ON THE DISTRIBUTION OF MULTIPLES OF REAL NUMBERS

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1. Introduction

We denote the integer part of a real number α by $\lfloor \alpha \rfloor$, its fractional part by $\{\alpha\}$ and the nearest integer to α by $\lfloor \alpha \rceil$, with the convention that it is $\lfloor \alpha \rfloor$ if $\alpha \in \mathbb{Z} + \frac{1}{2}$. The distance of α to $\mathbb{Z}$ is $||\alpha|| := \big|\alpha - \lfloor \alpha \rceil\big|$, whose value does not depend of the above convention.

The two main objects studied in this paper are the following sequences of continuous functions of α:

$$F_n(\alpha) := \sum_{k=1}^{n} \left| k\alpha - \frac{\lfloor kn\alpha \rceil}{n} \right| = \frac{1}{n} \sum_{k=1}^{n} ||nk\alpha||,$$

whose value is approximately $\frac{1}{4}$ (and the understanding of the word "approximately" is the aim of the present work), and its weighted average

$$G_{s,N}(\alpha) := \sum_{n=1}^{N} \frac{1}{n^s} F_n(\alpha). \tag{1.1}$$

A priori, α and s can be any real numbers but restriction will be made later.

In a certain sense, $F_n(\alpha)$ and $G_{s,N}(\alpha)$ are tools to measure how far the (multi)set $\big\{\{\alpha\}, \{2\alpha\}, \ldots, \{n\alpha\}\big\}$ is from being equal to a subset of $\big\{\frac{0}{n}, \frac{1}{n}, \ldots, \frac{n-1}{n}\big\}$. This is a problem related to uniform distribution of the sequence $(\{n\alpha\})_n$ and rational approximations of α. Before going into the core of the paper in Section 1.2, we set a few definitions and recall some basic facts.

1.1. **Lagrange constants and other diophantine statistics.** We will occasionaly use the notion of Lagrange constant $L(\alpha)$ of an irrational number α. It is defined as $L(\alpha) := \limsup_q \frac{1}{q||q\alpha||}$. The smallest value of $L(\alpha)$ is $\sqrt{5}$ and is achieved at $\frac{\sqrt{5}-1}{2}$ and numbers equivalent to it (in the sense of continued fractions, see below). The next smallest value $\sqrt{8}$ is achieved at $\sqrt{2}$ and numbers equivalent to it, etc. Furthermore, the lim sup defining $L(\alpha)$ is achieved along the subsequence $(q_n)_n$ of the denominators of the continued fraction of α. The set of values of $L(\alpha)$ forms the Lagrange spectrum; see [6] for a survey of its properties. Moreover, $L(\alpha) = +\infty$ if and only if, in the continued fraction $[a_0, a_1, a_2, \ldots]$ of α, the sequence of the partial quotients $(a_n)_n$ is unbounded; this is due to the inequality

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$\left|\alpha - \frac{p_n}{q_n}\right| \le \frac{1}{a_{n+1}q_n^2}$. In particular, $L(\alpha)$ is finite for α in a set of measure 0. The irrationality exponent $m(\alpha)$, another classical diophantine statistic defined in Section 1.3, is equal to 2 almost surely and therefore does not really distinguish irrational numbers. Baxa [2] showed that a statistic related to the discrepancy of $\{n\alpha\}$-sequences is minimal at $\sqrt{2}$, and another one (related to the discrepancy) is conjectured to be minimal at $\frac{\sqrt{5}-1}{2}$. In the context of irregularities of distribution of $\{n\alpha\}$-sequences, we can also mention the "dispersion constant" of Niederreiter [12] and the similar one of Jager-De Jong [7], both of which are minimal at $\frac{\sqrt{5}-1}{2}$ (and its equivalents). In this paper, we define in a natural way two functions $\mathscr{G}_s(\alpha)$ and $\Phi_s(\alpha)$ that also seem to be extremal at $\frac{\sqrt{5}-1}{2}$, at least for $s = 1$. This observation, which remains unproven, is the main motivation to the study undertaken here.

1.2. **Description of $F_n(\alpha)$ and $G_{s,N}(\alpha)$.** The behavior of $F_n(\alpha)$ strongly depends on the (ir)rationality of α and also on whether or not n is a denominator of a convergent of α. The function $G_{s,N}(\alpha)$ smoothens the dependence on n. In this respect, since $0 \le F_n(\alpha) \le \frac{1}{2}$, it is clear that the sequence $(G_{s,N}(\alpha))_{N\ge1}$ converges as $N \to +\infty$ for any real number α and any $s > 1$. We will no longer consider this case because it gives too much weight to the first values of $(F_n(\alpha))_{n\ge1}$ whereas we seek average results.

We will first study the sequence $(F_n(\alpha))_{n\ge1}$ whose behavior is not easy to understand. We will show in Theorem 4 in Section 2 that, in particular, $(F_n(\alpha))_{n\ge1}$ tends to be periodic when α is rational. When α is irrational, we will obtain lower and upper bounds for the lim inf and lim sup of $F_n(\alpha)$, in particular $\liminf_n F_n(\alpha) \le \big(2L(\alpha)\big)^{-1}$. We also observe that each function $F_n(\alpha)$ is 1-periodic in α and satisfies the equation $F_n(1-\alpha) = F_n(\alpha)$; these two properties are also inherited by $G_{s,N}(\alpha)$ and the limiting cases studied in the paper, the first one justifying that we limit ourselves to the case $\alpha \in [0, 1]$.

We will investigate in much more details the behavior of the sequence $(G_{s,N}(\alpha))_{N\ge1}$. We will focus on the case $s \le 1$ and in fact our results will be proved in the case $s \in (0, 1]$. See Theorem 2 (v), for the case $s \le 0$, which leads to results of a different nature that will not be investigated in depth. We set $H_N(s) := 1 + \frac{1}{2^s} + \cdots + \frac{1}{N^s}$ and denote $H_N(1)$ by H_N. When $0 \le s < 1$, $H_N(s) = \frac{N^{1-s}}{1-s} + \mathcal{O}(1)$, whereas $H_N = \log(N) + \mathcal{O}(1)$ as $N \to +\infty$.

We will show that, given $s \in (0, 1]$ and a rational number α, $\frac{1}{H_N(s)}\, G_{s,N}(\alpha)$ converges to a rational number $< \frac{1}{4}$ that depends on the denominator of α and not on s. On the other hand, this sequence converges to $\frac{1}{4}$ for almost all irrational numbers α, including for example the real irrational algebraic numbers, the numbers e and π – only conjecturally for the latter when $s < 1$. This does not seem to be the kind of result that helps to classify irrational numbers. However, our most striking results will concern the remainder

$$\mathscr{G}_{s,N}(\alpha) := G_{s,N}(\alpha) - \frac{1}{4}\, H_N(s) = \sum_{n=1}^{N} \frac{1}{n^s} \Big(F_n(\alpha) - \frac{1}{4}\Big),$$

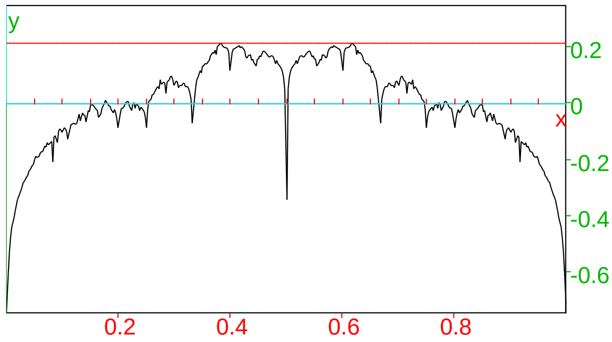

FIGURE 1. "Bear's pawprint" graph of $\mathscr{G}_{1,200}$ and the constant $\mathscr{G}_{1,200}\big(\frac{\sqrt{5}-1}{2}\big)$

when α is irrational. We will show that the sequence converges also almost surely as $N \to +\infty$ to a function $\mathscr{G}_s(\alpha)$ for ([1]) any given $s \in (0,1]$. (For the definition of the diophantine notions used from now on, see Section 1.3.) One of our results will be that the sequence $(\mathscr{G}_{1,N}(\alpha))_{N\geq 1}$ converges for all irrational numbers with finite $m(\alpha)$ exponent and diverges for all rational numbers, leaving mainly open the question of convergence or divergence of $(\mathscr{G}_{1,N}(\alpha))_{N\geq 1}$ for the rather sparse set of Liouville numbers. For $s \in (0,1)$, we will show that $(\mathscr{G}_{s,N}(\alpha))_{N\geq 1}$ diverges for all rational numbers and that it converges, resp. diverges, for all irrational numbers α with $m(\alpha) < 1 + \frac{1}{1-s}$, resp. $m(\alpha) > 2 + \frac{4}{1-s}$.

Figure 1 illustrates the case $s = 1$. It should be taken with precautions because our estimate for the speed of convergence of $(\mathscr{G}_{1,N}(\alpha))_{N\geq 1}$ to $\mathscr{G}_1(\alpha)$ is not uniform on $\mathbb{R}\setminus\mathbb{Q}$. Nonetheless, it is quite surprising to observe that $\mathscr{G}_{1,200}\big(\frac{\sqrt{5}-1}{2}\big) \approx 0.2169$ seems to be very close to the maximum of $\mathscr{G}_{1,200}$. We don't know if in the limit, $\mathscr{G}_1\big(\frac{\sqrt{5}-1}{2}\big)$ coincides with the maximum of $\mathscr{G}_1$; this would be a very interesting problem to solve.

1.3. **The results.** We denote by $(p_n/q_n)_{n\geq 0}$ the sequence of convergents of an irrational number α. Its partial quotients $(a_n)_{n\geq 0}$ are such that $q_{n+1} = a_{n+1}q_n + q_{n-1}$. Let us define a family $\mathscr{A}_s$, $s \in (0,1]$, of sets of irrational numbers $\alpha \in [0,1]$ such that, for $s \in (0,1)$,

$$\sum_m \frac{q_{m+1}^{1-s}}{q_m} < \infty$$

and, for $s = 1$,

$$\sum_m \frac{\log\big(\max(q_{m+1}/q_m, q_m)\big)}{q_m} < \infty.$$

We recall that an irrational number α is said to have *a* finite irrationality exponent $\mu(\alpha) \geq 2$ if there exists a constant $c(\alpha) > 0$ such that ([2]) $|\alpha - \frac{p}{q}| \geq \frac{1}{c(\alpha)q^{\mu(\alpha)}}$ for all integers p, q with

[1]Although this is not the point of view adopted in this paper, we can consider $\mathscr{G}_s(\alpha)$ as the Dirichlet series $\sum_{n=1}^{\infty} \frac{1}{n^s}\big(F_n(\alpha) - \frac{1}{4}\big)$ of the variable s, α being a parameter: our results will show the strong dependence of the abscissa of convergence on the diophantine properties of α, mainly $m(\alpha)$.

[2]When we will talk about an irrationality exponent for an irrational number α, one should understand the couple $(\mu(\alpha), c(\alpha))$.

$q \geq 1$. ($c(\alpha) \geq 1$ follows by putting $q = 1$ and $p = \lfloor \alpha \rfloor$.) We denote by $m(\alpha)$ *the* irrationality exponent of α, defined as the infinimum of all possible $\mu(\alpha)$, regardless of the value of $c(\alpha)$. By definition, Liouville numbers are precisely those irrational numbers which don't have a finite irrationality exponent; they are not only irrational but also transcendental.

We first state a lemma, whose proof is postponed to Section 3.

Lemma 1. *(i) The set $\mathscr{A}_1$ contains all irrational numbers with a finite irrationality exponent. Some Liouville numbers belong to $\mathscr{A}_1$, some do not.*

(ii) For any $s \in (0,1)$, the set $\mathscr{A}_s$ contains all irrational numbers with $m(\alpha) < 1 + \frac{1}{1-s}$ but no real number whose irrationality exponent $m(\alpha)$ is $> 1 + \frac{1}{1-s}$. In particular, it does not contain any Liouville number. Some irrational numbers with $m(\alpha) = 1 + \frac{1}{1-s}$ belong to $\mathscr{A}_s$, some do not.

(iii) The sets $\mathscr{A}_s$, $s \in (0,1]$, all have measure 1.

Set

$$\Phi_s(\alpha) := \sum_{n=1}^{\infty} \frac{1}{n^{s+1}} \sum_{m=1}^{n} \cos\big(2mn\pi\alpha\big) = \sum_{n=1}^{\infty} \frac{\cos\big(\pi n(n+1)\alpha\big)\sin\big(\pi n^2\alpha\big)}{n^{s+1}\sin\big(\pi n\alpha\big)} \tag{1.2}$$

and

$$\Phi_{s,N}(\alpha) := \sum_{n=1}^{N} \frac{1}{n^{s+1}} \sum_{m=1}^{n} \cos\big(2mn\pi\alpha\big) = \sum_{n=1}^{N} \frac{\cos\big(\pi n(n+1)\alpha\big)\sin\big(\pi n^2\alpha\big)}{n^{s+1}\sin\big(\pi n\alpha\big)} \tag{1.3}$$

the N-th partial sum. In (1.2), the second equality holds only for irrational numbers α and one has to use the definition in (1.2) if α is rational. In (1.3) the second equality holds for irrational numbers α as well as for some rational numbers; but if α is a rational number, we shall only use the first equality in (1.3).

We discard the case $s > 1$ because the series trivially converges for any α (as is clear from the first expression for $\Phi_s(\alpha)$). We consider $\Phi_s(\alpha)$ because of the relation

$$\mathscr{G}_{s,N}(\alpha) = -\frac{2}{\pi^2} \sum_{k=0}^{\infty} \frac{\Phi_{s,N}\big((2k+1)\alpha\big)}{(2k+1)^2}, \tag{1.4}$$

which will be proved later. Hence, $\Phi_{s,N}$ is a building block in the study of $\mathscr{G}_{s,N}$ and this explains why we study it in Theorem 1. Given some non-zero integers a and b, we denote the greatest common divisor of a and b by $\gcd(a,b)$.

We can now state our main results.

Theorem 1. *Let us fix $s \in (0,1]$.*

(i) For any rational number a/b with $\gcd(a,b) = 1$, $b \geq 1$, we have

$$\lim_{N\to+\infty} \frac{1}{H_N(s)}\,\Phi_{s,N}\Big(\frac{a}{b}\Big) = \frac{1}{b}.$$

In particular, $\lim_N \Phi_{s,N}(\frac{a}{b}) = +\infty$.

