SIEGEL'S PROBLEM FOR E-FUNCTIONS OF ORDER 2

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Abstract. E-functions are entire functions with algebraic Taylor coefficients at the origin satisfying certain arithmetic conditions, and solutions of linear differential equations with coefficients in $\overline{\mathbb{Q}}(z)$; they naturally generalize the exponential function. Siegel and Shidlovsky proved a deep transcendence result for their values at algebraic points. Since then, a lot of work has been devoted to apply their theorem to special E-functions, in particular the hypergeometric ones. In fact, Siegel asked whether any E-function can be expressed as a polynomial in z and generalized confluent hypergeometric series. As a first positive step, Shidlovsky proved that E-functions with order of the differential equation equal to 1 are in $\overline{\mathbb{Q}}[z]e^{\overline{\mathbb{Q}}z}$. In this paper, we give a new proof of a result of Gorelov that any E-function (in the strict sense) with order ≤ 2 can be written in the form predicted by Siegel with confluent hypergeometric functions ${}_1F_1[\alpha;\beta;\lambda z]$ for suitable $\alpha, \beta \in \mathbb{Q}$ and $\lambda \in \overline{\mathbb{Q}}$. Gorelov's result is in fact more general as it holds for E-functions in the large sense. Our proof makes use of André's results on the singularities of the minimal differential equations satisfied by E-functions, together with a rigidity criterion for (irregular) differential systems recently obtained by Bloch-Esnault and Arinkin. An *ad-hoc* version of this criterion had already been used by Katz in his study of confluent hypergeometric series. Siegel's question remains unanswered for orders ≥ 3.

1. Introduction

We fix an embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}$. An E-function (in the strict sense) is a power series

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n \in \overline{\mathbb{Q}}[[z]]$$

such that:

(1) $f(z)$ satisfies a non-zero linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$;
(2) there exists $C > 0$ such that for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we have $|\sigma(a_n)| \leq C^{n+1}$; and there exists a sequence of positive integers d_n such that $d_n \leq C^{n+1}$ and $d_n a_m$ is an algebraic integer for all $m \leq n$.

If $a_n \in \mathbb{Z}$, the conditions in (2) reduce to $|a_n| \leq C^{n+1}$. This class of arithmetic power series was defined by Siegel [18] (in a slightly more general way) to mimic the Diophantine properties of the exponential function, and his program was later completed by Shidlovsky [20]. Throughout the paper, we set $\theta = z\frac{d}{dz}$ and by "solution of a differential operator $\mathcal{L} \in \overline{\mathbb{Q}}(z)[\frac{d}{dz}]$", it must be understood "solution of the differential equation

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$\mathcal{L}y(z) = 0$". More recently, André [2] and Beukers [4] gave a new impulse to the Diophantine theory of E-functions, whose prototypical example is the generalized hypergeometric function

$$ {}_pF_q\left[\begin{matrix}\alpha_1, & \ldots, & \alpha_p \\ \beta_1, & \ldots, & \beta_q\end{matrix}; z^{q-p+1}\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{n!(\beta_1)_n \cdots (\beta_q)_n} z^{(q-p+1)n} $$

with $q \geq p \geq 1$, $\alpha_j, \beta_j \in \mathbb{Q}$, and none of the β's is a negative integer. If $p = q = 1$, it is a solution of the differential operator $\theta(\theta + \beta - 1) - z(\theta + \alpha)$ and when $\alpha = \beta$, this is simply $\exp(z)$.

The present paper is concerned with the following classical questions: What are E-functions? Are they related to generalized hypergeometric functions? In fact, Siegel [19, p. 58] asked the following question: can any E-function be represented as a multivariate polynomial, with coefficients in $\overline{\mathbb{Q}}[z]$, in finitely many confluent hypergeometric series of the form ${}_pF_q[\underline{a}; \underline{b}; \lambda z^{q-p+1}]$, for various $q \geq p \geq 1$, $\underline{a} \in \overline{\mathbb{Q}}^p$, $\underline{b} \in \overline{\mathbb{Q}}^q$ and $\lambda \in \overline{\mathbb{Q}}$? See also [20, p. 84]. ([1])

In the recent papers [16, 17], we studied the structural properties of differential equations satisfied by strict E-functions, in the light of [2]. In particular, as a consequence of the main result of [16], we proved that any strict E-function $f(z)$ solution of an inhomogeneous linear equation

$$ f'(z) = u(z)f(z) + v(z) \tag{1.1} $$

of order 1 is essentially hypergeometric, where $u(z) \in \overline{\mathbb{Q}}(z)^\times$ and $v(z) \in \overline{\mathbb{Q}}(z)$. More precisely, there exist some $a(z), b(z) \in \overline{\mathbb{Q}}[z, z^{-1}]$, $\beta \in \{1\} \cup \mathbb{Q} \setminus \mathbb{Z}$ and $\lambda \in \overline{\mathbb{Q}}$ such that

$$ f(z) = a(z)\,{}_1F_1(1; \beta; \lambda z) + b(z). \tag{1.2} $$

This solved a problem, raised by Shidlovsky [20, p. 184], and already partially solved by André in [2, p. 724]. If $v(z) = 0$, then in fact $b(z) = 0$, $\beta = 1$ and $a(z) \in \overline{\mathbb{Q}}[z]$, so that $f(z) = a(z)e^{\lambda z}$, a result due to Shidlovsky [20, p. 184]. When $v(z) \neq 0$, any solution of (1.1) is also solution of the differential operator of order 2 given by $\frac{d}{dz}(\frac{1}{v(z)}\frac{d}{dz} - \frac{u(z)}{v(z)})$. We were not aware at that time that Gorelov had already solved Shidlovsky's problem in [11, p. 139, Theorem 2] for E-functions in Siegel's original sense.

It is thus natural to wonder if something similar to (1.2) can be said of E-functions solutions of differential equations of order 2. Actually, a first case was considered in [16]. Indeed, the following result is a direct consequence of [16, Theorem 4]. Again, this result has been proved by Gorelov in [12, p. 515, Theorem 2] for E-functions in Siegel's original sense.

