots,K+1}$ with $\widetilde{a}_n = a_n$ for $n < K$ and $\widetilde{a}_K = a_k - 1$ and $\widetilde{a}_{K+1} = 1$" changes only marginally the discussion following Conjecture 4 for rational numbers and does not affect both conjectures which concern only irrational numbers.

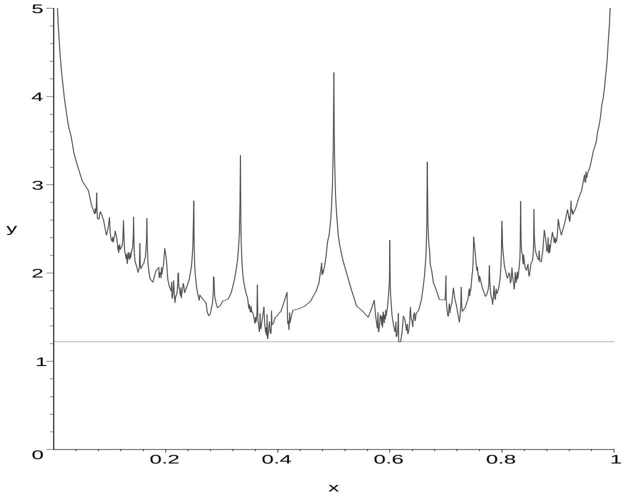

FIGURE 13. Graphs of $\mathcal{Q}_{5,1,1}$ and the constant $\mathcal{Q}_{5,1,1}(\frac{\sqrt{5}-1}{2})$ on $[0,1]$

as follows: if $N = 2k$, then $\mathcal{Q}_{N,s,t}(\alpha)$ is constant on the interval $\big(\frac{F_{2k+2}}{F_{2k+3}}, \frac{F_{2k+1}}{F_{2k+2}}\big)$, where it is minimal. If $N = 2k+1$, then $\mathcal{Q}_{N,s,t}(\alpha)$ is constant on the interval $\big(\frac{F_{2k+2}}{F_{2k+3}}, \frac{F_{2k+3}}{F_{2k+4}}\big)$, where it is minimal.

The conjecture is trivially true in the case $s = 0$ and $t = 1$ because then $\mathcal{Q}_{N,s,t}(\alpha) = \log(q_{N+1}(\alpha))$: that quantity is minimal if and only if $q_0 = 1$, $q_1 = 1$ and $q_{n+1} = q_n + q_{n-1}$ for any n such that $1 \leq n \leq N$.

A careful analysis of many graphs similar to those presented in Figures 13 to 17 led to Conjecture 5. The latter is easily proved for $N = 0, 1, 2$ and $s = t = 1$ by a direct computation (which could probably be extended to further values of N, s and t).

• $N = 0$: we have to show that $\frac{1}{q_0}\log(q_1/q_0) = \log(q_1)$ is minimal for $q_1 = 1$, which is obviously true.

• $N = 1$: we have to show that

$$\frac{\log(q_1/q_0)}{q_0} + \frac{\log(q_2/q_1)}{q_1} = \log(q_1)\Big(1 - \frac{1}{q_1}\Big) + \frac{\log(q_2)}{q_1}$$

is minimal for $q_1 = 1$ and $q_2 = 2$. Clearly, we must choose q_2 minimal, i.e, $q_2 = q_1 + q_0 = q_1 + 1$. To conclude, it remains to see that when $q_1 \geq 1$, the function of the *integer* q_1

$$\log(q_1)\Big(1 - \frac{1}{q_1}\Big) + \frac{\log(q_1+1)}{q_1} \tag{8.1}$$

is minimal for $q_1 = 1$.