(ii) for $\alpha \in \mathscr{A}_s$, the series $\Phi_s(\alpha)$ converges absolutely.

(iii) The sequence $(\Phi_{s,N})_{N\geq 1}$ *converges to* Φ_s *almost surely and in* $L^2(0,1)$.

(iv) If $s\in(0,1)$ *and if the irrational number* α *has an irrationality exponent* $\mu(\alpha) > 2+\frac{4}{1-s}$*, then for any* $\varepsilon>0$,

$$\limsup_{N\to+\infty}\frac{\Phi_{s,N}(\alpha)}{H_N(s)^{1-\frac{3-s}{\mu(\alpha)(1-s)}-\varepsilon}}=+\infty.$$

This also holds if α *is a Liouville number when we put* $1/\mu(\alpha)=0$.

If $s=1$*, there exists a dense set of Liouville numbers* α *such that, for any* $\varepsilon>0$,

$$\limsup_{N\to+\infty}\frac{\Phi_{1,N}(\alpha)}{H_N^{1-\varepsilon}}=+\infty.$$

In all these cases, the sequence $(\Phi_{s,N}(\alpha))_{N\geq 1}$ *does not converge.*

(v) For any real numbers α *and* $s\leq 0$*, the sequence* $(\Phi_{s,N}(\alpha))_{N\geq 1}$ *does not converge.*

Remarks 1. a) For $s\in(0,1)$, the result in (iv) is probably not optimal and in particular one may expect the divergence of the series $\Phi_s(\alpha)$ for all α such that $m(\alpha)>1+\frac{1}{1-s}$. Nonetheless, for Liouville numbers (where one puts $1/\mu(\alpha)=0$), it is essentially best possible, even when $s=1$, because $|\Phi_{s,N}(\alpha)|\leq H_N(s)$ for any real numbers α and s.

b) From (iv) and (ii), we deduce that $(\Phi_{s,N}(\alpha))_{N\geq 1}$ converges if $s=1$ for many more numbers that in the case $s\in(0,1)$, thereby including all Liouville except a few ones. The same dichotomy can also be obtained from statements (ii) and (iv) of Theorem 2 below for the functions $\mathscr{G}_s$.

c) We will use the identity

$$\int_0^1\Phi_s(\alpha)^2\,\mathrm{d}\alpha=\frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\gcd(m,n)}{(mn)^{s+1}},$$

which will be a consequence of the proof of (iii) where, in particular, the convergence of the double series will be proved.

The results of Theorem 1, as well as the methods of proof, will be useful to understand the behavior of $G_{s,N}(\alpha)$, which we now describe. We recall that $\mathscr{G}_{s,N}(\alpha)=G_{s,N}(\alpha)-\frac{1}{4}H_N(s)$. Here and in the sequel, $v_p(n)$ denotes the p-adic valuation of a positive integer n.

Theorem 2. *(i) Let us fix* $s\in(0,1]$. *For any rational number* a/b *with* $\gcd(a,b)=1$, $b\geq 1$, *we have*

$$\begin{aligned}\lim_{N\to+\infty}\frac{1}{H_N(s)}G_{s,N}\Big(\frac{a}{b}\Big)&=\frac{1}{4}-\frac{2}{\pi^2}\sum_{k=0}^{\infty}\frac{\gcd(b,2k+1)}{b(2k+1)^2}\\&=\frac{1}{4}-\frac{1}{4b}\prod_{\substack{p\geq 3\\p\mid b}}\left(\frac{(p^{v_p(b)+1}-1)(p+1)+p^{v_p(b)}}{p^{v_p(b)+2}}\right).\end{aligned}$$

This limit is a rational number $<\frac{1}{4}$. *In particular,* $\lim_N\mathscr{G}_{s,N}(\frac{a}{b})=-\infty$.

(ii) For any $s \in (0,1)$ and any $\alpha \in \mathscr{A}_s$ with $m(\alpha) < 1 + \frac{1}{1-s}$, or for $s = 1$ and any α with finite $m(\alpha)$, we have

$$\lim_{N\to+\infty} \mathscr{G}_{s,N}(\alpha) = -\frac{2}{\pi^2}\sum_{k=0}^{\infty} \frac{\Phi_s\big((2k+1)\alpha\big)}{(2k+1)^2} =: \mathscr{G}_s(\alpha),$$

where the series $\mathscr{G}_s(\alpha)$ converges absolutely.

(iii) The sequence $(\mathscr{G}_{s,N})_{N\geq 1}$ converges to $\mathscr{G}_s$ almost surely and in $L^2(0,1)$.

(iv) For any $s \in (0,1)$, any $\varepsilon > 0$ and any irrational number α with an irrationality exponent $\mu(\alpha) > 2 + \frac{4}{1-s}$ (in particular, if it is a Liouville number), then,

$$\liminf_{N\to+\infty} \frac{\mathscr{G}_{s,N}(\alpha)}{H_N(s)^{1-\frac{3-s}{\mu(\alpha)(1-s)}-\varepsilon}} = -\infty.$$

This also holds if α is a Liouville number by setting $1/\mu(\alpha) = 0$.

For $s = 1$, there exists a dense set of Liouville numbers such that, for any $\varepsilon > 0$,

$$\liminf_{N\to+\infty} \frac{\mathscr{G}_{1,N}(\alpha)}{H_N^{1-\varepsilon}} = -\infty.$$

In all these cases, the sequence $(\mathscr{G}_{s,N}(\alpha))_{N\geq 1}$ does not converge.

(v) For any real numbers α and $s \leq 0$, the sequence $(\mathscr{G}_{s,N}(\alpha))_{N\geq 1}$ does not converge.

Remark 2. It is useful to have in mind the trivial bound $|\mathscr{G}_{s,N}(\alpha)| \leq \frac{1}{4}\, H_N(s)$, which holds for any real numbers α and s.

We will also prove the following theorem, which is of independent interest. It provides examples of Fourier series that converge almost everywhere but at no rational point. For $s \in (0,1]$, we define $\mathscr{B}_s$ as the set of irrational numbers in $[0,1]$ such that $\sum_n q_{n+1}/q_n^{s+1}$ is convergent. This set is of measure 1 and contains all the irrational numbers with irrationality exponent $< s+2$ and no numbers with irrationality exponent $> s+2$; we always have $\mathscr{B}_s \subseteq \mathscr{A}_s$.

Theorem 3. *Let us fix $s \in (0,1]$.*

(i) The Fourier series of Φ_s is given by

$$S(\Phi_s)(\alpha) := \sum_{k=1}^{\infty}\Bigg(\sum_{n|k,\, n\geq\sqrt{k}} \frac{1}{n^{s+1}}\Bigg)\cos(2\pi k\alpha).$$

The series $S(\Phi_s)(\alpha)$ converges almost surely. More precisely, it is equal to $\Phi_s(\alpha)$ for all $\alpha \in \mathscr{B}_s$. It also converges to Φ_s in $L^2(0,1)$.

(ii) The series $S(\Phi_s)$ converges for no rational number. More precisely, let $S_{s,N}$ denote the N-th partial sum of $S(\Phi_s)$. Then, for any rational number a/b with $\gcd(a,b) = 1$, $b \geq 1$, we have

$$\lim_{N\to+\infty} \frac{1}{H_{\lfloor\sqrt{N}\rfloor}(s)}\, S_{s,N}\Big(\frac{a}{b}\Big) = \frac{2}{b(1+s)}.$$

(iii) There exists a dense set of Liouville numbers on which $S(\Phi_s)$ does not converge.

The initial impulse to prove these results is given by the Fourier series of the function $||\alpha||$, which converges normally on $\mathbb{R}$:

$$||\alpha|| = \frac{1}{4} - \frac{2}{\pi^2}\sum_{k=0}^{\infty} \frac{\cos(2(2k+1)\pi\alpha)}{(2k+1)^2}. \tag{1.5}$$

Its form already "explains" (1.4).

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2. Properties of $(F_n(\alpha))_{n\geq 1}$

In this section, we prove a few results concerning the sequence $(F_n(\alpha))_{n\geq 1}$. A more exhaustive study would be desireable.

Theorem 4. *(i) The sequence $(F_n)_{n\geq 1}$ converges to $\frac{1}{4}$ in $L^2(0,1)$.*

(ii) For any rational number a/b with $\gcd(a,b)=1$, we have

$$F_n\Big(\frac{a}{b}\Big) = \frac{1}{4} - \frac{2}{\pi^2}\sum_{\substack{\ell=0\\ b|n(2\ell+1)}}^{\infty} \frac{1}{(2\ell+1)^2} + \mathcal{O}\Big(\frac{b}{n}\Big),$$

where the constant is absolute. In particular, $\liminf_n F_n(\frac{a}{b}) = 0$ and $\limsup_n F_n(\frac{a}{b}) \leq \frac{1}{4}$.

(iii) For any irrational number α and any $\varepsilon > 0$, there exist infinitely many n such that $q_n||q_n\alpha|| \leq \frac{1}{L(\alpha)-\varepsilon}$ and simultaneously $F_{q_n}(\alpha) = \frac{q_n+1}{2}\,||q_n\alpha||$.

(iv) For any irrational number α, we have

$$\liminf_{n\to+\infty} F_n(\alpha) \leq \frac{1}{2L(\alpha)} \quad \text{and} \quad \limsup_{n\to+\infty} F_n(\alpha) \geq \frac{1}{4},$$

the latter only if $m(\alpha)$ is finite.

Remarks 3. a) Since the sequence $n \mapsto \sum_{\substack{\ell=0\\ b|n(2\ell+1)}}^{\infty} \frac{1}{(2\ell+1)^2}$ is positive and periodic of period b, we see that $F_n(a/b)$ oscillates nearly periodically without converging. The $\limsup_n F_n(a/b)$ can be $< \frac{1}{4}$ but it can also be equal to $\frac{1}{4}$ (consider $F_{2m+1}(a/b)$ when b is even, for example).

b) For irrational numbers α, we always have $\liminf_n F_n(\alpha) \leq \frac{1}{2\sqrt{5}} < \frac{1}{4}$ and the $\liminf$ is 0 when $L(\alpha) = +\infty$, i.e. for almost all real numbers. Concerning the $\limsup$, numerical experiments suggest that $\limsup_n F_n(\alpha) = \lim_n F_{2q_n}(\alpha) > \frac{1}{4}$ if α is equivalent to $\frac{\sqrt{5}+1}{2}$.

Proof of Theorem 4. We will use various properties of continued fractions in the the sequel. The reader is refered to Kintchine's classical book [8] on this subject.

(i) Using the Fourier expansion (1.5) of $||\alpha||$, we find that, for any real number α,

$$\begin{aligned} F_n(\alpha) - \frac{1}{4} &= \frac{1}{n}\sum_{k=1}^{n}\Big(||kn\alpha|| - \frac{1}{4}\Big) = -\frac{2}{\pi^2 n}\sum_{k=1}^{n}\sum_{\ell=0}^{\infty}\frac{\cos\big(2(2\ell+1)kn\pi\alpha\big)}{(2\ell+1)^2} \\ &= -\frac{2}{\pi^2 n}\sum_{\ell=0}^{\infty}\frac{1}{(2\ell+1)^2}\sum_{k=1}^{n}\cos\big(2(2\ell+1)kn\pi\alpha\big) \\ &= -\frac{2}{\pi^2 n}\sum_{\ell=0}^{\infty}\frac{1}{(2\ell+1)^2}\frac{\cos\big((2\ell+1)n(n+1)\pi\alpha\big)\sin\big((2\ell+1)n^2\pi\alpha\big)}{\sin\big((2\ell+1)n\pi\alpha\big)}, \end{aligned}$$

with standard conventions when $\sin\big((2\ell+1)n\pi\alpha\big) = 0$. Set $u_n(\alpha) := \sum_{k=1}^{n}\cos\big(2kn\pi\alpha\big)$. We claim that $||u_n(\alpha)||_2 = \sqrt{n/2}$. Indeed,

$$\int_0^1 u_n(\alpha)^2\,\mathrm{d}\alpha = \sum_{k=1}^{n}\sum_{\ell=1}^{n}\int_0^1\cos\big(2kn\pi\alpha\big)\cos\big(2\ell n\pi\alpha\big)\,\mathrm{d}\alpha = \frac{1}{2}\sum_{1\le k=\ell\le n}1 = \frac{n}{2}.$$

Moreover by 1-periodicity of $u_n(\alpha)$, we also have $||u_n((2\ell+1)\alpha)||_2 = \sqrt{n/2}$ for any integer $\ell \ge 0$. Hence,

$$\begin{aligned} \Big|\Big|F_n(\alpha) - \frac{1}{4}\Big|\Big|_2 &\le \frac{2}{\pi^2 n}\sum_{\ell=0}^{\infty}\frac{1}{(2\ell+1)^2}||u_n((2\ell+1)\alpha)||_2 \\ &\le \frac{2||u_n(\alpha)||_2}{\pi^2 n}\sum_{\ell=0}^{\infty}\frac{1}{(2\ell+1)^2} \le \frac{1}{4n}\,||u_n(\alpha)||_2 \ll \frac{1}{\sqrt{n}}. \end{aligned}$$

It follows that $(F_n)_{n\ge1}$ converges to $\frac{1}{4}$ in $L^2(0,1)$.

(ii) Let us fix a rational number a/b, with $\gcd(a,b) = 1$ and $b \ge 1$. We start again with the Fourier series (1.5):

$$F_n\Big(\frac{a}{b}\Big) - \frac{1}{4} = -\frac{2}{\pi^2}\sum_{\ell=0}^{\infty}\frac{1}{(2\ell+1)^2}\frac{1}{n}\sum_{k=1}^{n}\cos\Big(2\pi(2\ell+1)kn\frac{a}{b}\Big).$$

In order to use the periodicity of cos, we write $k = rb + j$ with $1 \le j \le b$ and $r \ge 0$, so that

$$\begin{aligned} &\sum_{k=1}^{n}\cos\Big(2(2\ell+1)\pi kn\frac{a}{b}\Big) \\ &\qquad = \sum_{j=1}^{b}\sum_{\substack{r\ge0\\ rb+j\le n}}\cos\Big(2\pi(2\ell+1)(rb+j)n\frac{a}{b}\Big) = \sum_{j=1}^{b}\cos\Big(2\pi(2\ell+1)jn\frac{a}{b}\Big)\sum_{\substack{r\ge0\\ rb+j\le n}}1 \\ &\qquad = \sum_{j=1}^{b}\cos\Big(2\pi(2\ell+1)jn\frac{a}{b}\Big)\Big\lfloor\frac{n-j}{b}+1\Big\rfloor = \frac{n}{b}\sum_{j=1}^{b}\cos\Big(2\pi(2\ell+1)jn\frac{a}{b}\Big) + \mathcal{O}(b), \end{aligned}$$

where the constant in the $\mathcal{O}$ is absolute.