Theorem 1. *Let $f(z)$ be a strict E-function solution of a non-zero linear differential operator of order 2 with coefficients in $\overline{\mathbb{Q}}(z)$ and reducible over $\overline{\mathbb{Q}}(z)$. Then, there exist $a(z), b(z) \in \overline{\mathbb{Q}}[z, z^{-1}]$, $\beta \in \{1\} \cup \mathbb{Q} \setminus \mathbb{Z}$ and $\lambda, \mu \in \overline{\mathbb{Q}}$ such that*

$$ f(z) = a(z)e^{\mu z}\,{}_1F_1(1; \beta; \lambda z) + b(z)e^{\mu z}. \tag{1.3} $$

[1] In this problem, Siegel referred to his original definition of E-functions, which are slightly more general than the strict ones used in this paper, which are themselves called E^*-functions in [20]. It is widely believed that both class are identical, but our proof holds only in the strict sense.

We now state the result if we remove the reducibility hypothesis in the previous result. We emphasize that Theorem 2 below is a particular case of a result of Gorelov [12, p. 514, Theorem 1], who in fact did not make the distinction between the reducible and irreducible case, which is in accordance with the remarks made after the theorem.

Theorem 2. *Let $f(z)$ be a strict E-function solution of a non-zero linear differential operator of order 2 with coefficients in $\overline{\mathbb{Q}}(z)$ and irreducible over $\overline{\mathbb{Q}}(z)$. Then, we have*

$$f(z) = a(z)e^{\mu z}{}_1F_1(\alpha;\beta;\lambda z) + b(z)e^{\mu z}{}_1F_1'(\alpha;\beta;\lambda z) \tag{1.4}$$

where $a(z), b(z) \in \overline{\mathbb{Q}}(z)$, $\lambda \in \overline{\mathbb{Q}}^{\times}$, $\mu \in \overline{\mathbb{Q}}$, and $\alpha \in \mathbb{Q}$, $\beta \in \mathbb{Q}\setminus\mathbb{Z}_{\leq 0}$ are such that $\alpha-\beta \notin \mathbb{Z}$.

Note that (1.3) is of the form (1.4) because we have the relation $z{}_1F_1'(1;\beta;z) = (z-\beta+1){}_1F_1(1;\beta;z)+\beta-1$. Moreover, ${}_1F_1'(\alpha;\beta;z) = \frac{\alpha}{\beta}{}_1F_1(\alpha+1;\beta+1;z)$. This explains why such results give a positive solution to Siegel's problem for orders ≤ 2. Gorelov proved a stronger version of Theorem 2: he showed that $f(z)$ can be assumed to be an E-function in Siegel's original sense and moreover that the conclusion holds with some $a(z), b(z) \in \overline{\mathbb{Q}}[z]$. On the other hand, he did not state that $\alpha-\beta \notin \mathbb{Z}$. We observe that in fact a strict E-function of order 2 may have a representation of the form (1.4) with $a(z)$ and $b(z)$ not necessarily polynomials, for instance $\varphi(z) := \frac{2}{z}{}_1F_1(1;1/2;z) - \frac{1}{z}{}_1F_1'(1;1/2;z) = \frac{4}{3}+\frac{16z}{15}+\frac{16z^2}{35}+\dots$ is such an E-function. This does not contradict Gorelov's stronger "polynomial coefficients" version, because a strict E-function does not necessarily admit a unique representation of the form (1.4); indeed, it is readily proved that $\varphi(z) = 2{}_1F_1'(1;3/2;z)$.

The main contribution of this paper is our proof of Theorem 2, which is quite different from that of Gorelov, even though he also used André's theory at some point. We did not try to reprove his version in full. We hope our point of view will be useful for further studies in this field.

We now illustrate Theorem 2 with the non-hypergeometric E-function

$$a(z) = \sum_{n=0}^{\infty} \frac{1}{n!}\Big(\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{n}\Big)z^n. \tag{1.5}$$

It was brought to our attention by F. Beukers during a lecture he gave in June 2016 at the conference *Automates and Number Theory* held at Porquerolles, where he asked if $a(z)$ was related in some way to hypergeometric series. Since it is solution of the irreducible differential operator $z(\frac{d}{dz})^2-(6z-1)\frac{d}{dz}+(z-3)$, Theorem 2 applies to it and the answer is yes. Let us give (1.4) in this case. Since $\sum_{n=0}^{\infty}\big(\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{n}\big)z^n = \frac{1}{\sqrt{1-6z+z^2}}$ (see [15, §3]), we have

$$a(z) = \frac{1}{2i\pi}\int_L \frac{e^{zx}}{\sqrt{1-6x+x^2}}dx$$

where L is a "vertical" straight line leaving the roots of $1-6x+x^2$ to its left; see [10, §§4.2-4.3]. With the change of variable $x = t+3$ and with L' a "vertical" straight line leaving the roots of t^2-8 to its left, we obtain

$$a(z)=\frac{1}{2i\pi}\int_{L'}\frac{e^{z(t+3)}}{\sqrt{t^2-8}}dt=e^{3z}\sum_{n=0}^{\infty}\frac{(2z^2)^n}{n!^2}=e^{3z}\cdot{}_0F_1(\cdot\,;1;2z^2) \tag{1.6}$$

$$=e^{(3-2\sqrt{2})z}\cdot{}_1F_1(1/2\,;1;4\sqrt{2}z). \tag{1.7}$$

The hypergeometric function on the right of (1.6) is not formally of the form suggested by Theorem 2, but this is the case of (1.7). The equality of both expressions is a consequence of an hypergeometric identity between Kummer M function and Bessel I_0 function [1, p. 509, 13.6.1]. In passing, we obtain the binomial identity

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{n}=\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\sqrt{2}^k(3-2\sqrt{2})^{n-k},\ n\geq 0, \tag{1.8}$$

after multiplication of the two (implicit) power series in (1.7). Conversely, (1.8) could be proved first (by means of Zeilberger's algorithm) and then (1.7) would follow again.

We don't know if results similar to Theorems 1 and 2 could be obtained for E-functions of higher order, even for the order 3; the methods of this paper do not give enough informations to conclude. In fact, Siegel's question might have a negative answer in general and as the order increases, one needs to add more and more special functions to the hypergeometric ones. For instance, $\sum_{n=0}^{\infty}\frac{1}{n!}(\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{n})z^n$ is solution of $z^2y'''(z)+(3z-11z^2)y''(z)+(1-22z-z^2)y'(z)-(3+z)y(z)=0$. Although not hypergeometric, can it be expressed using confluent hypergeometric series as requested in Siegel's problem, like $a(z)$ above, or is it of a different nature? We refer to [13, 14] for further results on this problem.

The paper is organized as follows. In Sections 2 (hypergeometric operators), 3 (Fuchs relation) and 4 (rigidity), we collect some results from the literature that we need for the proof of Theorem 2, which is given in Section 5. Some aspects of the proof are similar with certain results of Katz in [8], though the methods are not written in the same language; we include these computations for the sake of completeness. Finally, in Section 6, we make some remarks on E-operators and G-operators. From now on, any E-function is understood to be in the strict sense.