• $N = 2$: we have to show that

$$\frac{\log(q_1/q_0)}{q_0} + \frac{\log(q_2/q_1)}{q_1} + \frac{\log(q_3/q_2)}{q_2}$$

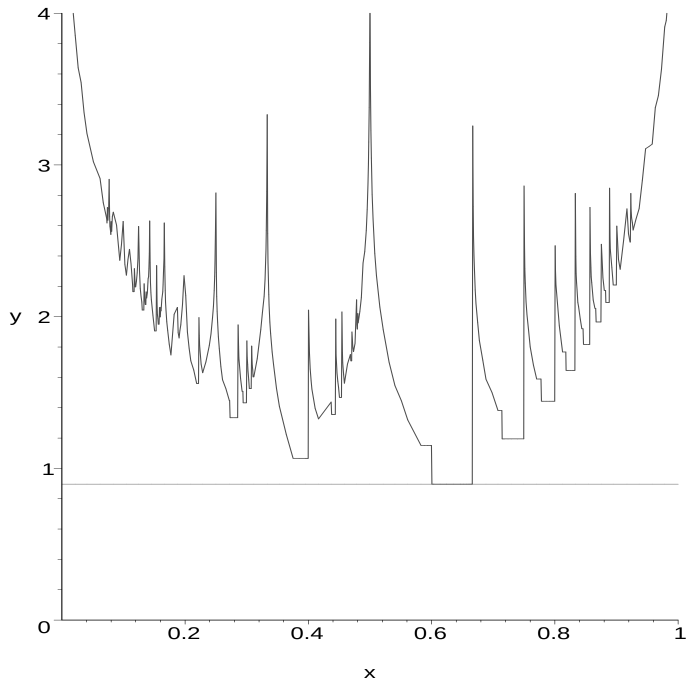

FIGURE 14. Graphs of $\mathcal{Q}_{2,1,1}$ and the constant $\mathcal{Q}_{2,1,1}(\frac{\sqrt{5}-1}{2})$ on $[0,1]$

is minimal for $q_1 = 1$, $q_2 = 2$ and $q_3 = 3$. Again, we must choose q_3 minimal, i.e, $q_3 = q_2 + q_1$. When $q_2 \geq q_1 + 1$, the function of the *integer* q_2

$$\frac{\log(q_2/q_1)}{q_1} + \frac{\log((q_2+q_1)/q_2)}{q_2}$$

is minimal for $q_2 = q_1 + 1$. It remains therefore to find the minimum of the function

$$\log(q_1) + \frac{\log(\frac{q_1+1}{q_1})}{q_1} + \frac{\log(\frac{2q_1+1}{q_1+1})}{q_1+1} \tag{8.2}$$

as a function of the integer $q_1 \geq 1$ and again it is attained at $q_1 = 1$. ([6]) This proves this case too.

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[6]The functions (8.1) and (8.2) viewed as functions of the *real* variable $q_1 \geq 1$ are not minimal at $q_1 = 1$ but somewhere between 1 and 2.

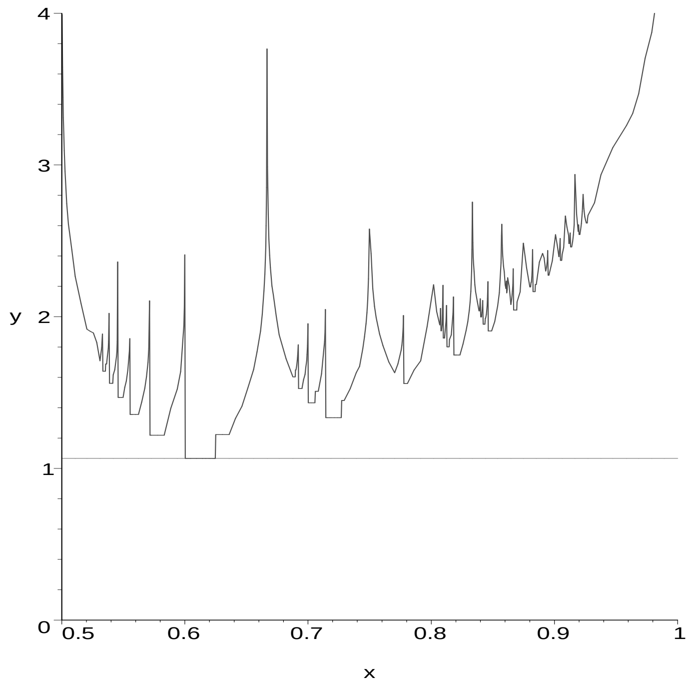

FIGURE 15. Graphs of $\mathcal{Q}_{3,1,1}$ and the constant $\mathcal{Q}_{3,1,1}(\frac{\sqrt{5}-1}{2})$ on $[1/2, 1]$