Hence,

$$F_n\left(\frac{a}{b}\right)-\frac{1}{4}=-\frac{2}{\pi^2}\sum_{\ell=0}^{\infty}\frac{1}{(2\ell+1)^2}\frac{1}{b}\sum_{j=1}^{b}\cos\left(2\pi(2\ell+1)jn\frac{a}{b}\right)+\mathcal{O}\left(\frac{b}{n}\right)$$
$$=-\frac{2}{\pi^2}\sum_{\substack{\ell=0\\ b\mid n(2\ell+1)}}^{\infty}\frac{1}{(2\ell+1)^2}+\mathcal{O}\left(\frac{b}{n}\right),$$

where we have used the fact that, for any integer k and any rational u/v, with $\gcd(u,v)=1$ and $v\geq 1$,

$$\sum_{j=1}^{v}\cos\left(2\pi\frac{jku}{v}\right)=\begin{cases}0 & \text{if} \quad v\nmid k\\ v & \text{if} \quad v\mid k.\end{cases} \tag{2.1}$$

The estimates for the lim inf and lim sup follow from the two obvious facts: $F_{bn}(\frac{a}{b})=0$ and $F_n(\frac{a}{b})\leq\frac{1}{4}+\mathcal{O}(\frac{b}{n})$ respectively.

(iii) By definition, $L(\alpha)=\limsup\limits_{q\to+\infty}\dfrac{1}{q||q\alpha||}$, hence for any ε, there exist infinitely many positive integers b_n (depending on α and ε) such that $b_n||b_n\alpha||\leq\frac{1}{L(\alpha)-\varepsilon}$. (Without loss of generality, we can even assume that $a_n:=\lfloor b_n\alpha\rceil$ and b_n are coprime.) Since $L(\alpha)\geq\sqrt{5}$ for any irrational number α, we can choose ε small enough such that $L(\alpha)>2+\varepsilon$ (and thus $b_n||b_n\alpha||<\frac{1}{2}$, which implies that a_n/b_n are convergents to α by a classical property of continued fractions).

Now, for any integer $k\in\{1,2,\ldots,b_n\}$, we have

$$|kb_n\alpha-ka_n|=k|b_n\alpha-a_n|\leq\frac{k}{b_n(L(\alpha)-\varepsilon)}\leq\frac{1}{L(\alpha)-\varepsilon}<\frac{1}{2}.$$

This forces that $ka_n=\lfloor kb_n\alpha\rceil$ and therefore

$$F_{b_n}(\alpha)=\frac{1}{b_n}\sum_{k=1}^{b_n}||b_nk\alpha||=\frac{1}{b_n}\sum_{k=1}^{b_n}|kb_n\alpha-ka_n|=\frac{||b_n\alpha||}{b_n}\sum_{k=1}^{b_n}k=\frac{b_n+1}{2}\,||b_n\alpha||,$$

as claimed.

(iv) For any $\varepsilon>0$ such that $L(\alpha)>2+\varepsilon$, we have

$$\liminf_{n\to+\infty}F_n(\alpha)\leq\liminf_{n\to+\infty}F_{b_n}(\alpha)=\liminf_{n\to+\infty}\frac{b_n+1}{2}\,||b_n\alpha||=\frac{1}{2}\liminf_{n\to+\infty}b_n||b_n\alpha||\leq\frac{1}{2(L(\alpha)-\varepsilon)}.$$

Since $\liminf\limits_{n\to+\infty}F_n(\alpha)$ does not depend on ε, we get that $\liminf_n F_n(\alpha)\leq\frac{1}{2L(\alpha)}$.

If we now assume that α has a finite irrationality exponent, then $\alpha\in\mathscr{A}_1$ and by Theorem 2, ([3]) the series $\sum_{n=1}^{\infty}\frac{\frac{1}{4}-F_n(\alpha)}{n}$ is convergent. Let us assume that $\limsup_n F_n(\alpha)<\frac{1}{4}$. Then there exists a constant $c>0$ such that $\frac{\frac{1}{4}-F_n(\alpha)}{n}\geq\frac{c}{n}$ for all n large enough and

[3]Statement (iv) of Theorem 4 will not be used in the proof of Theorem 2.

the above series cannot converge. This contradiction proves that $\limsup_n F_n(\alpha) \geq \frac{1}{4}$ and finishes the proof of Theorem 4. □

3. Proof of Lemma 1

(*i*) Let α be an irrational number with a finite irrationality exponent $\mu(\alpha) \geq 2$, so that $|\alpha - \frac{p}{q}| \geq \frac{1}{c(\alpha)q^{\mu(\alpha)}}$ for all integers p, q with $q \geq 1$. In particular, if $p/q = p_n/q_n$ is the n-th convergent of α, classical properties of continued fractions imply that

$$\frac{1}{c(\alpha)q_n^{\mu(\alpha)}} \leq \Big|\alpha - \frac{p_n}{q_n}\Big| \leq \frac{1}{q_n q_{n+1}}.$$

Thus, $q_{n+1} \leq c(\alpha)q_n^{\mu(\alpha)-1}$. Moreover, it is known that $q_n \geq (\frac{1+\sqrt{5}}{2})^{n-1}$ for all $n \geq 1$ and all irrational number α. It follows that both series

$$\sum_n \frac{\log\big(\max(q_{n+1}/q_n, q_n)\big)}{q_n} \ll \sum_n \frac{\log(q_n)}{q_n}$$

are convergent (at geometric rate). The real number β whose partial quotients are $a_n = 2^{(n-1)!^2}$ is a Liouville number and the inequalities $2^{n!^2}q_n \leq q_{n+1} \leq 2^{n!^2+1}q_n$ ensure that $\beta \in \mathscr{A}_1$. On the other hand, the number $\kappa = \sum_{n=1}^{\infty} 1/b_n$ with $b_{n+1} = 2^{b_n}$, $b_1 = 1$, is also a Liouville number but it has infinitely many convergents such that $q_{n+1} = 2^{q_n}$, so that $\kappa \notin \mathscr{A}_1$.

(*ii*) Similarly as above, we prove that

$$\sum_n \frac{q_{n+1}^{1-s}}{q_n} \ll \sum_n \frac{1}{q_n^{1-(1-s)(\mu(\alpha)-1)}}$$

and both series are convergent (at geometric rate) when $\mu(\alpha) < 1 + \frac{1}{1-s}$.

The convergence of the series $\sum_m q_{m+1}^{1-s}/q_m$ implies that $q_{m+1} = o\big(q_m^{\frac{1}{1-s}}\big)$ and this in turn implies that $\alpha \in \mathcal{A}_s$ implies $m(\alpha) \leq 1+\frac{1}{1-s}$, hence cannot be a Liouville number. Examples of continued fractions can be contructed that have exact irrationality exponent $1 + \frac{1}{1-s}$ for which the series $\sum_m q_{m+1}^{1-s}/q_m$ converges or not.

(*iii*) Almost all real numbers have $m(\alpha) = 2$ (see [8, page 69, Theorem 32]). Hence, for any $s \in (0, 1]$, almost all real numbers belong to $\mathscr{A}_s$.

4. Some diophantine estimates

In this section, we prove side results which we use in the proofs of Theorems 1 and 2. They are interesting in themselves and therefore we prove them separetely. We state them explicitely so that they can be used for numerical computations. But the value of the constants is not essential for the proofs of Theorem 1, 2 and 3, for we only need to know what they depend on. The sets $\mathscr{A}_s$ have been defined in the Introduction. The function ζ that appears below is the Riemann zeta function.

Proposition 1. *(i) Let us fix $s \in (0,1]$. For any irrational number $\alpha \in \mathscr{A}_s$, the series*

$$\sum_{n=1}^{\infty} \frac{||n^2\alpha||}{n^{s+1}||n\alpha||} \tag{4.1}$$

is convergent. In particular, it converges almost everywhere.

(ii) For any irrational number $\alpha \in \mathscr{A}_s$, we have the following estimate for the speed of convergence: for any integer $m \geq 2$,

$$\sum_{n=q_m}^{\infty} \frac{||n^2\alpha||}{n^{s+1}||n\alpha||} \leq \begin{cases} 2\zeta(s+1)\displaystyle\sum_{k=m}^{\infty} \frac{1+\log(q_k)}{q_k^s} + \frac{2}{1-s}\sum_{k=m}^{\infty} \frac{1+q_{k+1}^{1-s}}{q_k} & \text{if } 0<s<1 \\ 2\big(1+\zeta(2)\big)\displaystyle\sum_{k=m}^{\infty} \frac{1+\log\big(\max(q_{k+1}/q_k, q_k)\big)}{q_k} & \text{if } s=1. \end{cases} \tag{4.2}$$

(iii) If $\alpha \in \mathscr{A}_s$ (for some $s \in (0,1)$) and α has an irrationality exponent $\mu(\alpha) < 1 + \frac{1}{1-s}$, then for any $m \geq 2$ such that $q_m \geq 3$ and $\log(q_m) \leq q_m^{s/2}$, we have

$$\sum_{n=q_m}^{\infty} \frac{||n^2\alpha||}{n^{s+1}||n\alpha||} \leq 4\zeta(s+1)\cdot\frac{2^{s/4+1}-1}{2^{s/4}-1}\cdot\frac{\log(q_m)}{q_m^s} + \frac{9}{1-s}\cdot\frac{1}{q_m} + \frac{2c(\alpha)^{1-s}}{1-s}\cdot\frac{2\sqrt{2}^{1-(\mu(\alpha)-1)(1-s)}-1}{\sqrt{2}^{1-(\mu(\alpha)-1)(1-s)}-1}\cdot\frac{1}{q_m^{1-(\mu(\alpha)-1)(1-s)}}. \tag{4.3}$$

(iv) If $\alpha \in \mathscr{A}_1$ has a finite irrationality exponent $\mu(\alpha)$, then for any $m \geq 7$, we have

$$\sum_{n=q_m}^{\infty} \frac{||n^2\alpha||}{n^2||n\alpha||} \leq 2(1+\zeta(2))\cdot(3+\sqrt{2})\cdot\log(e\cdot c(\alpha))\cdot\frac{1}{q_m} + 2(1+\zeta(2))\cdot\frac{2^{5/4}-1}{2^{1/4}-1}\cdot(\mu(\alpha)-1)\cdot\frac{\log(q_m)}{q_m}. \tag{4.4}$$

Remarks 4. a) The series (4.1) also converges for all real number $\alpha \in [0,1]$ when $s > 1$, a result that follows immediately from Lemma 2. It is interesting to compare the upper bounds obtained in *(ii)* with the following ones, due to Kruse [9]: for any $s \geq 0$ and any irrational number α, we have

$$\sum_{n=q_m}^{q_\ell - 1} \frac{1}{n^{s+1}||n\alpha||} \ll \sum_{k=m}^{\ell-1} \frac{q_{k+1}}{q_k^{s+1}}. \tag{4.5}$$

The upper bound in (4.5) (which is optimal) displays the influence of our term $||n^2\alpha||$, in particular when $s = 1$.

b) In a preprint version of the paper [13], the equation analogous to (4.2) was incorrectly stated in the case $0 < s < 1$. The estimate analogous to (4.3) was thus wrong. This has no influence on the numerical computations done at the end the paper because they are only presented in the case $s = 1$: indeed, the estimate for $s = 1$ in (4.2) was the same in [13], and (4.4) is even an improvement on the analogous estimate in [13]. This improvement was suggested by the referee (see Lemma 4 below). However, the analogue of Proposition 1 of [13] was quoted and used in [14] for $s \in (0,1]$: there, one must use Proposition 1 above and, again, one can check that the numerical results in [14] are correct.

A few lemmas are necessary for the proof of Proposition 1.

Lemma 2. *For any real number $\alpha \in [0,1]$ and any integer $n \geq 0$, we have*

$$\frac{||n^2\alpha||}{||n\alpha||} \leq \frac{n}{2\lfloor n||n\alpha||\rfloor + 1}.$$

Proof. Let j be an integer such that $0 \leq j < n$. The function $D_n(\alpha) := \frac{||n\alpha||}{||\alpha||}$ is defined and continuous on $[0,1]$, with $D_n(0) = D_n(1) = n$. It is increasing on $[\frac{j}{n}, \frac{j+1/2}{n}]$ and decreasing on $[\frac{j+1/2}{n}, \frac{j+1}{n}]$.

Since $D_n(j/n) = D_n((j+1)/n) = 0$, we deduce that for all $\alpha \in [\frac{j}{n}, \frac{j+1}{n}[$, we have $0 \leq D_n(\alpha) \leq D_n\big(\frac{j+1/2}{n}\big)$ If $\alpha \leq 1/2$, $||\alpha|| = \alpha$ whereas if $\alpha > 1/2$, $||\alpha|| = 1 - \alpha$, whence

$$D_n\Big(\frac{j+1/2}{n}\Big) = \frac{n}{2\lfloor n||\alpha||\rfloor + 1}$$

for any $\alpha \in [\frac{j}{n}, \frac{j+1}{n}[$.