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2. The differential operators $H_{\alpha;\beta,\gamma}$ and $L_{\alpha,\beta,\gamma,\lambda,\mu}$

In this section, we gather some results on two explicit hypergeometric operators. They will be used in the proof of Theorem 2 later on.

2.1. **The confluent hypergeometric operator** $H_{\alpha;\beta,\gamma}$**.** We recall that $\theta=z\frac{d}{dz}$. The confluent hypergeometric operator with parameters $\alpha,\beta,\gamma\in\mathbb{C}$ is the linear differential operator given by

$$H_{\alpha;\beta,\gamma}=(\theta+\beta-1)(\theta+\gamma-1)-z(\theta+\alpha).$$

It has at most two singularities on $\mathbb{P}^1(\mathbb{C})$, namely 0 and ∞. The former is a regular singular point, the latter is an irregular singular point. More precisely, the slopes of the Newton polygon of $H_{\alpha;\beta,\gamma}$ at ∞ are 0 and 1, both with multiplicity 1. We denote by

$$Y' = A_{\alpha;\beta,\gamma}Y \tag{2.1}$$

the differential system associated to $H_{\alpha;\beta,\gamma}$.

For later use, we shall now describe the general form of the formal solutions of the differential system (2.1) as predicted by Turittin's theorem [21, Theorem 3.54].

We first assume that $\beta-\gamma \notin \mathbb{Z}$. Then, at 0, the differential system (2.1) admits a fundamental matrix of formal solutions of the form $F_0(z)z^{\Gamma_0}$ where

$$F_0(z) \in \mathrm{GL}_2(\mathbb{C}((z))) \quad \text{and} \quad \Gamma_0 = \begin{pmatrix} 1-\beta & 0 \\ 0 & 1-\gamma \end{pmatrix}.$$

At ∞, the differential system (2.1) admits a fundamental matrix of formal solutions of the form $F_\infty(z)z^{\Gamma_\infty}e^{\Delta z}$ where

$$F_\infty(z) \in \mathrm{GL}_2(\mathbb{C}((z^{-1}))), \quad \Gamma_\infty = \begin{pmatrix} \alpha-\beta-\gamma+1 & 0 \\ 0 & -\alpha \end{pmatrix} \quad \text{and} \quad \Delta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

We now assume that $\beta-\gamma \in \mathbb{Z}$. If $\alpha-\beta \notin \mathbb{Z}$, then, at 0, the differential system (2.1) admits a fundamental matrix of formal solutions of the form $F_0(z)z^{\Gamma_0}$ where

$$F_0(z) \in \mathrm{GL}_2(\mathbb{C}((z))) \quad \text{and} \quad \Gamma_0 = \begin{pmatrix} 1-\beta & 1 \\ 0 & 1-\beta \end{pmatrix}.$$

Remark. If the condition $\alpha-\beta \notin \mathbb{Z}$ is not satisfied, then Γ_0 may be diagonalizable. Consider for instance the case $\alpha=\gamma-1$ and $\beta-1=\alpha-1$.

We shall now consider a special case with $\alpha-\beta \in \mathbb{Z}$, namely $\alpha=\beta-1=\gamma-1$. In this case, at 0, the differential system (2.1) admits a fundamental matrix of formal solutions of the form $F_0(z)z^{\Gamma_0}$ where

$$F_0(z) \in \mathrm{GL}_2(\mathbb{C}((z))) \quad \text{and} \quad \Gamma_0 = \begin{pmatrix} 1-\beta & 1 \\ 0 & 1-\beta \end{pmatrix}.$$

This can be seen by direct calculation using that, in the present case, we have $H_{\alpha;\beta,\gamma} = (\theta+\alpha-z)(\theta+\alpha)$. At ∞, the differential system (2.1) admits a fundamental matrix of formal solutions of the form $F_\infty(z)z^{\Gamma_\infty}e^{\Delta z}$ where

$$F_\infty(z) \in \mathrm{GL}_2(\mathbb{C}((z^{-1}))), \quad \Gamma_\infty = \begin{pmatrix} 1-\beta & 0 \\ 0 & 1-\beta \end{pmatrix} \quad \text{and} \quad \Delta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

2.2. **The operator $L_{\alpha,\beta,\gamma,\lambda,\mu}$.** For any $\alpha,\beta,\gamma,\lambda,\mu \in \mathbb{C}$, with $\lambda \neq 0$, we consider the linear differential operator given by

$$L_{\alpha;\beta,\gamma;\lambda,\mu} = (\theta+\beta-1-\mu z)(\theta+\gamma-1-\mu z) - \lambda z(\theta+\alpha-\mu z).$$

There is a simple relation between the operators $L_{\alpha;\beta,\gamma;\lambda,\mu}$ and $H_{\alpha;\beta,\gamma}$: we have

$$H_{\alpha;\beta,\gamma}\big(y(z)\big) = 0 \Longleftrightarrow L_{\alpha;\beta,\gamma;\lambda,\mu}\big(y(\lambda z)e^{\mu z}\big) = 0.$$

We denote by

$$Y' = A_{\alpha;\beta,\gamma;\lambda,\mu}Y \tag{2.2}$$

the differential system associated to $L_{\alpha;\beta,\gamma;\lambda,\mu}$.

The results of Section 2.1 imply the following facts.

We first assume that $\beta - \gamma \notin \mathbb{Z}$. Then, at 0, the differential system (2.2) admits a fundamental matrix of formal solutions of the form $F_0(z)z^{\Gamma_0}$ where

$$F_0(z) \in \mathrm{GL}_2(\mathbb{C}((z))) \quad \text{and} \quad \Gamma_0 = \begin{pmatrix} 1-\beta & 0 \\ 0 & 1-\gamma \end{pmatrix}.$$

At ∞, the differential system (2.2) admits a fundamental matrix of formal solutions of the form $F_\infty(z)z^{\Gamma_\infty}e^{\Delta z}$ where

$$F_\infty(z) \in \mathrm{GL}_2(\mathbb{C}((z^{-1}))), \quad \Gamma_\infty = \begin{pmatrix} \alpha-\beta-\gamma+1 & 0 \\ 0 & -\alpha \end{pmatrix} \quad \text{and} \quad \Delta = \begin{pmatrix} \lambda+\mu & 0 \\ 0 & \mu \end{pmatrix}.$$