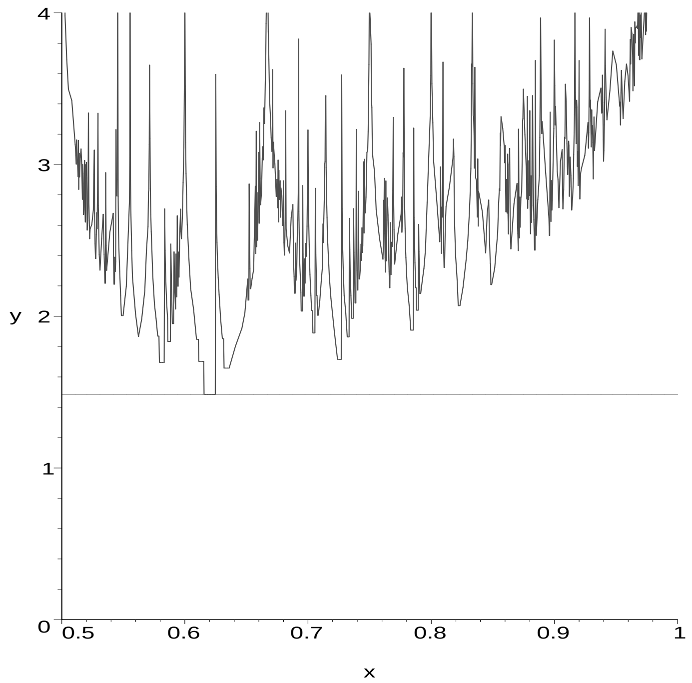

FIGURE 16. Graphs of $\mathcal{Q}_{4,1/2,1}$ and the constant $\mathcal{Q}_{4,1/2,1}(\frac{\sqrt{5}-1}{2})$ on $[1/2, 1]$

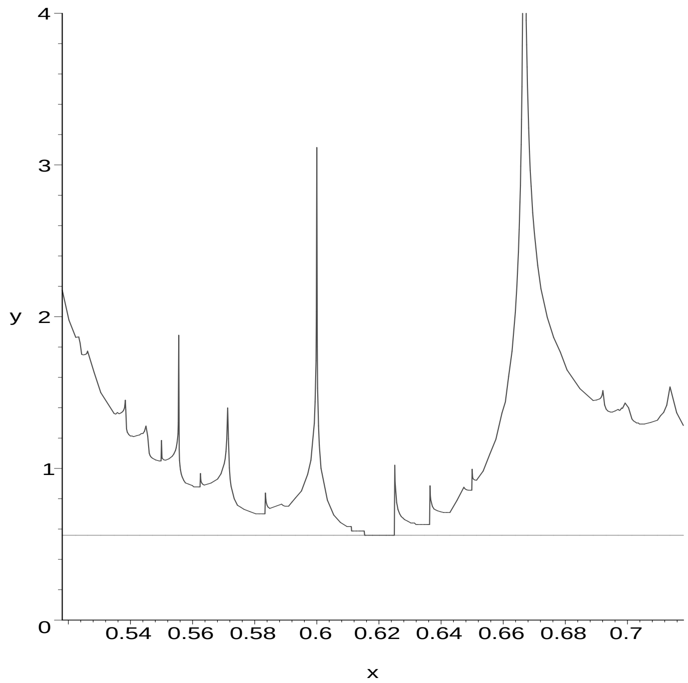

FIGURE 17. Graphs of $\mathcal{Q}_{4,2,2}$ and the constant $\mathcal{Q}_{4,2,2}(\frac{\sqrt{5}-1}{2})$ on $[\frac{\sqrt{5}-1}{2} - 0.1, \frac{\sqrt{5}-1}{2} + 0.1]$

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