We note that the right-hand side of the previous formula does not explicitly use the variable j. Therefore, we have shown that for any $\alpha \in [0,1]$, we have

$$0 \leq D_n(\alpha) \leq \frac{n}{2\lfloor n||\alpha||\rfloor + 1}. \tag{4.6}$$

This formula enables to bound not only $D_n(\alpha)$ for $\alpha \in [0,1]$ but also for all $\alpha \in \mathbb{R}$ because $||\alpha||$ and $D_n(\alpha)$ are 1-periodic. Therefore, the upper bound (4.6) holds for any real number α and the lemma follows when replacing α by $n\alpha$. □

Lemma 3. *For any $\alpha \in [0,1]$ and any integer N such that $q_m \leq N < q_{m+1}$, with $m \geq 2$, we have*

$$\sum_{k=q_m}^{N} \frac{||k^2\alpha||}{k^{s+1}||k\alpha||} \leq \begin{cases} 2\big(1+\zeta(2)\big) \cdot \dfrac{1 + \log\big(\max(q_{m+1}/q_m, q_m)\big)}{q_m} & \text{if} \quad s = 1 \\ 2\zeta(s+1)\dfrac{1+\log(q_m)}{q_m^s} + \dfrac{2}{1-s} \cdot \dfrac{1 + q_{m+1}^{1-s}}{q_m} & \text{if} \quad 0 < s < 1. \end{cases}$$

Proof. By Lemma 2, it is enough to show that the same bound holds for the sum

$$R_N := \sum_{k=q_m}^{N} \frac{1}{k^s\big(2\lfloor k||k\alpha||\rfloor + 1\big)}.$$

Since

$$0 \leq \frac{1}{2\lfloor k||k\alpha||\rfloor + 1} \leq \frac{1}{k||k\alpha||}, \tag{4.7}$$

it is tempting to bound R_N by $\sum_{k=q_m}^N \frac{1}{k^{s+1}||k\alpha||}$ and then use Kruse's bound (4.5). But then we would lose the benefit of the inequalities

$$0 \leq \frac{1}{2\lfloor k||k\alpha||\rfloor + 1} \leq 1 \tag{4.8}$$

because the quantity $1/||k\alpha||$ can take on very large values. We will consider three cases: in the first one, we will use (4.7) only, while for the remaining two, (4.8) will be used.

To study R_N, we adapt Kruse's ideas and cut the sum R_N in three parts : $k \not\equiv 0, q_{m-1}[q_m]$, $k \equiv 0[q_m]$ and $k \equiv q_{m-1}[q_m]$. Remember that we suppose $q_m \leq N < q_{m+1}$, with $q_{m+1} = a_{m+1}q_m + q_{m-1}$. Set $Q = \lfloor N/q_m \rfloor$, $r_h = q_m - 1$ if $0 \leq h < Q$ and $r_h = N - Qq_m$ if $h = Q$. In particular, $0 \leq r_h \leq q_m - 1$ and $Q < q_{m+1}/q_m$. The assumption that $m \geq 2$ ensures that $q_m \geq 2$ (a necessary assumption for Kruse's estimates) because $q_m \geq q_2 \geq \sqrt{2} > 1$.

• *first step.* We use (4.7):

$$\begin{aligned}\sum_{\substack{k=q_m \\ k\not\equiv 0, q_{m-1}[q_m]}}^{N} \frac{1}{k^s}\frac{1}{2\lfloor k||k\alpha||\rfloor + 1} &= \sum_{h=1}^{Q} \sum_{\substack{j=1 \\ j\neq q_{m-1}}}^{r_h} \frac{1}{(hq_m+j)^s\big(2\lfloor (hq_m+j)||(hq_m+j)\alpha||\rfloor + 1\big)} \\ &\leq \sum_{h=1}^{Q} \sum_{\substack{j=1 \\ j\neq q_{m-1}}}^{r_h} \frac{1}{(hq_m+j)^{s+1}||(hq_m+j)\alpha||}.\end{aligned}$$

To deal with the sum over j, we again follow Kruse [9, pp. 240-241] to get (even when the sums are empty, in which case their values are 0):

$$\begin{aligned}\sum_{\substack{j=1 \\ j\neq q_{m-1}}}^{r_h} \frac{1}{(hq_m+j)^{s+1}||(hq_m+j)\alpha||} &\leq \frac{1}{(hq_m)^{s+1}} \sum_{\substack{j=1 \\ j\neq q_{m-1}}}^{r_h} \frac{1}{||(hq_m+j)\alpha||} \\ &\leq \frac{2}{(hq_m)^{s+1}} \sum_{k=1}^{r_h} \frac{1}{\frac{k}{q_m}} \leq \frac{2q_m(1+\log(r_h+1))}{(hq_m)^{s+1}} \leq 2\frac{1+\log(q_m)}{q_m^s h^{s+1}}.\end{aligned}$$

Hence, finally,

$$\begin{aligned}\sum_{\substack{k=q_m \\ k\not\equiv 0, q_{m-1}[q_m]}}^{N} \frac{1}{k^s}\frac{1}{2\lfloor k||k\alpha||\rfloor + 1} &= \sum_{h=1}^{Q} \sum_{\substack{j=1 \\ j\neq q_{m-1}}}^{r_h} \frac{1}{(hq_m+j)^s\big(2\lfloor (hq_m+j)||(hq_m+j)\alpha||\rfloor + 1\big)} \\ &\leq 2\frac{1+\log(q_m)}{q_m^s} \sum_{h=1}^{Q} \frac{1}{h^{s+1}} \leq 2\zeta(s+1)\frac{1+\log(q_m)}{q_m^s}.\end{aligned} \tag{4.9}$$

- *Second and third steps.* We now use the inequality (4.8) twice. We have

$$\sum_{\substack{k=q_m \\ k\equiv 0[q_m]}}^{N} \frac{1}{k^s}\frac{1}{2\lfloor k||k\alpha||\rfloor+1} = \sum_{h=1}^{Q} \frac{1}{(hq_m)^s}\frac{1}{2\lfloor hq_m||hq_m\alpha||\rfloor+1} \le \frac{1}{q_m^s}\sum_{h=1}^{Q}\frac{1}{h^s}$$

$$\le \begin{cases} \dfrac{1+\log(q_{m+1}/q_m)}{q_m} & \text{if} \quad s=1 \\ \dfrac{1}{1-s}\cdot\dfrac{1+q_{m+1}^{1-s}}{q_m} & \text{if} \quad 0<s<1. \end{cases} \tag{4.10}$$

Similarily, we have

$$\sum_{\substack{k=q_m \\ k\equiv q_{m-1}[q_m]}}^{N} \frac{1}{k^s}\frac{1}{2\lfloor k||k\alpha||\rfloor+1} \le \sum_{h=1}^{a_{m+1}-1} \frac{1}{(hq_m+q_{m-1})^s} \le \frac{1}{q_m^s}\sum_{h=1}^{a_{m+1}}\frac{1}{h^s}$$

$$\le \begin{cases} \dfrac{1+\log(q_{m+1}/q_m)}{q_m} & \text{if} \quad s=1 \\ \dfrac{1}{1-s}\cdot\dfrac{1+q_{m+1}^{1-s}}{q_m} & \text{if} \quad 0<s<1, \end{cases} \tag{4.11}$$

Adding (4.9), (4.10) and (4.11), whose sum is R_N, we immediately obtain the claimed inequalities. □

We need another lemma.

Lemma 4. *Fix an irrational number α. For any $\omega>0$ and any integer $m\ge 1$, we have*

$$\sum_{j=m}^{\infty}\frac{1}{q_j^\omega} \le \frac{2^{\omega/2+1}-1}{2^{\omega/2}-1}\cdot\frac{1}{q_m^\omega}. \tag{4.12}$$

For any $\omega>0$ and any integer $m\ge 1$ such that $q_m\ge 2$ and $\log(q_m)\le q_m^{\omega/2}$, we have

$$0\le\sum_{j=m}^{\infty}\frac{\log(q_m)}{q_j^\omega} \le \frac{2^{\omega/4+1}-1}{2^{\omega/4}-1}\cdot\frac{\log(q_m)}{q_m^\omega}. \tag{4.13}$$

If $\omega=1$, (4.13) holds for any $m\ge 7$.

Remark 5. The inequality (4.14) below and its proof were suggested by the referee, leading to the upper bounds stated in lemma, which are better than those previously obtained in [13].

Proof. The inequality $q_m\ge 2^{(m-1)/2}$ is well-known: the proof given in [8] can be modified to give

$$q_{m+k}\ge 2^{(k-1)/2}q_m, \quad m\ge 0,\ k\ge 1. \tag{4.14}$$

Indeed, we have $q_{k+2}\ge q_{k+1}+q_k\ge 2q_k$, so that $q_{m+2k}\ge 2^k q_m$ and $q_{m+2k+1}\ge 2^k q_{m+1}\ge 2^k q_m$, and (4.14) follows.

We have

$$\sum_{j=m}^{\infty}\frac{1}{q_j^\omega}=\frac{1}{q_m^\omega}\Big(1+\sum_{j=1}^{\infty}\frac{q_m^\omega}{q_{m+j}^\omega}\Big)\le\frac{1}{q_m^\omega}\Big(1+\sum_{j=1}^{\infty}\frac{1}{2^{(j-1)\omega/2}}\Big)=\frac{2^{\omega/2+1}-1}{2^{\omega/2}-1}\cdot\frac{1}{q_m^\omega}.$$

For $m\ge 2$, let us define $\varepsilon_m:=\frac{\log\log(q_m)}{\log q_m}$, that is by $\log(q_m)=q_m^{\varepsilon_m}$. For all $j\ge m$, $\log(q_j)\le q_j^{\varepsilon_m}$, hence (since the assumption ensures that $\varepsilon_m\le\omega/2$), we have

$$0\le\sum_{j=m}^{\infty}\frac{\log(q_j)}{q_j^\omega}\le\sum_{j=m}^{\infty}\frac{1}{q_j^{\omega-\varepsilon_m}}\le\frac{2^{(\omega-\varepsilon_m)/2+1}-1}{2^{(\omega-\varepsilon_m)/2}-1}\cdot\frac{1}{q_m^{\omega-\varepsilon_m}}\le\frac{2^{\omega/4+1}-1}{2^{\omega/4}-1}\cdot\frac{\log(q_m)}{q_m^\omega}.$$

If $\omega=1$, the inequality $\varepsilon_m\le 1/2$ holds for $m\ge 7$. Indeed, the function $\log\log(x)/\log(x)$ is decreasing for $x\ge 17$ and we have $q_m\ge((1+\sqrt{5})/2)^{m-1}\ge 17$ for all $m\ge 7$. □

Proof of Proposition 1. Let us assume that $s=1$ and fix $\alpha\in\mathscr{A}_1$. By Lemma 3, for any integers n,m such that $2\le n<m$, we have

$$\sum_{k=q_n}^{q_m-1}\frac{||k^2\alpha||}{k^2||k\alpha||}=\sum_{j=n}^{m-1}\sum_{k=q_j}^{q_{j+1}-1}\frac{||k^2\alpha||}{k^2||k\alpha||}\le 2\big(1+\zeta(2)\big)\sum_{j=n}^{m-1}\frac{1+\log\big(\max(q_{j+1}/q_j,q_j)\big)}{q_j}.$$

Since both series converge when $m\to+\infty$, we deduce that

$$\sum_{k=q_n}^{\infty}\frac{||k^2\alpha||}{k^2||k\alpha||}\le 2\big(1+\zeta(2)\big)\sum_{j=n}^{\infty}\frac{1+\log\big(\max(q_{j+1}/q_j,q_j)\big)}{q_j}.$$

This proves (i) and (ii) in the case $s=1$. The proof can immediately be adapted to the case $0<s<1$.

Let us prove (iii). Consider $\alpha\in\mathscr{A}_s$ with an irrationality exponent $\mu:=\mu(\alpha)<1+\frac{1}{1-s}$, with the associated constant $c(\alpha)\ge 1$. We have already shown that $q_{j+1}\le c(\alpha)q_j^{\mu-1}$. Hence, $q_{j+1}^{1-s}/q_j\le c(\alpha)^{1-s}/q_j^{1-(\mu-1)(1-s)}$. It also follows that

$$\sum_{j=m}^{\infty}\frac{q_{j+1}^{1-s}}{q_j}\le c(\alpha)^{1-s}\sum_{j=m}^{\infty}\frac{1}{q_j^{1-(\mu-1)(1-s)}}.$$

For $m\ge 2$, we have $q_m\ge 3$ and $\log(q_m)\le q_m^{s/2}$. We use (4.12) with $\omega=s$ and $\omega=1-(\mu-1)(1-s)$ and (4.13) with $\omega=1$ and $\omega=s$ on the right hand side of the inequality in (ii). This gives (4.3) after some simple computations.

It remains to prove (iv). If $\alpha\in\mathscr{A}_1$ has a finite irrationality exponent $\mu(\alpha)$, we have $q_{j+1}\le c(\alpha)q_j^{\mu-1}$. Hence $\log(q_{j+1})\le(\mu(\alpha)-1)\log(q_j)+\log c(\alpha)$. It follows that

$$\sum_{k=q_m}^{\infty}\frac{||k^2\alpha||}{k^2||k\alpha||}\le 2\big(1+\zeta(2)\big)\left((1+\log c(\alpha))\sum_{j=m}^{\infty}\frac{1}{q_j}+(\mu(\alpha)-1)\sum_{j=m}^{\infty}\frac{\log(q_j)}{q_j}\right).$$

Equation (4.4) follows (4.12) and (4.13) with $\omega=1$ in both cases, when $m\ge 7$. □

5. Proof of Theorem 1

5.1. **Proof of** (i)**.** We fix a rational a/b such that $\gcd(a,b)=1$ and $b\geq 1$. We have

$$\Phi_{s,N}\Big(\frac{a}{b}\Big)=\sum_{n=1}^{N}\frac{1}{n^{s+1}}\sum_{m=1}^{n}\cos\big(2mn\pi\frac{a}{b}\big)$$

As in the proof of Theorem 4 (ii), we write $m=rb+j$ with $1\leq j\leq b$ and $r\geq 0$ and get

$$\Phi_{s,N}\Big(\frac{a}{b}\Big)=\frac{1}{b}\sum_{j=1}^{b}\sum_{n=1}^{N}\frac{1}{n^s}\cos\Big(2\pi jn\frac{a}{b}\Big)+\mathcal{O}\big(bH_N(s+1)\big),$$

where the implicit constant is absolute. Similarily we treat the sum over n to get

$$\Phi_{s,N}\Big(\frac{a}{b}\Big)=\frac{1}{b}\sum_{j=1}^{b}\sum_{k=1}^{b}\cos\Big(2\pi jk\frac{a}{b}\Big)\sum_{\substack{n\leq N\\ n\equiv k[b]}}\frac{1}{n^s}+\mathcal{O}\big(b\zeta(s+1)\big).$$

Hence, since $\sum_{\substack{n\leq N\\ n\equiv k[b]}}\frac{1}{n^s}=\frac{1}{b}H_N(s)+\mathcal{O}(1)$ where the constant depends at most on s, we deduce that

$$\Phi_{s,N}\Big(\frac{a}{b}\Big)=H_N(s)\frac{1}{b^2}\sum_{j=1}^{b}\sum_{k=1}^{b}\cos\Big(2\pi jk\frac{a}{b}\Big)+\mathcal{O}\big(b\big),$$

where the constant depends at most on s. We now need a lemma that will be used again later.