We now assume that $\beta - \gamma \in \mathbb{Z}$. If $\alpha - \beta \notin \mathbb{Z}$, then, at 0, the differential system (2.2) admits a fundamental matrix of formal solutions of the form $F_0(z)z^{\Gamma_0}$ where

$$F_0(z) \in \mathrm{GL}_2(\mathbb{C}((z))) \quad \text{and} \quad \Gamma_0 = \begin{pmatrix} 1-\beta & 1 \\ 0 & 1-\beta \end{pmatrix}.$$

If $\alpha = \beta - 1 = \gamma - 1$, then, at 0, the differential system (2.2) admits a fundamental matrix of formal solutions of the form $F_0(z)z^{\Gamma_0}$ where

$$F_0(z) \in \mathrm{GL}_2(\mathbb{C}((z))) \quad \text{and} \quad \Gamma_0 = \begin{pmatrix} 1-\beta & 1 \\ 0 & 1-\beta \end{pmatrix}.$$

At ∞, the differential system (2.2) admits a fundamental matrix of formal solutions of the form $F_\infty(z)z^{\Gamma_\infty}e^{\Delta z}$ where

$$F_\infty(z) \in \mathrm{GL}_2(\mathbb{C}((z^{-1}))), \quad \Gamma_\infty = \begin{pmatrix} \alpha-\beta-\gamma+1 & 0 \\ 0 & -\alpha \end{pmatrix} \quad \text{and} \quad \Delta = \begin{pmatrix} \lambda+\mu & 0 \\ 0 & \mu \end{pmatrix}.$$

3. Fuchs' relation

In this section, we state a "Fuchs relation" between the exponents of certain (possibly irregular) differential systems.

Proposition 1. *Let us consider a differential system $Y' = AY$ with $A \in M_2(\mathbb{C}(z))$. We assume that the following properties are satisfied :*

1. *$Y' = AY$ has only apparent singularities on $\mathbb{C}^\times$;*
2. *$Y' = AY$ has a basis of solutions at 0 of the form $F_0(z)z^{\Gamma_0}$ where $F_0(z) \in \mathrm{GL}_2(\mathbb{C}((z)))$ and $\Gamma_0 \in M_2(\mathbb{C})$ is upper-triangular.*
3. *$Y' = AY$ admits a basis of formal solutions at ∞ of the form $F_\infty(z)z^{\Gamma_\infty}e^{\Delta z}$ where $F_\infty(z) \in \mathrm{GL}_2(\mathbb{C}((z^{-1})))$, $\Gamma_\infty \in M_2(\mathbb{C})$ is upper-triangular, $\Delta = \mathrm{diag}(\theta_1, \theta_2) \in M_2(\mathbb{C})$ is diagonal, and Γ_∞ and Δ commute.*

Then, the trace of $\Gamma_0 - \Gamma_\infty$ belongs to $\mathbb{Z}$.

Proof. The monodromy of $Y' = AY$ at 0 with respect to the fundamental matrix of solutions $F_0(z)z^{\Gamma_0}$ is given by $M_0 = e^{2\pi i \Gamma_0}$. Its monodromy at ∞ with respect to the same fundamental matrix of solutions is of the form $M_\infty = Pe^{-2\pi i \Gamma_\infty} S_\infty P^{-1}$ where $P \in \mathrm{GL}_2(\mathbb{C})$ and where $S_\infty \in \mathrm{GL}_2(\mathbb{C})$ is a product of Stokes matrices (see [21, Proposition 8.12]). In particular, S_∞ is unipotent and, hence, $\det(S_\infty) = 1$. But, we have $M_0 M_\infty = I_2$ because $Y' = AY$ has only apparent singularities on $\mathbb{C}^\times$ and, hence, trivial monodromies around each point of $\mathbb{C}^\times$. It follows that $\det(M_0 M_\infty) = \det(I_2) = 1$ *i.e.* $e^{2\pi i \,\mathrm{tr}(\Gamma_0 - \Gamma_\infty)} = 1$. Whence the result. □

4. A Reminder on rigidity

This section is a reminder about the notion of rigidity of (possibly irregular) differential systems. We start by recalling the notions of formal and rational equivalences.

4.1. **Formal equivalence.** Recall that two differential systems

$$Y' = AY \text{ and } Y' = BY \text{ with } A, B \in M_n(\mathbb{C}(z)) \tag{4.1}$$

are formally equivalent at 0 if there exists $R \in \mathrm{GL}_n(\mathbb{C}((z)))$ such that

$$B = R^{-1}AR - R^{-1}R'. \tag{4.2}$$

This means that one gets $Y' = BY$ from $Y' = AY$ by replacing Y by RY.

More generally, there is a notion of formal equivalence at any $s \in \mathbb{P}^1(\mathbb{C})$. More precisely, we denote by $\widehat{K_s}$ the field of formal Laurent series at $s \in \mathbb{P}^1(\mathbb{C})$. We say that the differential systems (4.1) are formally equivalent at s if there exists $R \in \mathrm{GL}_n(\widehat{K_s})$ such that equation (4.2) holds.

4.2. **Rational equivalence.** The differential systems (4.1) are called rationally equivalent if there exists $R \in \mathrm{GL}_n(\mathbb{C}(z))$ such that equation (4.2) holds.

Of course, "rationally equivalent" implies "formally equivalent at any $s \in \mathbb{P}^1(\mathbb{C})$", but the converse is not true in general. This is where rigidity comes into play.

4.3. **Rigidity.** We say that a given differential system

$$Y' = AY \text{ with } A \in M_n(\mathbb{C}(z)) \tag{4.3}$$

is rigid if, for any differential system $Y' = BY$ with $B \in M_n(\mathbb{C}(z))$, the fact that $Y' = AY$ is *formally equivalent* to $Y' = BY$ at each $s \in \mathbb{P}^1(\mathbb{C})$ implies that $Y' = AY$ and $Y' = BY$ are *rationally equivalent*.

If $Y' = AY$ is irreducible over $\mathbb{C}(z)$, there is a numerical rigidity criterion, which can be stated as follows. We denote by $\otimes$ the Kronecker tensor product on $M_n(\mathbb{C})$, *i.e.*, for $C, D \in M_n(\mathbb{C}(z))$, the tensor product $C \otimes D \in M_{n^2}(\mathbb{C})$ is defined by

$$C \otimes D = \begin{pmatrix} c_{1,1}D & \cdots & c_{1,n}D \\ \vdots & \ddots & \vdots \\ c_{n,1}D & \cdots & c_{n,n}D \end{pmatrix}.$$

We define the "internal End" of $Y' = AY$ as the differential system $Y' = \mathcal{E}(A)Y$ with

$$\mathcal{E}(A) = A \otimes I_n - I_n \otimes A^t \in M_{n^2}(\mathbb{C}).$$

We set

$$\mathrm{rig}(A) = 2n^2 - \sum_{s \in \mathbb{P}^1(\mathbb{C})} \big(\mathrm{irr}(\mathcal{E}(A), s) + n^2 - \dim_{\mathbb{C}} \mathrm{sol}(\mathcal{E}(A), s)\big)$$

where $\mathrm{irr}(\mathcal{E}(A), s)$ is Malgrange irregularity of $Y' = \mathcal{E}(A)Y$ at s, and where $\mathrm{sol}(\mathcal{E}(A), s)$ is the $\mathbb{C}$-vector space of the solutions in $M_{n,1}(\widehat{K_s})$ of $Y' = \mathcal{E}(A)Y$. We recall that the Malgrange irregularity $\mathrm{irr}(B, s)$ at s of a given differential system $Y' = BY$, with $B \in M_n(\mathbb{C}(z))$, is the height of its Newton polygon. Equivalently, it is equal to the sum of the slopes (counted with multiplicities) of the Newton polygon at s of $Y' = BY$.

Theorem 3 ([3, Proposition 3.4], [5, Theorems 4.7 and 4.10])**.** *Assume that $Y' = AY$ is irreducible. Then, it is rigid if and only if* $\mathrm{rig}(A) = 2$.

We will also use the following inequality.

Theorem 4 ([3, Remark following Proposition 3.4])**.** *Assume that $Y' = AY$ is irreducible. Then, we have* $\mathrm{rig}(A) \leq 2$.

5. Proof of Theorem 2

We are now in position to prove our main result. For simplicity, we split the proof into two steps.

5.1. **First step.** We first prove the following result.

Theorem 5. *Let us consider a differential system $Y' = AY$ with $A \in M_2(\mathbb{C}(z))$. We assume that this differential system is irreducible and that the following properties are satisfied :*

1. *$Y' = AY$ has at most apparent singularities on $\mathbb{C}^\times$;*
2. *$Y' = AY$ has at most a regular singularity at 0;*
3. *the slopes at ∞ of $Y' = AY$ are included in $\{0, 1\}$.*

Then, the differential system $Y' = AY$ is rationally equivalent to $Y' = A_{\alpha;\beta,\gamma;\lambda,\mu}Y$ for some $\alpha, \beta, \gamma, \lambda, \mu \in \mathbb{C}$ with $\lambda \neq 0$.

Proof. Turittin's theorem yields the following facts :

1. $Y' = AY$ has a basis of solutions at 0 of the form $F_0(z)z^{\Gamma_0}$ where $F_0(z) \in \mathrm{GL}_2(\mathbb{C}((z)))$ and $\Gamma_0 \in M_2(\mathbb{C})$ is upper-triangular;
2. $Y' = AY$ admits a basis of formal solutions at ∞ of the form $F_\infty(z)z^{\Gamma_\infty}e^{\Delta z}$ where $F_\infty(z) \in \mathrm{GL}_2(\mathbb{C}((z^{-1})))$, $\Gamma_\infty \in M_2(\mathbb{C})$ is upper-triangular, $\Delta = \mathrm{diag}(\theta_1, \theta_2) \in M_2(\mathbb{C})$ is diagonal, and Γ_∞ and Δ commute.

Hence, we have:

- at 0, the differential system $Y' = AY$ is formally equivalent to $Y' = \frac{\Gamma_0}{z}Y$;
- at ∞, the differential system $Y' = AY$ is formally equivalent to $Y' = \left(\frac{\Gamma_\infty}{z} + \Delta\right)Y$.

Therefore, setting $B=\mathcal{E}(A)$, we see that

- at 0, the differential system $Y'=BY$ is formally equivalent to $Y'=B_0Y$ with
$$B_0=\frac{\Gamma_0\otimes I_2-I_2\otimes\Gamma_0^t}{z}.$$
- at ∞, the differential system $Y'=BY$ is formally equivalent to $Y'=B_\infty Y$ with
$$B_\infty=\left(\frac{\Gamma_\infty}{z}+\Delta\right)\otimes I_2-I_2\otimes\left(\frac{\Gamma_\infty}{z}+\Delta\right)^t.$$

Note that $\theta_1\neq\theta_2$. Indeed, assume at the contrary that $\theta_1=\theta_2$. Then, the differential system $Y'=AY$ is rationally equivalent to $Y'=\left(\frac{\Gamma_0}{z}+\theta_1I_2\right)Y$ (because the differential system satisfied by $F_0(z)z^{\Gamma_0}e^{-\theta_1z}$ is regular singular on $\mathbb{P}^1(\mathbb{C})$, with at most apparent singularities on $\mathbb{C}^\times$ and, hence, is of the form $R(z)z^{\Gamma_0}$ for some $R(z)\in\mathrm{GL}_2(\mathbb{C}(z))$). Therefore, the differential system $Y'=AY$ is reducible. This is a contradiction.

Since Δ and Γ_∞ commute, the fact that $\theta_1\neq\theta_2$ implies that Γ_∞ is diagonal :
$$\Gamma_\infty=\begin{pmatrix}\gamma_{\infty,1}&0\\0&\gamma_{\infty,2}\end{pmatrix}.$$

We now split our study in several cases, but the idea of the proof will be the same in any cases : we will prove that the differential system $Y'=AY$ is rigid and formally equivalent to $Y'=A_{\alpha;\beta,\gamma;\lambda,\mu}Y$ for some $\alpha,\beta,\gamma,\lambda,\mu\in\mathbb{C}$, with $\lambda\neq0$, at any $s\in\mathbb{P}^1(\mathbb{C})$. We will conclude that these systems are actually rationally equivalent by rigidity.