Lemma 5. *Let u,v be integers such that $v\geq 1$. We have*

$$\frac{1}{v}\sum_{j=1}^{v}\sum_{k=1}^{v}\cos\Big(2\pi jk\frac{u}{v}\Big)=\gcd(u,v). \tag{5.1}$$

Proof. We denote by S the quantity on the left hand side of (5.1). The inner sum over k equals v if v divides ju and 0 otherwise. Thus,

$$S=\sum_{\substack{j=1\\ v|ju}}^{v}1=\sum_{\substack{j=1,\dots,v\\ \frac{v}{\gcd(u,v)}|j\frac{u}{\gcd(u,v)}}}1=\sum_{\substack{j=1,\dots,v\\ \frac{v}{\gcd(u,v)}|j}}1=\sum_{j=1}^{\gcd(u,v)}1=\gcd(u,v).$$

□

Applying the lemma with $u=a$ and $v=b$ (which are coprime), we obtain that

$$\Phi_{s,N}\Big(\frac{a}{b}\Big)=\frac{1}{b}H_N(s)+\mathcal{O}\big(b\big), \tag{5.2}$$

(the constant depends on s), which proves the assertion, in an even more precise form that will be used later.

5.2. **Proof of** (ii)**.** We remark that

$$\left|\frac{\cos\big(\pi n(n+1)\alpha\big)\sin\big(\pi n^2\alpha\big)}{\sin\big(\pi n\alpha\big)}\right| \le \frac{|\sin\big(\pi n^2\alpha\big)|}{|\sin\big(\pi n\alpha\big)|} = \frac{\sin\big(\pi||n^2\alpha||\big)}{\sin\big(\pi||n\alpha||\big)} \le \frac{\pi}{2}\frac{||n^2\alpha||}{||n\alpha||}.$$

(We have used the inequalities $2x \le \sin(\pi x) \le \pi x$, which hold for any x such that $0 \le x \le 1/2$.)

From the definition of $\Phi_{s,N}(\alpha)$, we get

$$\Big|\Phi_{s,N}(\alpha)\Big| \le \sum_{n=1}^{N}\left|\frac{\cos\big(\pi n(n+1)\alpha\big)\sin\big(\pi n^2\alpha\big)}{n^{s+1}\sin\big(\pi n\alpha\big)}\right| \le \frac{\pi}{2}\sum_{n=1}^{N}\frac{||n^2\alpha||}{n^{s+1}||n\alpha||}. \tag{5.3}$$

We can now use Proposition 1 (i), to conclude that both series converge for any $\alpha \in \mathscr{A}_s$.

5.3. **Proof of** (iii)**.** • *Almost sure convergence of* $\Phi_{s,N}$*.* We have proved that $\Phi_{s,N}(\alpha)$ converges to $\Phi_s(\alpha)$ for all $\alpha \in \mathscr{A}_s$ and moreover, that $\mathscr{A}_s$ is of measure 1.

• *Convergence of* $\Phi_{s,N}$ *to* Φ_s *in* $L^2(0,1)$*.*

Let us first show that $(\Phi_{s,N})_{n\ge1}$ converges in L^2. For this, it is enough to show that the sequence is Cauchy. For any integers $N \ge M \ge 1$, we have

$$\begin{aligned}
&\int_0^1 \big(\Phi_{s,N}(\alpha) - \Phi_{s,M}(\alpha)\big)^2\,\mathrm{d}\alpha \\
&\qquad = \sum_{m=M+1}^{N}\sum_{n=M+1}^{N}\frac{1}{(mn)^{s+1}}\sum_{k=1}^{m}\sum_{\ell=1}^{n}\int_0^1 \cos\big(2km\pi\alpha\big)\cos\big(2\ell n\pi\alpha\big)\,\mathrm{d}\alpha \\
&\qquad = \frac12\sum_{m=M+1}^{N}\sum_{n=M+1}^{N}\frac{1}{(mn)^{s+1}}\sum_{\substack{1\le k\le m,1\le \ell\le n\\ \ell m=kn}} 1 = \frac12\sum_{m=M+1}^{N}\sum_{n=M+1}^{N}\frac{1}{(mn)^{s+1}}\sum_{\substack{1\le \ell\le n\\ n|\ell m}} 1 \\
&\qquad = \frac12\sum_{m=M+1}^{N}\sum_{n=M+1}^{N}\frac{\gcd(m,n)}{(mn)^{s+1}}.
\end{aligned}$$

(In the last line, we have used an identity already obtained in the proof of Lemma 5.) For any integer $m \ge 1$, the Dirichlet series with positive terms $A_m(s+1) := \sum_{n=1}^{\infty}\frac{\gcd(m,n)}{n^{s+1}}$ is convergent and thus

$$\int_0^1 \big(\Phi_{s,N}(\alpha) - \Phi_{s,M}(\alpha)\big)^2\,\mathrm{d}\alpha \le \frac12\sum_{m=M+1}^{N}\frac{A_m(s+1)}{m^{s+1}}.$$

It remains to prove that the series with term $A_m(s+1)/m^{s+1}$ converges. For this, we proceed as follows. The arithmetic function $n \mapsto \gcd(m,n)$ is multiplicative and bounded by m. Therefore, $A_m(t)$ converges for any t such that $\mathrm{Re}(t) > 1$ and we have

$$\begin{aligned} A_m(t) &= \prod_p \Big(\sum_{k=0}^{\infty} \frac{\gcd(m,p^k)}{p^{kt}} \Big) = \prod_{p|m} \Big(\sum_{k=0}^{v_p(m)} \frac{p^k}{p^{kt}} + \sum_{k=v_p(m)+1}^{\infty} \frac{p^{v_p(m)}}{p^{kt}} \Big) \cdot \prod_{p\nmid m} \Big(\sum_{k=0}^{\infty} \frac{1}{p^{kt}} \Big) \\ &= \zeta(t) \prod_{p|m} \left(\frac{1}{p^{(t-1)v_p(m)}} \left(\frac{p^{(t-1)(v_p(m)+1)}-1}{p^{t-1}-1} + \frac{1}{p^t-1} \right) \left(1 - \frac{1}{p^t} \right) \right). \end{aligned}$$

In particular, substituting $s+1$ for t with $s > 0$, we obtain the bound

$$A_m(s+1) \ll \prod_{p|m} \left(1 + \frac{1}{p^s} \right),$$

where the implicit constant depends on s. For any $s > 0$, we have

$$\prod_{p|m} \left(1 + \frac{1}{p^s} \right) = \sum_{d|m} \frac{|\mu(d)|}{d^s} \leq \sum_{d|m} 1 \leq e^{\mathcal{O}(\log(m)/\log\log(m))}$$

for some absolute constant (See Tenenbaum [16, p. 84]). Hence we have proved that

$$\left| \frac{A_m(s+1)}{m^{s+1}} \right| \ll \frac{e^{\mathcal{O}(\log(m)/\log\log(m))}}{m^{s+1}},$$

where the right-hand side is the term of a convergent series, as was to be proved.

Therefore, $||\Phi_{s,N} - \Phi_{s,M}||_2$ tends to 0 when $N \geq M \to +\infty$, so that the sequence $(\Phi_{s,N})_{N\geq 0}$ converges in $L^2(0,1)$ to a certain function $\widehat{\Phi}_s$. By a classical property (see [15, p.68, Theorem 3.12]), we deduce the existence of a subsequence $(\Phi_{N_k,s})_k$ such that $\Phi_{N_k,s} \to \widehat{\Phi}_s$ almost surely, which implies that $\Phi_s = \widehat{\Phi}_s$ almost surely.

5.4. **Proof of** *(iv)***.** Given some irrational number α and $s \in (0,1]$, that will be further specified later, we consider a sequence of coprime rational numbers $(a_m/b_m)_m$ that converges to α.

By the mean value theorem, we have

$$\begin{aligned} \Big| \Phi_{s,N}(\alpha) - \Phi_{s,N}\Big(\frac{a_m}{b_m}\Big) \Big| &\leq \sum_{n=1}^{N} \frac{1}{n^{s+1}} \sum_{k=1}^{n} \Big| \cos(2\pi k n \alpha) - \cos\left(2\pi k n \frac{a_m}{b_m}\right) \Big| \\ &\leq \sum_{n=1}^{N} \frac{1}{n^{s+1}} \sum_{k=1}^{n} \Big| 2\pi k n \alpha - 2\pi k n \frac{a_m}{b_m} \Big| \leq 2\pi \Big| \alpha - \frac{a_m}{b_m} \Big| \cdot \sum_{n=1}^{N} \frac{1}{n^s} \sum_{k=1}^{n} k \\ &\ll N^{3-s} \Big| \alpha - \frac{a_m}{b_m} \Big|, \end{aligned} \tag{5.4}$$

where the implicit constant depends only on s. Hence,

$$\Phi_{s,N}(\alpha) = \Phi_{s,N}\Big(\frac{a_m}{b_m}\Big) + \mathcal{O}\Big(N^{3-s}\Big|\alpha - \frac{a_m}{b_m}\Big|\Big) = \frac{H_N(s)}{b_m} + \mathcal{O}(b_m) + \mathcal{O}\Big(N^{3-s}\Big|\alpha - \frac{a_m}{b_m}\Big|\Big) \tag{5.5}$$

by (5.2) and where the implicit constants depend on s only. We now distinguish the two cases $s = 1$ and $s \in (0, 1)$.

• *Case* $s = 1$. Since $H_N = \log(N) + \mathcal{O}(1)$, Equation (5.5) becomes

$$\Phi_{s,N}(\alpha) = \frac{\log(N)}{b_m} + \mathcal{O}(b_m) + \mathcal{O}\Big(N^2\Big|\alpha - \frac{a_m}{b_m}\Big|\Big). \tag{5.6}$$

Let us now assume that α is such that

$$0 < \Big|\alpha - \frac{a_m}{b_m}\Big| \leq \frac{1}{e^{\delta_m b_m^2}}, \tag{5.7}$$

where δ_m is some function tending to $+\infty$ with m. We can take for example $\alpha = \sum_{n\geq 1} 1/b_n$ with $b_{n+1} = 2^{b_n^3}$ and $b_1 = 1$: for any $m \geq 1$, we have $\sum_{n=1}^m 1/b_n =: \frac{a_m}{b_m}$ and obviously $(a_m, b_m) = 1$ because a_m is odd and b_m is a pure power of 2. Of course, the diophantine condition (5.7) implies that α is a Liouville number.

We choose $N = N_m = \lfloor e^{\frac{1}{2}\delta_m b_m^2} \rfloor$, so that (5.6) becomes

$$\Phi_{1,N_m}(\alpha) = \frac{1}{2}\,\delta_m b_m + \mathcal{O}(b_m) + \mathcal{O}(1). \tag{5.8}$$

From this follows that $\limsup_N \Phi_{1,N}(\alpha) = +\infty$. In fact, (5.8) even implies that $\Phi_{1,N_m}(\alpha) \gg \sqrt{\delta_m \log(N_m)}$, with $N_m \approx e^{\frac{1}{2}\delta_m b_m^2}$. We can choose δ_m much larger than b_m and would get $b_m = \mathcal{O}(\log(N_m)^{\varepsilon})$ for any given $\varepsilon \in (0, 1)$. Therefore, there exist infinitely many choices for the sequence $(\delta_m)_m$ (corresponding to infinitely many α) such that $\delta_m \gg \log(N_m)/b_m^2 \gg \log(N_m)^{1-2\varepsilon}$, in which case we have

$$\limsup_{N\to+\infty} \frac{\Phi_{1,N}(\alpha)}{\log(N)^{1-\varepsilon}} > 0.$$

Since $\varepsilon \in (0, 1)$ is arbitrary, this is in fact equivalent to the fact that

$$\limsup_{N\to+\infty} \frac{\Phi_{1,N}(\alpha)}{\log(N)^{1-\varepsilon}} = +\infty,$$

for all $\varepsilon \in (0, 1)$, as claimed. Clearly, we can make this contruction for a dense set of Liouville numbers.

• *Case* $s \in (0, 1)$. Since $H_N(s) = \frac{N^{1-s}}{1-s} + \mathcal{O}(1)$, Equation (5.5) becomes

$$\Phi_{s,N}(\alpha) = \frac{N^{1-s}}{(1-s)b_m} + \mathcal{O}(b_m) + \mathcal{O}\Big(N^{3-s}\Big|\alpha - \frac{a_m}{b_m}\Big|\Big). \tag{5.9}$$

Let us assume for the moment that α is not a Liouville number and is such that $0 < |\alpha - \frac{a_m}{b_m}| \leq \frac{1}{b_m^{\mu}}$, where $(a_m, b_m) = 1$, for some $\mu > 2 + \frac{4}{1-s}$, which implies that the irrationality exponent of α is $\geq 2 + \frac{4}{1-s}$.

We choose $N = N_m = \big\lfloor b_m^{\frac{\mu}{3-s}} \big\rfloor$ so that Eq (5.9) becomes

$$\Phi_{s,N_m}(\alpha) = \frac{1}{1-s}\, b_m^{\frac{(1-s)\mu}{3-s}-1} + \mathcal{O}(b_m) + \mathcal{O}(1). \tag{5.10}$$

The condition on μ ensures that $\frac{(1-s)\mu}{3-s} - 1 > 1$ and thus $\limsup_N \Phi_{s,N}(\alpha) = +\infty$. In fact, (5.10) even implies that for any $\varepsilon > 0$, we have

$$\limsup_{N\to+\infty} \frac{\Phi_{s,N}(\alpha)}{N^{(1-s)-\frac{3-s}{\mu}-\varepsilon}} = +\infty.$$

The reader will easily adapt this argument to the case of Liouville numbers where she will replace μ by a sequence $(\mu_m)_m$ that tends to $+\infty$ with m. In this case, we obtain

$$\limsup_{N\to+\infty} \frac{\Phi_{s,N}(\alpha)}{N^{(1-s)-\varepsilon}} = +\infty.$$

Again, this contruction can be done for a dense set of Liouville numbers.

5.5. **Proof of** (v)**.** To prove the divergence of $\Phi_{s,N}(\alpha)$ for any α and $s \leq 0$, it is enough to prove that that $\sum_{m=1}^{n} \cos(2\pi mn\alpha)$ does not tend to 0 as $n \to +\infty$. For a rational $\alpha = a/b$, this is immediate because, for any integer n, we have $\sum_{m=1}^{bn} \cos(2\pi mnb\alpha) = \sum_{m=1}^{bn} \cos(2\pi mna) = bn$, hence

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{m=1}^{n} \cos\left(2\pi mn\frac{a}{b}\right) = 1.$$

For α irrational, this is a consequence of the following lemma.