The case Γ_0 diagonal non resonant. We assume that Γ_0 is a diagonal matrix
$$\Gamma_0=\begin{pmatrix}\gamma_{0,1}&0\\0&\gamma_{0,2}\end{pmatrix}$$
such that $\gamma_{0,2}-\gamma_{0,1}\notin\mathbb{Z}$. It follows that
$$B_0=\begin{pmatrix}0&0&0&0\\0&\frac{\gamma_{0,1}-\gamma_{0,2}}{z}&0&0\\0&0&\frac{\gamma_{0,2}-\gamma_{0,1}}{z}&0\\0&0&0&0\end{pmatrix}$$
and
$$B_\infty=\begin{pmatrix}0&0&0&0\\0&\frac{\gamma_{\infty,1}-\gamma_{\infty,2}}{z}+\theta_1-\theta_2&0&0\\0&0&-(\frac{\gamma_{\infty,1}-\gamma_{\infty,2}}{z}+\theta_1-\theta_2)&0\\0&0&0&0\end{pmatrix}.$$
Then, we have
$$\mathrm{irr}(B,0)=\mathrm{irr}(B_0,0)=0\text{ and }\dim_{\mathbb{C}}\mathrm{sol}(B,0)=\dim_{\mathbb{C}}\mathrm{sol}(B_0,0)=2.$$
Moreover, we have
$$\mathrm{irr}(B,\infty)=\mathrm{irr}(B_\infty,\infty)=2\text{ and }\dim_{\mathbb{C}}\mathrm{sol}(B,\infty)=\dim_{\mathbb{C}}\mathrm{sol}(B_\infty,\infty)=2.$$

So, we get

$$\mathrm{rig}(A) = 2 \cdot 4 - (0 + 4 - 2) - (2 + 4 - 2) = 2.$$

Therefore, $Y' = AY$ is rigid in virtue of Theorem 3.

We now consider $\alpha, \beta, \gamma, \lambda, \mu \in \mathbb{C}$ such that

$$\begin{cases} 1-\beta & \equiv & \gamma_{0,1} \mod \mathbb{Z} \\ 1-\gamma & \equiv & \gamma_{0,2} \mod \mathbb{Z} \\ \alpha-\beta-\gamma+1 & \equiv & \gamma_{\infty,1} \mod \mathbb{Z} \\ -\alpha & \equiv & \gamma_{\infty,2} \mod \mathbb{Z} \\ \lambda+\mu & = & \theta_1 \\ \mu & = & \theta_2 \end{cases}.$$

We can indeed solve this system of equations because, in virtue of Proposition 1, we have $\gamma_{0,1} + \gamma_{0,2} - \gamma_{\infty,1} - \gamma_{\infty,2} \equiv 0 \mod \mathbb{Z}$. Note that $\lambda \neq 0$ because $\theta_1 \neq \theta_2$. Note also that $\beta - \gamma \equiv \gamma_{0,2} - \gamma_{0,1} \not\equiv 0 \mod \mathbb{Z}$. Using Section 2, we see that $Y' = AY$ is formally equivalent at 0 and ∞ to $Y' = A_{\alpha;\beta,\gamma;\lambda,\mu} Y$. Therefore, by rigidity, these two systems are rationally equivalent.

The case Γ_0 diagonal and resonant is impossible. We assume that Γ_0 is a diagonal matrix

$$\Gamma_0 = \begin{pmatrix} \gamma_{0,1} & 0 \\ 0 & \gamma_{0,2} \end{pmatrix}$$

such that $\gamma_{0,2} - \gamma_{0,2} \in \mathbb{Z}$. Then, the equality $\mathrm{rig}(A) \leq 2$ (see Theorem 4 above) reads as follows :

$$8 - (0 + 4 - 4) - (2 + 4 - \dim_{\mathbb{C}} \mathrm{sol}(B, \infty)) \leq 2$$

i.e. $\dim_{\mathbb{C}} \mathrm{sol}(B, \infty) \leq 0$. This is a contradiction because $\dim_{\mathbb{C}} \mathrm{sol}(B, \infty) \geq 2$.

The case Γ_0 non diagonal. Up to conjugation, we can assume that

$$\Gamma_0 = \begin{pmatrix} \gamma_{0,1} & 1 \\ 0 & \gamma_{0,1} \end{pmatrix}.$$

Then, we have

$$B_0 = \begin{pmatrix} 0 & 0 & 1/z & 0 \\ -1/z & 0 & 0 & 1/z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1/z & 0 \end{pmatrix},$$

whose Jordan normal form is given by

$$\begin{pmatrix} 0 & 1/z & 0 & 0 \\ 0 & 0 & 1/z & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Moreover, we have

$$B_\infty = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\gamma_{\infty,1}-\gamma_{\infty,2}}{z} + \theta_1 - \theta_2 & 0 & 0 \\ 0 & 0 & -(\frac{\gamma_{\infty,1}-\gamma_{\infty,2}}{z} + \theta_1 - \theta_2) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then, we have

$$\operatorname{irr}(B,0) = \operatorname{irr}(B_0,0) = 0 \text{ and } \dim_{\mathbb{C}} \operatorname{sol}(B,0) = \dim_{\mathbb{C}} \operatorname{sol}(B_0,0) = 2.$$

Moreover, we have

$$\operatorname{irr}(B,\infty) = \operatorname{irr}(B_\infty,\infty) = 2 \text{ and } \dim_{\mathbb{C}} \operatorname{sol}(B,\infty) = \dim_{\mathbb{C}} \operatorname{sol}(B_\infty,\infty) = 2.$$

So, we have

$$\operatorname{rig}(A) = 2 \cdot 4 - (0+4-2) - (2+4-2) = 2.$$

Therefore, $Y' = AY$ is rigid in virtue of Theorem 3.

We consider $\alpha, \beta, \gamma, \lambda, \mu \in \mathbb{C}$ such that

$$\begin{cases} 1-\beta & \equiv \gamma_{0,1} \mod \mathbb{Z} \\ 1-\gamma & \equiv \gamma_{0,1} \mod \mathbb{Z} \\ \alpha-\beta-\gamma+1 & \equiv \gamma_{\infty,1} \mod \mathbb{Z} \\ -\alpha & \equiv \gamma_{\infty,2} \mod \mathbb{Z} \\ \lambda+\mu & = \theta_1 \\ \mu & = \theta_2 \end{cases}.$$

We can indeed solve this system of equations because, in virtue of Proposition 1, we have $\gamma_{0,1} + \gamma_{0,1} - \gamma_{\infty,1} - \gamma_{\infty,2} \equiv 0 \mod \mathbb{Z}$. Note that $\lambda \neq 0$ because $\theta_1 \neq \theta_2$. Note also that $\beta - \gamma \in \mathbb{Z}$. If $\alpha - \beta \in \mathbb{Z}$, then the congruence $\alpha - \beta - \gamma + 1 \equiv \gamma_{\infty,1} \mod \mathbb{Z}$ implies that $\beta \equiv \gamma \equiv -\gamma_{\infty,1} \mod \mathbb{Z}$ but $\beta \equiv \gamma \equiv -\gamma_{0,1} \mod \mathbb{Z}$ so $\gamma_{0,1} \equiv \gamma_{\infty,1} \mod \mathbb{Z}$. Now, the congruence $\gamma_{0,1} + \gamma_{0,1} - \gamma_{\infty,1} - \gamma_{\infty,2} \equiv 0 \mod \mathbb{Z}$ implies that $\gamma_{\infty,2} \equiv \gamma_{\infty,1} \mod \mathbb{Z}$ hence $\alpha \equiv \beta \equiv \gamma \mod \mathbb{Z}$. Therefore, we can and will assume that $\alpha = \beta - 1 = \gamma - 1$. Using Section 2, we see that $Y' = AY$ is formally equivalent at 0 and ∞ to $Y' = A_{\alpha;\beta,\gamma;\lambda,\mu} Y$. By rigidity, these two systems are rationally equivalent. □

5.2. **Second step.** Let $f(z)$ be an E-function as in Theorem 2. By hypothesis, $f(z)$ satisfies a non-zero linear differential equation of order ≤ 2 with coefficients in $\overline{\mathbb{Q}}(z)$.

If $f(z)$ satisfies a non-zero differential equation of order 1 with coefficients in $\overline{\mathbb{Q}}(z)$, then it is well-known that $f(z) = a(z)e^{\alpha z}$ with $a(z) \in \overline{\mathbb{Q}}(z)$ and $\alpha \in \overline{\mathbb{Q}}$. Whence the result.

We shall now assume that $f(z)$ does not satisfy a differential equation of order 1 with coefficients in $\overline{\mathbb{Q}}(z)$. Then, according to [2, Theorem 4.3], the vector $F(z) = (f(z), f'(z))^t$ satisfies some linear differential system $F'(z) = A(z)F(z)$ for some $A(z) \in M_2(\overline{\mathbb{Q}}(z))$, with the following properties :

1. $Y' = AY$ has only apparent singularities on $\mathbb{C}^\times$;
2. $Y' = AY$ has a basis of solutions at 0 of the form $F_0(z)z^{\Gamma_0}$ where $F_0(z) \in \mathrm{GL}_2(\overline{\mathbb{Q}}((z)))$ and $\Gamma_0 \in M_2(\mathbb{Q})$ is upper-triangular;

(3) $Y' = AY$ admits a basis of formal solutions at ∞ of the form $F_\infty(z)z^{\Gamma_\infty}e^{\Delta z}$ where $F_\infty(z) \in \mathrm{GL}_2(\overline{\mathbb{Q}}((z)))$, $\Gamma_\infty \in M_2(\mathbb{Q})$ is upper-triangular, $\Delta \in M_2(\overline{\mathbb{Q}})$ is diagonal and such that $\Gamma_\infty \Delta = \Delta \Gamma_\infty$.

If $Y' = AY$ is reducible over $\mathbb{C}(z)$ or, equivalently, over $\overline{\mathbb{Q}}(z)$, then we are in the situation of Theorem 1, which is a consequence of [16, Theorem 4]. We observe here that the series ${}_1F_1(1;\gamma;\lambda z)$ is a solution of $L_{1;1,\gamma;\lambda,0}$.

We shall now assume that $Y' = AY$ is irreducible over $\mathbb{C}(z)$ or, equivalently, over $\overline{\mathbb{Q}}(z)$. Then, according to Theorem 5, there exists $R(z) \in \mathrm{GL}_2(\mathbb{C}(z))$ such that

$$A = R^{-1}A_{\alpha;\beta,\gamma;\lambda,\mu}R - R^{-1}R'.$$

But both β and γ are congruent to one of the eigenvalues of Γ_0 modulo $\mathbb{Z}$. Therefore, β and γ belong to $\mathbb{Q}$. A similar argument yields that $\alpha \in \mathbb{Q}$, $\lambda \in \overline{\mathbb{Q}}^\times$ and $\mu \in \overline{\mathbb{Q}}$. In particular, $A_{\alpha;\beta,\gamma;\lambda,\mu}$ has coefficients in $\overline{\mathbb{Q}}(z)$ and, hence, one can assume that $R(z)$ has coefficients in $\overline{\mathbb{Q}}(z)$.

We observe that RF is a vector solution of $Y' = A_{\alpha;\beta,\gamma;\lambda,\mu}Y$, and any such solutions are of the form $(h(z), h'(z))^t$ where $h(z)$ is a solution of $L_{\alpha;\beta,\gamma;\lambda,\mu}$. Hence, there exist $a(z), b(z) \in \overline{\mathbb{Q}}(z)$ such that

$$f(z) = a(z)h(z) + b(z)h'(z).$$

Since the entries of RF belong to $\overline{\mathbb{Q}}((z))$, we see that $h(z)$ also belongs to $\overline{\mathbb{Q}}((z))$. As noted at the beginning of Section 2.2, there exist a solution $g(z) \in \overline{\mathbb{Q}}((z))$ of $H_{\alpha;\beta,\gamma}$ such that

$$h(z) = g(\lambda z)e^{\mu z}.$$

By assumption, the differential system $Y' = AY$ is irreducible over $\overline{\mathbb{Q}}(z)$, so that the differential operators $L_{\alpha;\beta,\gamma;\lambda,\mu}$ and, hence, $H_{\alpha;\beta,\gamma}$ are irreducible over $\overline{\mathbb{Q}}(z)$. This implies that $\alpha - \beta \notin \mathbb{Z}$ and $\alpha - \gamma \notin \mathbb{Z}$. Using Section 2.1, we see that $H_{\alpha;\beta,\gamma}$ has a nonzero solution in $\overline{\mathbb{Q}}((z))$ if and only if $\beta \in \mathbb{Z}$ or $\gamma \in \mathbb{Z}$. Assume for instance that $\beta \in \mathbb{Z}$. Moreover, if $\beta - \gamma \in \mathbb{Z}$, we can assume that $\gamma \geq \beta$. Then, the $\overline{\mathbb{Q}}$-vector space of solutions of $H_{\alpha;\beta,\gamma}$ in $\overline{\mathbb{Q}}((z))$ is generated by

$$z^{1-\beta}\sum_{n=0}^{\infty}\frac{(\alpha-\beta+1)_n}{n!(\gamma-\beta+1)_n}z^n = z^{1-\beta}{}_1F_1(\alpha-\beta+1;\gamma-\beta+1;z).$$

Hence, up to a multiplicative constant in $\overline{\mathbb{Q}}^\times$, we have $g(z) = z^{1-\beta}{}_1F_1(\alpha-\beta+1;\gamma-\beta+1;z)$. This completes the proof of Theorem 2.