Lemma 6. *For any irrational number α, we have*

$$\limsup_{n\to+\infty} \left| \frac{1}{n} \sum_{m=1}^{n} \cos(2\pi mn\alpha) \right| \geq \frac{L(\alpha)}{2\pi} \sin\Big(\frac{2\pi}{L(\alpha)}\Big) > 0,$$

where, by convention, $\frac{L(\alpha)}{2\pi} \sin\big(\frac{2\pi}{L(\alpha)}\big) = 1$ if $L(\alpha) = +\infty$.

Remark 6. We believe that equality holds in Lemma 6. This is in fact the case for any α such that $L(\alpha) = +\infty$ (hence for almost all irrational numbers), because the right-hand side equals 1 while the left-hand side is obviously ≤ 1.

Proof. We have

$$\begin{aligned} \frac{1}{n} \sum_{m=1}^{n} \cos(2\pi mn\alpha) &= \frac{\cos(\pi n(n+1)\alpha) \sin(\pi n^2\alpha)}{n \sin(\pi n\alpha)} \\ &= \frac{\cos(\pi n\alpha) \sin(2\pi n^2\alpha)}{2n \sin(\pi n\alpha)} - \frac{\big(\sin(\pi n^2\alpha)\big)^2}{n}. \end{aligned}$$

Hence

$$\left| \frac{1}{n} \sum_{m=1}^{n} \cos(2\pi mn\alpha) \right| = \frac{\big|\cos(\pi||n\alpha||)\big| \cdot \big|\sin(2\pi n||n\alpha||)\big|}{2n \sin(\pi||n\alpha||)} + \mathcal{O}\Big(\frac{1}{n}\Big).$$

Let us denote by $(Q_k)_k$ a subsequence of the denominators $(q_k)_k$ of the convergents to α such that

$$\lim_{k\to+\infty} Q_k||Q_k\alpha|| = \frac{1}{L(\alpha)}.$$

We have

$$\lim_{k\to+\infty} \big|\cos(\pi||Q_k\alpha||)\big| = 1, \qquad \lim_{k\to+\infty} \big|\sin(2\pi Q_k||Q_k\alpha||)\big| = \big|\sin(2\pi/L(\alpha))\big|,$$

$$\lim_{k\to+\infty} 2Q_k \sin(\pi||Q_k\alpha||) = 2\pi/L(\alpha).$$

Moreover, since $L(\alpha) \geq \sqrt{5}$, we have $0 < 2\pi/L(\alpha) < \pi$ and thus $\sin(2\pi/L(\alpha)) > 0$. Therefore,

$$\limsup_{n\to+\infty} \left|\frac{1}{n}\sum_{m=1}^{n} \cos(2\pi m n\alpha)\right| \geq \limsup_{k\to+\infty} \left|\frac{1}{Q_k}\sum_{m=1}^{Q_k} \cos(2\pi m Q_k\alpha)\right| = \frac{L(\alpha)}{2\pi}\,\sin\Big(\frac{2\pi}{L(\alpha)}\Big) > 0$$

and this completes the proof of the lemma. □

6. Proof of Theorem 2

6.1. **Proof of** (i)**.** We recall that $G_{s,N}(\alpha)$ is defined by (1.1). We fix a rational number a/b with $\gcd(a,b)=1$ and $b=1$. Using the Fourier expansion of $||\alpha||$ as for $F_N(\alpha)$, we obtain

$$\frac{1}{H_N(s)}\, G_{s,N}(\alpha) = \frac{1}{4} - \frac{2}{\pi^2}\sum_{\ell=0}^{\infty} \frac{1}{(2\ell+1)^2}\frac{1}{H_N(s)}\,\Phi_{s,N}\big((2\ell+1)\alpha\big). \tag{6.1}$$

We now use Equation (5.2) which says for coprime $u,v\geq 1$ that $\Phi_{s,N}(\frac{u}{v}) = \frac{1}{v}\,H_N(s) + \mathcal{O}(v)$, where the constant depends only on s. Hence, if u and v are not necessarily coprime, we have

$$\Phi_{s,N}\Big(\frac{u}{v}\Big) = \frac{\gcd(u,v)}{v}\,H_N(s) + \mathcal{O}\Big(\frac{v}{\gcd(u,v)}\Big).$$

It follows from this and (6.1) that

$$\frac{1}{H_N(s)}\, G_{s,N}\Big(\frac{a}{b}\Big) = \frac{1}{4} - \frac{2}{\pi^2}\sum_{\ell=0}^{\infty} \frac{\gcd(b,2\ell+1)}{b(2\ell+1)^2} + \mathcal{O}\Big(\frac{b}{H_N(s)}\Big), \tag{6.2}$$

where the constant depends only on s.

We now represent the main term as a product. The principle is very similar to what was done earlier. We observe that the arithmetic function $\ell \mapsto \gcd(b,\ell)$ is multiplicative and that, for any complex number t such that $\mathrm{Re}(t) > 1$,

$$\sum_{\ell=0}^{\infty}\frac{\gcd(b,2\ell+1)}{(2\ell+1)^t} = \prod_{p\geq 3}\Big(\sum_{k=0}^{\infty}\frac{\gcd(b,p^k)}{p^{kt}}\Big) = \prod_{\substack{p\geq 3\\ p|b}}\Big(\sum_{k=0}^{v_p(b)}\frac{p^k}{p^{kt}} + \sum_{k=v_p(b)+1}^{\infty}\frac{p^{v_p(b)}}{p^{kt}}\Big)\cdot\prod_{\substack{p\geq 3\\ p\nmid b}}\Big(\sum_{k=0}^{\infty}\frac{1}{p^{kt}}\Big)$$

$$= \zeta(t)\Big(1-\frac{1}{2^t}\Big)\prod_{\substack{p\geq 3\\ p|b}}\Bigg(\frac{1}{p^{(t-1)v_p(b)}}\Big(\frac{p^{(t-1)(v_p(b)+1)}-1}{p^{t-1}-1} + \frac{1}{p^t-1}\Big)\Big(1-\frac{1}{p^t}\Big)\Bigg).$$

Using this for $t = 2$, we find

$$\lim_{N\to+\infty} \frac{1}{H_N(s)}\, G_{s,N}\left(\frac{a}{b}\right) = \frac{1}{4} - \frac{1}{4b}\prod_{\substack{p\geq 3\\ p\mid b}}\left(\frac{(p^{v_p(b)+1}-1)(p+1)+p^{v_p(b)}}{p^{v_p(b)+2}}\right),$$

which is a rational number $< \frac{1}{4}$.

6.2. **Proof of** *(ii)***.** From (6.1), we have

$$\mathscr{G}_{s,N}(\alpha) := G_{s,N}(\alpha) - \frac{1}{4}\,H_N(s) = -\frac{2}{\pi^2}\sum_{\ell=0}^{\infty}\frac{\Phi_{s,N}\big((2\ell+1)\alpha\big)}{(2\ell+1)^2}.$$

To justify that we can pass to the limit $N \to +\infty$ under the sum sign, we will use *Tannery's theorem*, which is a special case of Lebesgue dominated convergence theorem: *Let $(A_n(k))_{n\geq 0}$ be a sequence of complex numbers that depends on an integer parameter $k \geq 0$. Let us assume that*

- *for all $n \geq 0$, $\lim_{k\to+\infty} A_n(k)$ exists and is finite;*
- *for all $n \geq 0$, there exists M_n such that $|A_n(k)| \leq M_n$ for all $k \geq 0$ and such that $\sum_n M_n$ is convergent.*

Then, $\displaystyle\lim_{k\to+\infty}\sum_{n=0}^{\infty} A_n(k) = \sum_{n=0}^{\infty}\lim_{k\to+\infty} A_n(k) < \infty.$

We will also need the following lemma.

Lemma 7. *(i) For any $\alpha \in \mathscr{A}_1$ with a finite irrationality exponent $\mu(\alpha)$, any integer $N \geq 1$ and any integer $k \geq 1$, we have $|\Phi_{1,N}(k\alpha)| \ll \log(k+1)$, where the constant depends on α.*

(ii) Given $s \in (0,1)$, for any $\alpha \in \mathscr{A}_s$ with an irrationality exponent $\mu(\alpha) < 1 + \frac{1}{1-s}$, any integer $N \geq 1$ and any integer $k \geq 1$, we have $|\Phi_{s,N}(k\alpha)| \ll k^{(\mu(\alpha)-1)(1-s)}$, where the constant depends at most on α and s.

Proof. Using the inequality (5.3) and statements *(iii)* and *(iv)* of Proposition 1, we find that $|\Phi_{1,N}(\alpha)| \ll \mu(\alpha) + \log c(\alpha)$ and $|\Phi_{s,N}(\alpha)| \ll c(\alpha)^{1-s}\cdot\frac{2\sqrt{2}^{1-(\mu(\alpha)-1)(1-s)}-1}{\sqrt{2}^{1-(\mu(\alpha)-1)(1-s)}-1}$, where the first constant is absolute and the second one depends at most on s. Furthermore, it is straitghforward to see that if α has a finite irrationality exponent, then for all integer $k \geq 1$, the irrational number $k\alpha$ also has a finite irrationality exponent and that we can take $\mu(k\alpha)$ and $c(k\alpha)$) such that $\mu(k\alpha) = \mu(\alpha)$ and $c(k\alpha) \leq c(\alpha)k^{\mu(\alpha)-1}$. Hence,

$$|\Phi_{1,N}(k\alpha)| \ll \mu(k\alpha) + \log c(k\alpha) \leq \mu(\alpha) + (\mu(\alpha)-1)\log(k) + \log c(\alpha) \ll \log(k)$$

and similarly

$$\begin{aligned}|\Phi_{s,N}(k\alpha)| &\ll c(k\alpha)^{1-s}\cdot\frac{2\sqrt{2}^{1-(\mu(k\alpha)-1)(1-s)}-1}{\sqrt{2}^{1-(\mu(k\alpha)-1)(1-s)}-1}\\ &\leq c(\alpha)^{1-s}k^{(\mu(\alpha)-1)(1-s)}\cdot\frac{2\sqrt{2}^{1-(\mu(\alpha)-1)(1-s)}-1}{\sqrt{2}^{1-(\mu(\alpha)-1)(1-s)}-1} \ll k^{(\mu(\alpha)-1)(1-s)},\end{aligned}$$

where both constants depend at most on α and s, respectively. □

We can now easily finish the proof of statement (ii) of Theorem 2. Indeed, in the case $s = 1$, for any $\alpha \in \mathscr{A}_1$ with finite irrationality exponent and any integer $\ell \geq 0$, $(2\ell + 1)\alpha$ also has a finite irrationality exponent and therefore belongs to $\mathscr{A}_1$ (by Lemma 1, (i)). Hence $\Phi_{1,N}((2\ell+1)\alpha)$ converges to $\Phi_1((2\ell+1)\alpha)$ and by (i) of Lemma 7, we have

$$\left|\frac{\Phi_{1,N}((2\ell+1)\alpha)}{(2\ell+1)^2}\right| \ll \frac{\log(\ell+1)}{(2\ell+1)^2},$$

where the right-hand side is the term of a convergent series. By Tannery's theorem, we have therefore

$$\lim_{N\to+\infty} \mathscr{G}_{s,N}(\alpha) = -\lim_{N\to+\infty} \frac{2}{\pi^2}\sum_{\ell=0}^{\infty} \frac{\Phi_{1,N}\big((2\ell+1)\alpha\big)}{(2\ell+1)^2} = -\frac{2}{\pi^2}\sum_{\ell=0}^{\infty}\frac{\Phi_1\big((2\ell+1)\alpha\big)}{(2\ell+1)^2}.$$

In the case $s \in (0,1)$, for any $\alpha \in \mathscr{A}_s$ with an irrationality exponent $\mu(\alpha) < 1+\frac{1}{1-s}$ and for any integer $\ell \geq 0$, $(2\ell+1)\alpha$ also has an irrationality exponent $\mu((2\ell+1)\alpha) = \mu(\alpha) < 1+\frac{1}{1-s}$ and therefore belongs to $\mathscr{A}_s$ (by Lemma 1, (ii)). Hence $\Phi_{s,N}((2\ell+1)\alpha)$ converges to $\Phi_s((2\ell+1)\alpha)$ and by (ii) of Lemma 7, we have

$$\left|\frac{\Phi_{s,N}((2\ell+1)\alpha)}{(2\ell+1)^2}\right| \ll \frac{1}{(2\ell+1)^{2-(\mu(\alpha)-1)(1-s)}},$$

where the right-hand side is the term of convergent series, which is implied by $(\mu(\alpha)-1)(1-s) < 1$. Again, by Tannery's theorem, we therefore have

$$\lim_{N\to+\infty} \mathscr{G}_{s,N}(\alpha) = -\lim_{N\to+\infty} \frac{2}{\pi^2}\sum_{\ell=0}^{\infty} \frac{\Phi_{s,N}\big((2\ell+1)\alpha\big)}{(2\ell+1)^2} = -\frac{2}{\pi^2}\sum_{\ell=0}^{\infty}\frac{\Phi_s\big((2\ell+1)\alpha\big)}{(2\ell+1)^2}.$$

6.3. **Proof of** (iii)**.** The almost sure convergence of $\mathscr{G}_{s,N}$ to $\mathscr{G}_s$ is a consequence of (ii) because the sets $\mathscr{A}_s$ all have measure 1. It remains to prove the convergence in $L^2(0,1)$. We note first that for any integer $k \geq 0$ and any integers $M \geq N \geq 1$, we have $\big|\big|\Phi_{s,M}\big((2k+1)\alpha\big) - \Phi_{s,N}\big((2k+1)\alpha\big)\big|\big|_2 = \big|\big|\Phi_{s,M}(\alpha) - \Phi_{s,N}(\alpha)\big|\big|_2$ by the 1-periodicity of the $\Phi_{s,N}$. Therefore,

$$\begin{aligned}
\big|\big|\mathscr{G}_{s,M}(\alpha) - \mathscr{G}_{s,N}(\alpha)\big|\big|_2 &\leq \frac{2}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\,\big|\big|\Phi_{s,M}\big((2k+1)\alpha\big) - \Phi_{s,N}\big((2k+1)\alpha\big)\big|\big|_2 \\
&\leq \frac{2}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\,\big|\big|\Phi_{s,M}(\alpha) - \Phi_{s,N}(\alpha)\big|\big|_2 \\
&\leq \frac{1}{4}\,\big|\big|\Phi_{s,M}(\alpha) - \Phi_{s,N}(\alpha)\big|\big|_2.
\end{aligned}$$

Since the right hand side converges to 0 by Theorem 1 when $M \geq N \to +\infty$, we have obtained the convergence of the $\mathscr{G}_{s,N}$ in $L^2(0,1)$ to a function which is necessarily equal to $\mathscr{G}_s$.