6. Some remarks on E-operators and G-operators

Finally, though this is not the subject of this paper, we mention that similar problems have been formulated for G-functions.

A G-function at $z = 0$ is a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[[z]]$ such that $\sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$ is an E-function. Both classes of functions have been first introduced by Siegel, and recently André [2] showed the deep relations that exist between E and G-functions. A non-zero minimal equation in $\overline{\mathbb{Q}}[z, \frac{d}{dz}]$ satisfied by a G-function is called a G-operator. From results of André, Chudnovky and Katz (see [2, 6, 7]), we know that a G-operator is Fuchsian with rational exponents at its singularities, and that all its solutions at any point $\alpha \in \overline{\mathbb{Q}} \cup \{\infty\}$ are essentially G-functions of the variable $z - \alpha$ or $1/z$ if $\alpha = \infty$. André [2] defined an E-operator as a differential operator in $\overline{\mathbb{Q}}[z, \frac{d}{dz}]$ such that its Fourier-Laplace transform is a G-operator. We recall that the Fourier-Laplace transform $\widehat{\mathcal{L}} \in \overline{\mathbb{Q}}[z, \frac{d}{dz}]$ of an operator $\mathcal{L} \in \overline{\mathbb{Q}}[z, \frac{d}{dz}]$ is the image of $\mathcal{L}$ by the automorphism of the Weyl algebra $\overline{\mathbb{Q}}[z, \frac{d}{dz}]$ defined by $z \mapsto -\frac{d}{dz}$ and $\frac{d}{dz} \mapsto z$. Any E-function is solution of an E-operator, which is not necessarily minimal for the degree in $\frac{d}{dz}$ but is minimal for the degree in z. André proved that the leading polynomial of an E-operator is z^m for some integer $m \geq 0$, i.e, that 0 is its only possible finite singularity. It follows that the minimal non-zero differential equation of a given E-function has only apparent finite non-zero singularities. That property was crucial for the results proved here.

Using the André-Chudnovky-Katz Theorem, it is easy to prove that G-functions of differential order 1 are algebraic functions over $\overline{\mathbb{Q}}(z)$ of the form $z^m \prod_{j=1}^{d}(z-\alpha_j)^{e_j}$, $m \in \mathbb{N}$, $\alpha_j \in \overline{\mathbb{Q}}^{\times}, e_j \in \mathbb{Z}$. Dwork conjectured that a globally nilpotent differential operator in $\overline{\mathbb{Q}}(z)[\frac{d}{dz}]$ of order 2 either has algebraic solutions or there exists an algebraic pullback to Gauss's hypergeometric equation with rational parameters. This conjecture was disproved by Krammer [9]. We shall not define the notion of "global nilpotence" here. Let us simply say that G-operators are conjectured to be exactly the globally nilpotent operators in $\overline{\mathbb{Q}}(z)[\frac{d}{dz}]$, and that G-operators coming from geometry are known to be globally nilpotent (Katz [7]). Hence, Krammer's result rules out the possibility to describe all G-functions of order 2 with algebraic functions and algebraic pullbacks of Gauss's hypergeometric series only. This is in clear contrast with Gorelov's results.

Bibliography

[1] M. Abramowitz, I. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 9th edition, 1970.

[2] Y. André, *Séries Gevrey de type arithmétique I. Théorèmes de pureté et de dualité*, Ann. of Math. **151** (2000), 705–740.

[3] D. Arinkin, *Rigid irregular connections on* $\mathbb{P}^1$, Compositio Math. **146** (2010), 1323–1338.

[4] F. Beukers, *A refined version of the Siegel-Shidlovskii theorem*, Annals of Math. **163**.1 (2006), 369–379.

[5] S. Bloch, H. Esnault, *Local Fourier Transforms and Rigidity*, Asian J. Math., Vol. 8, no. 4 (2004), 587–606.

[6] G. V. Chudnovsky, *On applications of diophantine approximations*, Proc. Natl. Acad. Sci. USA **81** (1984), 7261–7265.

[7] N. Katz, *Nilpotent connections and the monodromy theorem: applications of a result of Turritin*, Publ. Math. IHES, **32** (1970), 232–355.

[8] N. Katz, *Exponential sums and differential equations*, Annals of Mathematical Studies, Princeton 1990.

[9] D. Krammer, *An example of an arithmetic Fuchsian group*, J. reine angew. Math. **473** (1996), 69–85.

[10] S. Fischler, T. Rivoal, *Arithmetic theory of E-operators*, Journal de l'École polytechnique - Mathématiques **3** (2016), 31–65.

[11] V. A. Gorelov, *On the algebraic independence of values of E-functions at singular points and the Siegel conjecture*, Mat. Notes **67**.2 (2000), 174—190.

[12] V. A. Gorelov, *On the Siegel conjecture for second-order homogeneous linear differential equations*, Math. Notes **75**.4 (2004), 513–529.

[13] V. A. Gorelov, *On the structure of the set of E-functions satisfying linear differential equations of second order*, Math. Notes **78**.3 (2005), 304–319.

[14] V. A. Gorelov, *On the weakened Siegel conjecture*, Prikl. Mat. **11**.6 (2005), 33–39, in russian; translation in J. Math. Sci. (N.Y.) **146**.2 (2007), 5649–5654.

[15] É. Reyssat, *Irrationalité de $\zeta(3)$ selon Apéry*, Séminaire Delange-Pisot-Poitou, Théorie des nombres, **20** (1978-1979), Exposé No. 6, 6 p. Available at `http://archive.numdam.org`

[16] T. Rivoal, J. Roques, *On the algebraic dependence of E-functions*, Bull. London Math. Soc. **48**.2 (2016), 271–279.

[17] T. Rivoal, J. Roques, *Holomorphic solutions of E-operators*, Israel J. Math., 6 pages, to appear.

[18] C. L. Siegel, *Über einige Anwendungen Diophantischer Approximationen*, Abh. Preuss. Akad.Wiss., Phys.-Math. Kl. (1929-30), no. 1, 1–70.

[19] C. L. Siegel, *Transcendental Numbers*, Annals of Mathematical Studies, Princeton 1949.

[20] A. B. Shidlovsky, *Transcendental Numbers*, W. de Gruyter Studies in Mathematics **12**, 1989.

[21] M. van der Put, M. F. Singer, *Galois Theory of Linear Differential Equations*, Grundlehren der mathematischen Wissenschaften, Vol. 328, Springer, 2003.

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