6.4. **Proof of** (iv)**.** We will use the same method as in the proof of statement (iv) in Theorem 1. The proof is based on the "identity" (6.3) given in Lemma 8 below. We will then leave most of the details to the reader as no new idea will be involved.

Lemma 8. *Let us fix* $s \in (0,1]$*, an irrational number* α *and a rational number* a/b *with* $\gcd(a,b)=1$ *and* $b \geq 1$*. For any integers* $M \geq 1, N \geq 1$ *we have*

$$G_{s,N}(\alpha) = \left(\frac{1}{4} - \frac{2}{\pi^2}\sum_{\ell=0}^{\infty}\frac{\gcd(b,2\ell+1)}{b(2\ell+1)^2}\right)H_N(s) + \mathcal{O}(b) + \mathcal{O}\left(\frac{H_N(s)}{M}\right) + \mathcal{O}\left(H_M \cdot N^{3-s}\left|\alpha - \frac{a}{b}\right|\right), \quad (6.3)$$

where the implicit constants depend at most on s*.*

Proof. As the reader will check, all the implicit constants below depend at most on s. Firstly, starting from the definition, we find that for any integer $M \geq 1$

$$\left|G_{s,N}(\alpha) - G_{s,N}\left(\frac{a}{b}\right)\right| \leq \frac{2}{\pi^2}\left(\sum_{\ell=0}^{M} + \sum_{\ell=M+1}^{\infty}\right)\frac{1}{(2\ell+1)^2}\left|\Phi_{s,N}\big((2\ell+1)\alpha\big) - \Phi_{s,N}\left((2\ell+1)\frac{a}{b}\right)\right|.$$

Secondly, using the upper bound (5.4), we have

$$\sum_{\ell=0}^{M}\frac{1}{(2\ell+1)^2}\left|\Phi_{s,N}\big((2\ell+1)\alpha\big) - \Phi_{s,N}\left((2\ell+1)\frac{a}{b}\right)\right| \ll N^{3-s}\left|\alpha - \frac{a}{b}\right|\sum_{\ell=0}^{M}\frac{1}{2\ell+1} \ll H_M \cdot N^{3-s}\left|\alpha - \frac{a}{b}\right|.$$

Thirdly, we know that $|\Phi_{s,N}(\alpha)| \leq H_N(s)$ in all cases. Hence,

$$\sum_{\ell=M+1}^{\infty}\frac{1}{(2\ell+1)^2}\left|\Phi_{s,N}\big((2\ell+1)\alpha\big) - \Phi_{s,N}\left((2\ell+1)\frac{a}{b}\right)\right| \ll H_N(s)\sum_{\ell=M+1}^{\infty}\frac{1}{(2\ell+1)^2} \ll \frac{H_N(s)}{M}.$$

Therefore, we have obtained

$$G_{s,N}(\alpha) = G_{s,N}\left(\frac{a}{b}\right) + \mathcal{O}\left(\frac{H_N(s)}{M}\right) + \mathcal{O}\left(H_M \cdot N^{3-s}\left|\alpha - \frac{a}{b}\right|\right)$$

and the lemma follows by using (6.2). □

To conclude the proof of statement (iv), we first choose a sequence of rational $(a_m/b_m)_m$ that converges to α, with $(a_m, b_m) = 1$, $b_m \geq 1$. Then, we take $M = \lfloor H_N(s) \rfloor \ll N^{1-s}$, so that $H_M \ll \log(N)$. After some simplifications, (6.3) becomes

$$G_{s,N}(\alpha) = \left(\frac{1}{4} - \frac{2}{\pi^2}\sum_{\ell=0}^{\infty}\frac{\gcd(b_m,2\ell+1)}{b_m(2\ell+1)^2}\right)H_N(s) + \mathcal{O}(b_m) + \mathcal{O}\left(\log(N)N^{3-s}\left|\alpha - \frac{a_m}{b_m}\right|\right).$$

Since $\sum_{\ell=0}^{\infty}\frac{\gcd(b_m,2\ell+1)}{b_m(2\ell+1)^2}\geq\frac{1}{b_m}$, it follows that

$$G_{s,N}(\alpha)-\frac{1}{4}H_N(s)\leq-\frac{2H_N(s)}{\pi^2 b_m}+\mathcal{O}(b_m)+\mathcal{O}\bigg(\log(N)\cdot N^{3-s}\Big|\alpha-\frac{a_m}{b_m}\Big|\bigg). \tag{6.4}$$

For $s=1$ or $s\in(0,1)$, we can take for α the same reals as those used during the proof of statement (v) of Theorem 1 and with the same choices for N in function of b_m. In (6.4), the main term is $-\frac{2H_N(s)}{\pi^2 b_m}$ and we obtain the divergence at the rate indicated for irrational numbers with irrationality exponent $m(\alpha)>2+\frac{4}{1-s}$ when $s\in(0,1)$, or for a dense set of Liouville numbers when $s=1$.

6.5. **Proof of** (iv)**.** Since $L(\alpha)\geq\sqrt{5}$ for any irrational number α, it follows from statement (iv) of Theorem 4 that $\liminf_n F_n(\alpha)<\frac{1}{4}$. Thus, for any $s\leq 0$, the sequence $n^{-s}\big(F_n(\alpha)-\frac{1}{4}\big)$ does not tend to zero and a fortiori the series $\sum_n n^{-s}\big(F_n(\alpha)-\frac{1}{4}\big)$ does not converge.

The divergence for α a rational number is a consequence of the oscillating (and nearly periodic) behavior of $F_n(\alpha)-\frac{1}{4}$ as shown in statement (ii) of Theorem 4.

7. Proof of Theorem 3

7.1. **Proof of** (i)**.** The Fourier expansion of $\Phi_{s,N}(\alpha)$ is

$$\begin{aligned}\Phi_{s,N}(\alpha)&=\sum_{k=1}^{N^2}\cos(2\pi k\alpha)\sum_{\substack{1\leq m\leq n\leq N\\ mn=k}}\frac{1}{n^{s+1}}=\sum_{k=1}^{N^2}\cos(2\pi k\alpha)\sum_{\substack{n|k\\ \sqrt{k}\leq n\leq N}}\frac{1}{n^{s+1}}\\ &=\sum_{k=1}^{\infty}\cos(2\pi k\alpha)\sum_{\substack{n|k\\ \sqrt{k}\leq n\leq N}}\frac{1}{n^{s+1}}\end{aligned}$$

and the difficulty is to justify that we can pass to the limit under the sum sign.

By Carleson's theorem [4], we know that the Fourier expansion of Φ_s converges almost surely to Φ_s because $\Phi_s\in L^2(0,1)$. We will give a direct proof of the almost sure convergence that avoids this deep theorem.

Set $\widehat{S}_{s,N}(\alpha):=\sum_{k=1}^{N}\widehat{\phi}_{s,k}\cos(2\pi k\alpha)$, where $\widehat{\phi}_{s,k}:=\displaystyle\sum_{n|k,n\geq\sqrt{k}}\frac{1}{n^{s+1}}$. We will show that, for all $\alpha\in\mathscr{B}_s$, $\lim_N\widehat{S}_{s,N}(\alpha)=\Phi_s(\alpha)$. This will prove that the trigonometric series $\widehat{S}_s(\alpha):=\sum_{k=1}^{\infty}\widehat{\phi}_{s,k}\cos(2\pi k\alpha)$ converges almost surely to $\Phi_s(\alpha)$. It will remain to show that $\widehat{S}_s$ is the Fourier series of Φ_s, i.e. that the coefficients $\widehat{\phi}_{k,s}$ coincide with the Fourier coefficients $\phi_{k,s}$ of Φ_s by the usual integrals.

• *Almost sure convergence of a trigonometric series to* Φ_s. For any real number α, we observe that

$$\begin{aligned}\widehat{S}_{s,N}(\alpha) &= \sum_{n=1}^{N} \frac{1}{n^{s+1}} \sum_{1\le k\le n^2, k\le N, n|k} \cos(2\pi k\alpha) = \sum_{n=1}^{N} \frac{1}{n^{s+1}} \sum_{1\le k\le n,\, k\le N/n} \cos(2\pi kn\alpha) \\ &= \sum_{1\le n\le \sqrt{N}} \frac{1}{n^{s+1}} \sum_{k=1}^{n} \cos(2\pi kn\alpha) + \sum_{\sqrt{N}<n\le N} \frac{1}{n^{s+1}} \sum_{1\le k\le N/n} \cos(2\pi kn\alpha) \qquad (7.1) \\ &= \sum_{1\le n\le \sqrt{N}} \frac{1}{n^{s+1}} \sum_{k=1}^{n} \cos(2\pi kn\alpha) + \mathcal{O}\Big(\sum_{\sqrt{N}<n\le N} \frac{1}{n^{s+1}||n\alpha||}\Big).\end{aligned}$$

In the last step, we have used the following fact: we have

$$\sum_{1\le k\le N/n} \cos(2\pi kn\alpha) = \mathcal{O}\Big(\frac{1}{|\sin(\pi n\alpha)|}\Big) = \mathcal{O}\Big(\frac{1}{||n\alpha||}\Big),$$

where the constants are absolute, so that

$$\sum_{\sqrt{N}<n\le N} \frac{1}{n^{s+1}} \sum_{1\le k\le N/n} \cos(2\pi kn\alpha) = \mathcal{O}\Big(\sum_{\sqrt{N}<n\le N} \frac{1}{n^{s+1}||n\alpha||}\Big).$$

We have thus obtained the equality

$$\widehat{S}_{s,N}(\alpha) = \Phi_{\lfloor\sqrt{N}\rfloor,s}(\alpha) + \mathcal{O}\Big(\sum_{\sqrt{N}<n\le N} \frac{1}{n^{s+1}||n\alpha||}\Big).$$

If $\alpha \in \mathscr{A}_s$, then $\Phi_{\lfloor\sqrt{N}\rfloor,s}(\alpha)$ converges to $\Phi_s(\alpha)$ while if $\alpha \in \mathscr{B}_s$, then the series $\sum_n \frac{1}{n^{s+1}||n\alpha||}$ is convergent by Kruse's inequality (4.5). Therefore, $\lim_N \widehat{S}_{s,N}(\alpha)$ exists and is equal to $\Phi_s(\alpha)$ for all irrational numbers $\alpha \in \mathscr{A}_s \cap \mathscr{B}_s = \mathscr{B}_s$ (at least).

• *Convergence of* $\widehat{S}_s$ *to* Φ_s *in* $L^2(0,1)$. It is clear that, for any $s \in (0,1]$,

$$|\widehat{\phi}_{s,k}| \le \frac{1}{k^{(s+1)/2}} \sum_{n|k} 1 \ll \frac{e^{\mathcal{O}(\log(k)/\log\log(k))}}{k^{(s+1)/2}}.$$

Hence $\sum_k \widehat{\phi}_{k,s}^2 < \infty$ and by the Riesz-Fischer Theorem the series $\widehat{S}_s$ converges in $L^2(0,1)$. Its L^2-sum is Φ_s because it converges to it also almost surely (again, by [15, p. 68, Theorem 3.12]).

• *Fourier coefficients of* Φ_s. Let $(c_{s,k})_{k\in\mathbb{Z}}$ and $(\widehat{c}_{s,k})_{k\in\mathbb{Z}}$ denote the Fourier coefficients of Φ_s and $\widehat{S}_s$ respectively (which are already known to be 0 for $k = 0$ and k odd). Since both functions belong to $L^2(0,1)$, the coefficients can be computed by the usual integrals, and by the Cauchy-Schwarz inequality, for any $k \in \mathbb{Z}$, $|c_{s,k} - \widehat{c}_{k,s}| \le ||\Phi_s - \widehat{S}_s||_2 = 0$. Hence, $\widehat{S}_s$ is the Fourier series of Φ_s.

7.2. **Proof of** (*ii*)**.** We now know that $\widehat{S}_{s,N}(\alpha) = S_{s,N}(\alpha)$ and we start again with the identity (7.1):

$$S_{s,N}(\alpha) = \sum_{1\le n\le\sqrt{N}} \frac{1}{n^{s+1}} \sum_{k=1}^{n} \cos(2\pi kn\alpha) + \sum_{\sqrt{N}<n\le N} \frac{1}{n^{s+1}} \sum_{1\le k\le N/n} \cos(2\pi kn\alpha).$$

We now specify $\alpha = a/b \in \mathbb{Q}$ with $\gcd(a,b) = 1$, $b \ge 1$.

We use again a computation done during the proof of Theorem 4, (*ii*):

$$\sum_{1\le k\le N/n} \cos\Big(2\pi kn\frac{a}{b}\Big) = \frac{N}{bn}\sum_{j=1}^{b} \cos\Big(2\pi jn\frac{a}{b}\Big) + \mathcal{O}(b).$$

Hence, using the fact that $\sum_{j=1}^{b}\cos(2\pi jn\frac{a}{b}) = 0$ and $= b$ according to whether $b \nmid n$ or not, we obtain that

$$\begin{aligned} S_{s,N}\Big(\frac{a}{b}\Big) &= \Phi_{\lfloor\sqrt{N}\rfloor,s}\Big(\frac{a}{b}\Big) + N\sum_{\substack{\sqrt{N}<n\le N\\ b\mid n}} \frac{1}{n^{s+2}} + \mathcal{O}\Big(b\sum_{\sqrt{N}<n\le N}\frac{1}{n^{s+1}}\Big) \qquad (7.2)\\ &= \frac{1}{b}H_{\lfloor\sqrt{N}\rfloor}(s)(1+o(1)) + \frac{1}{b(s+1)}N^{\frac{1-s}{2}}(1+o(1)) + \mathcal{O}(b) \end{aligned}$$

where the implicit constant is absolute.

For $s \in (0,1)$, $H_{\lfloor\sqrt{N}\rfloor}(s) \sim \frac{1}{1-s}N^{\frac{1-s}{2}}$ and therefore

$$\lim_{N\to+\infty} \frac{1}{H_{\lfloor\sqrt{N}\rfloor}(s)} S_{s,N}\Big(\frac{a}{b}\Big) = \frac{2}{b(1+s)}. \qquad (7.3)$$

For $s = 1$, the term $\frac{1}{b(s+1)}N^{\frac{1-s}{2}}(1+o(1))$ is $\mathcal{O}(1)$ and hence

$$\lim_{N\to+\infty} \frac{1}{H_{\lfloor\sqrt{N}\rfloor}} S_{1,N}\Big(\frac{a}{b}\Big) = \frac{1}{b}$$

and (7.3) also holds for $s = 1$. This completes the proof of (*ii*).

7.3. **Proof of** (*iii*)**.** Statement (*iii*) is proved by the same method as the one used to show that the sequence $\Phi_{1,N}$ does not converge for certain Liouville numbers: for this we use the approximations (5.2) and (7.2).

8. Numerical values

In this section, we present approximate values of the function $\Phi_1(\alpha)$ for some values of α, obtained using the freeware GP-Pari. For this, we need to find an upper bound for the speed of convergence of the sequence $\Phi_{s,N}(\alpha)$ to its limit. We were not bale to compute values of $\mathscr{G}_s(\alpha)$ despite of the fact that explicit bounds for the speed of convergence of $\mathscr{G}_{s,N}(\alpha)$ are available. We explain this in more detail below.

8.1. **Speed of convergence.**

Proposition 2. *(i) If $\alpha \in \mathscr{A}_1$ has a finite irrationality exponent $\mu(\alpha)$, we have*

$$\left|\Phi_1(\alpha) - \Phi_{1,N}(\alpha)\right| \leq \pi(1+\zeta(2)) \cdot (3+\sqrt{2}) \cdot \log(e \cdot c(\alpha)) \cdot \frac{1}{q_m} + \pi(1+\zeta(2)) \cdot \frac{2^{5/4}-1}{2^{1/4}-1} \cdot (\mu(\alpha)-1) \cdot \frac{\log(q_m)}{q_m}.$$

for any integer N such that $N \geq q_m$ with $m \geq 7$.

(ii) If $s \in (0,1)$, $\alpha \in \mathscr{A}_s$ has an irrationality exponent $\mu(\alpha) < 1 + \frac{1}{1-s}$ and $N \geq q_m \geq 3$, $\log(q_m) \leq q_m^{s/2}$ and $m \geq 2$, then

$$\left|\Phi_s(\alpha) - \Phi_{s,N}(\alpha)\right| \leq 2\pi\zeta(s+1) \cdot \frac{2^{s/4+1}-1}{2^{s/4}-1} \cdot \frac{\log(q_m)}{q_m^s} + \frac{9\pi/2}{1-s} \cdot \frac{1}{q_m} + \frac{\pi c(\alpha)^{1-s}}{1-s} \cdot \frac{2\sqrt{2}^{1-(\mu(\alpha)-1)(1-s)}-1}{\sqrt{2}^{1-(\mu(\alpha)-1)(1-s)}-1} \cdot \frac{1}{q_m^{1-(\mu(\alpha)-1)(1-s)}}$$

for any integer N such that $N \geq q_m$ with $m \geq 2$, $q_m \geq 3$ and $\log(q_m) \leq q_m^{s/2}$.

Proof. During the proof of Theorem 1(ii), we saw that $\left|\frac{\cos(\pi n(n+1)\alpha)\sin(\pi n^2\alpha)}{\sin(\pi n\alpha)}\right| \leq \frac{\pi}{2}\frac{||n^2\alpha||}{||n\alpha||}$ for any $n \geq 1$. Hence,

$$\left|\Phi_s(\alpha) - \Phi_{s,N}(\alpha)\right| \leq \sum_{n=N}^{\infty} \left|\frac{\cos\left(\pi n(n+1)\alpha\right)\sin\left(\pi n^2\alpha\right)}{n^{s+1}\sin\left(\pi n\alpha\right)}\right| \leq \frac{\pi}{2}\sum_{n=N}^{\infty}\frac{||n^2\alpha||}{n^{s+1}||n\alpha||}$$

and the conclusion follows by Proposition 1(iii) and (iv). □

The bounds given by Proposition 2 are good enough to provide a few digits of $\Phi_s(\alpha)$ with a computer for any given α and s and with N of a reasonable size. The situation is somewhat different for the computation of $\mathscr{G}_s(\alpha)$. It is possible to obtain explicit bounds for the difference $\left|\mathscr{G}_s(\alpha) - \mathscr{G}_{s,N}(\alpha)\right|$, using the fact that

$$\left|\mathscr{G}_s(\alpha) - \mathscr{G}_{s,N}(\alpha)\right| \leq \frac{2}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\left|\Phi_s\big((2k+1)\alpha\big) - \Phi_{s,N}\big((2k+1)\alpha\big)\right|.$$

We now use Proposition 2 to bound $\Phi_s\big((2k+1)\alpha\big) - \Phi_{s,N}\big((2k+1)\alpha\big)$ but we have to be careful that the bound depends on $q_m = q_m\big((2k+1)\alpha\big)$. There does not seem to exist general results providing a simple link between the sequences $(q_m(\alpha))_m$ and $(q_m(\ell\alpha))_m$, where ℓ is any positive integer. Therefore, we uniformize the bounds of Proposition 2 by means of the (already multiply used) inequality $q_m(\alpha) \geq 2^{(m-1)/2}$. We don't note the exact result which seems useless in practice because, for example, to compute the first digit of $\mathscr{G}_1(\sqrt{2})$ one needs to compute $\mathscr{G}_{1,259717522849}(\sqrt{2})$.

8.2. **Explicit irrationality measures.** Our remaining problem is to find numerical expressions of $\mu(\alpha)$ and $c(\alpha)$ for a given α. There is no general recipe. In the proposition below, we present a small collection of known explicite values of of $\mu(\alpha)$ and $c(\alpha)$ for interesting irrational numbers. The values of $\mu(\alpha)$ below are not necessarily the best known (see [11] for more recent results) but, usually, the authors of these refinements take $c(\alpha) = 1$ and state their results for large enough q, which is not the kind of result we need. In the case of algebraic numbers, bounds for $\mu(\alpha)$ close to Roth's result have been obtained, but at the cost of too large values of $c(\alpha)$.

Proposition 3. (*i*) *(Liouville's inequality* [11]*) Assume that α is a real algebraic irrational number of degree d, with minimal polynomial $P(X) = \sum_{j=0}^{d} c_j X^j \in \mathbb{Z}[X]$. For any rational number p/q, $q \geq 1$, we have $|\alpha - \frac{p}{q}| \geq \frac{1}{c(\alpha)q^{\mu(\alpha)}}$ with $\mu(\alpha) = d$ and $c(\alpha) = (|\alpha| + 1)^{d-1} \sum_{j=1}^{d} j|a_j|$.*

(*ii*) *(Baker's inequality* [1]*) For any rational number p/q with $q \geq 1$, we have $|\sqrt[3]{2} - \frac{p}{q}| \geq \frac{10^{-6}}{q^{2.955}}$. Hence we can take $\mu(\sqrt[3]{2}) = 2.955$ and $c(\sqrt[3]{2}) = 10^6$.*

(*iii*) *(Mignotte's inequalities* [10]*) For any rational number p/q with $q \geq 2$, we have $|\pi - \frac{p}{q}| \geq \frac{1}{q^{21}}$ and $|\pi^2 - \frac{p}{q}| \geq \frac{1}{q^{18}}$. Hence we can take $\mu(\pi) = 21$, $\mu(\pi^2) = 18$ and $c(\pi) = c(\pi^2) = 1$.*

(*iv*) *(Bundschuh's inequality* [3]*) For any rational number p/q with $q \geq 1$, we have $|e - \frac{p}{q}| \geq \frac{\log\log(4q)}{18q^2 \log(4q)}$. Hence we can take $\mu(e) = 2.1$ and $c(e) = 77$.*

Proof. For (*ii*) to (*iv*), we refer to the cited references. Statement (*i*) is very classical (in this or another form) and its proof is left to the reader. ☐

8.3. **Numerical values of $\Phi_1(\alpha)$.** In this section, we present approximations of various values of $\Phi_1(\alpha)$. For this, we computed $\Phi_{1,N}(\alpha)$ with GP-Pari with N and the reader will check using Propositions 2 and 3 that we get three, four or five significant digits as indicated.

α	$\Phi_1(\alpha)$	N
$\frac{\sqrt{5}-1}{2}$	-1.11153	2.05×10^9
$\sqrt{2}-1$	-1.08588	2.08×10^9
$\frac{\sqrt{7565}-53}{82}$	-1.08589	2.27×10^9
$\sqrt[3]{2}$	$-0.1419(0)$	3.39×10^8
e	$-0.3666(3)$	1.92×10^8
π	$0.357(10)$	3.20×10^8
π^2	$0.370(67)$	2.67×10^8

The digits between parentheses are stable for a long time before reaching the indicated values of N, but our bound does not prove that they are correct. The choice of the quadratic number $\frac{\sqrt{7565}-53}{82}$ is not arbitrary. Indeed, in the Lagrange spectrum, it is the number with the fifth smallest Lagrange constant: we have

$$L\left(\frac{\sqrt{5}-1}{2}\right) < L(\sqrt{2}-1) < L\left(\frac{\sqrt{221}-9}{14}\right) < L\left(\frac{\sqrt{1517}-23}{38}\right) < L\left(\frac{\sqrt{7565}-53}{82}\right)$$

(see [6, p. 10]). This order is not respected by Φ_1:

$$\Phi_1\Big(\frac{\sqrt{5}-1}{2}\Big) < \Phi_1\Big(\frac{\sqrt{7565}-53}{82}\Big) < \Phi_1\big(\sqrt{2}-1\big).$$

Furthermore, it seems that $\Phi_1(\alpha)$ is minimal on $[0,1]$ at $\alpha = \frac{\sqrt{5}-1}{2}$.

8.4. **Numerical values of $\mathscr{G}_s(\alpha)$.** We present approximations to four digits for a few values of $\mathscr{G}_1(\alpha)$; they are computed from $\mathscr{G}_{1,30000}(\alpha)$. However, we cannot garantee that even a single digit after the decimal point is correct, even though our computations in Pari-GP suggest this is the case. It seems that

$$\mathscr{G}_1\Big(\frac{\sqrt{5}-1}{2}\Big) = 0.2169\ldots,\ \mathscr{G}_1\big(\sqrt{2}-1\big) = 0.2103\ldots,\ \mathscr{G}_1\Big(\frac{\sqrt{7565}-53}{82}\Big) = 0.2105\ldots$$

and again we observe that the order of the Lagrange spectrum does not seem to be respected.

Finally, let us mention that the related series (amongst many others)

$$\sum_{n=1}^{\infty} \frac{\sin(2\pi n||n\alpha||)}{n^2 \sin(\pi ||n\alpha||)},\quad \sum_{n=1}^{\infty} \frac{\sin(2\pi n||n\alpha||)}{n^2||n\alpha||},\quad \sum_{n=1}^{\infty} \frac{||n^2\alpha|| \sin^2(\pi n^2\alpha)}{n^2||n\alpha||}$$

all seem to be extremal at $\alpha = \frac{\sqrt{5}-1}{2}$. The series $\sum_{n=1}^{\infty} \frac{||n^2\alpha||}{n^2||n\alpha||}$ seems to be minimal at $\sqrt{5}-2$; it is studied in detail in [14] along with the series $\sum_{n=1}^{\infty}(-1)^n \frac{||n^2\alpha||}{n^2||n\alpha||}$ which seems to be minimal at $\sqrt{2}/2$.

References

[1] A. Baker, *Rational approximations to $\sqrt[3]{2}$ and other algebraic numbers*, Quart. J. Math. Oxford Ser. (2) **15** (1964), 375–383.

[2] C. Baxa, *Comparing the distribution of $(n\alpha)$-sequences*, Acta Arith. **94** (2000), 345–363.

[3] P. Bundschuh, *Irrationalitätsmaße für e^a, $a \neq 0$ rational oder Liouville-Zahl*, Math. Ann. **192** (1971), 229–242.

[4] L. Carleson, *On convergence and growth of partial sums of Fourier series*, Acta Math. **116** (1966), 135–157.

[5] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, D. E. Knuth, *On the Lambert W function*, Adv. Computational Maths. **5** (1996), 329–359.

[6] T. W. Cusick, M. E. Flahive, *The Markoff and Lagrange spectra*, Mathematical Surveys and Monographs, **30**. American Mathematical Society, Providence, RI, 1989.

[7] H. Jager, J. de Jonge, *The circular dispersion spectrum*, J. Number Theory **49** (1994), no. 3, 360–384.

[8] A. Ya. Khintchine, *Continued fractions*, Dover Publications, 1997.

[9] A. H. Kruse, *Estimates of $\sum_{k=1}^{N} k^{-s}\langle kx\rangle^{-t}$*, Acta Arith. **12** (1966/1967), 229–261.

[10] M. Mignotte, *Approximations rationnelles de π et quelques autres nombres*, Journées Arithmétiques de Grenoble 1973, Bull. Soc. Math. France, Mem. **37** (1974), 121–132.

[11] N. I. Fel'dman, Yu. V. Nesterenko, *Transcendental numbers*, translated in English from the Russian by Neal Koblitz. Encyclopaedia of Mathematical Sciences **44**. Berlin: Springer.

[12] H. Niederreiter, *On a measure of denseness for sequences*, Topics in classical number theory, Vol. I, II (Budapest, 1981), 1163–1208, Colloq. Math. Soc. Janos Bolyai, **34**, North-Holland, Amsterdam, 1984.

[13] T. Rivoal, *On the distribution of multiples of real numbers*, preprint version (2010), 42 pages, available at http://hal.archives-ouvertes.fr/hal-00526980/fr/

[14] T. Rivoal, *Extremality properties of some diophantine series*, Experiment. Math. **19**.4 (2010), 481–494.

[15] W. Rudin, *Real and complex analysis*, third edition, McGraw-Hill international editions, 1987.

[16] G. Tenenbaum, *Introduction à la théorie analytique et probabiliste des nombres*, Deuxième édition. Cours Spécialisés, **1**. Société Mathématique de France, Paris, 1995.

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