IS EULER'S CONSTANT A VALUE OF AN ARITHMETIC SPECIAL FUNCTION?

TANGUY RIVOAL

Abstract. Euler's constant γ is one of the mathematical constants with the most different analytic representations, probably on par with π. Yet, none of these representations proves that γ is a value of an E-function, a G-function or an M-function at an algebraic point. In fact, it is plausible that no such representation of γ exists with these three arithmetic special functions, and thus the arithmetic nature of γ might not be determined by the powerful Diophantine theorems of Siegel-Shidlovsky, Chudnovsky and Nishioka. Nonetheless, we explain here why certain of these representations show that γ is not far from being a special value of these functions. We also present a new family of series summing to γ, which generalize an identity of Vacca.

1. *Representations of Euler's constant.*

The usual definition of Euler's constant is

$$\gamma := \lim_{k\to+\infty}\Big(\sum_{n=1}^{k}\frac{1}{n}-\log(k)\Big)=\sum_{n=1}^{\infty}\Big(\frac{1}{n}-\log\big(1+\frac{1}{n}\big)\Big), \tag{1}$$

where the second equality is a trivial reformulation of the first one. There exist a lot of seemingly different analytic representations of γ. We list below thirty three expressions, all equal to γ. See [4, 8, 10, 13, 16, 17, 18, 19, 26, 29, 15, 21, 36, 37, 38, 39, 40] for proofs, other identities and references.

$$\int_1^\infty\Big(\frac{1}{\lfloor x\rfloor}-\frac{1}{x}\Big)dx,\quad 1-\int_1^\infty\frac{x-\lfloor x\rfloor}{x^2}dx,\quad \int_0^\infty\frac{\log(1+x)}{x^2(\log(x)^2+\pi^2)}dx, \tag{2}$$

$$-\int_0^\infty\log(x)e^{-x}dx,\quad -\int_1^\infty\frac{\log\log(x)}{x^2}dx,\quad \int_0^1\Big(\frac{1}{\log(x)}+\frac{1}{1-x}\Big)dx, \tag{3}$$

$$\int_0^1\int_0^1\frac{x-1}{(1-xy)\log(xy)}dxdy,\ \int_0^\infty\frac{1}{x}\Big(\frac{1}{1+x^2}-\cos(x)\Big)dx,\quad \frac{ab}{a-b}\int_0^\infty\frac{e^{-x^a}-e^{x^b}}{x}dx, \tag{4}$$

$$\frac{1}{2}+\int_0^\infty\frac{2xdx}{(1+x^2)(e^{2\pi x}-1)},\quad \int_0^1\frac{1-e^x-e^{-1/x}}{x}dx,\quad \int_0^\infty\Big(\frac{1}{e^x-1}-\frac{1}{xe^x}\Big)dx, \tag{5}$$

$$\int_0^\infty\frac{1}{x}\Big(\frac{1}{1+x}-e^{-x}\Big)dx,\quad \int_0^1\frac{1}{x(1+x)}\sum_{n=1}^\infty x^{2^n}dx,\quad \int_0^1\frac{2+x}{x(1+x+x^2)}\sum_{n=1}^\infty x^{3^n}dx, \tag{6}$$

$$\sum_{n=1}^\infty\frac{|\int_0^1\binom{t}{n}dt|}{n},\quad \sum_{n=1}^\infty\frac{(-1)^{n-1}}{n!n}-\frac{1}{e}\int_0^\infty\frac{e^{-x}}{1+x}dx,\quad \sum_{n=2}^\infty\frac{1}{n^2}\Big(\sum_{k=1}^n(-1)^k\binom{n}{k}k\log(k)\Big), \tag{7}$$

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$$\sum_{n=1}^{\infty}(-1)^n\frac{\lfloor\log_2(n)\rfloor}{n}, \quad 1-\sum_{n=2}^{\infty}(-1)^n\frac{\lfloor\log_2(n)\rfloor}{n+1}, \quad 2\sum_{n=1}^{\infty}\frac{\cos(\frac{2\pi n}{3})\lfloor\log_3(n)\rfloor}{n}, \tag{8}$$

$$\frac{1}{2}+\sum_{n=1}^{\infty}\frac{\lfloor\log_2(2n)\rfloor}{2n(2n+1)(2n+2)}, \quad \frac{1}{2}+\sum_{n=1}^{\infty}\frac{(12n+6)\lfloor\log_3(3n)\rfloor}{3n(3n+1)(3n+2)(3n+3)}, \tag{9}$$

$$\log(\pi)-4\log\Gamma\Big(\frac{3}{4}\Big)+\frac{4}{\pi}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\log(2k+1)}{2k+1}, \quad \frac{1}{2}+\sum_{n=2}^{\infty}\frac{1}{2^{n+1}}\sum_{k=0}^{n-1}\frac{1}{\binom{2^{n-k}+k}{k}}, \tag{10}$$

$$\sum_{n=1}^{\infty}\frac{N_0(n)+N_1(n)}{2n(2n+1)}, \quad N_j(n)=\text{number of } j\text{'s in } n \text{ written in base 2}, \tag{11}$$

$$\sum_{n=1}^{\infty}\Big(\frac{1}{\lfloor\sqrt{n}\rfloor^2}-\frac{1}{n}-\frac{1}{n^2}\Big), \quad \frac{1}{s-1}\sum_{n=1}^{\infty}\Big(\frac{s}{n}-\sum_{j=n^s}^{(n+1)^s-1}\frac{1}{j}\Big), \quad s\geq 2, \tag{12}$$

$$\sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)}{n}, \quad 1+\sum_{n=2}^{\infty}\frac{1-\zeta(n)}{n}, \quad 1+\log\Big(\frac{2}{3}\Big)+\sum_{n=1}^{\infty}\frac{1-\zeta(2n+1)}{(2n+1)2^{2n}}, \tag{13}$$

$$\sum_{n=1}^{\infty}\frac{\rho(n)}{n}, \quad \rho(n)=-(2k-1) \text{ if } n=k^2,\ \rho(n)=1 \text{ otherwise}, \tag{14}$$

$$\sum_{n=2}^{\infty}\frac{\Lambda(n)-1}{n}, \quad \Lambda(n)=\log(p) \text{ if } n=p^k,\ \Lambda(n)=0 \text{ otherwise}. \tag{15}$$

Some of these expressions are merely variations of one another, but it is difficult to distinguish a common pattern because of the diversity of elementary functions involved: floor, square root, exponential, cosine, logarithm, zeta, digital, arithmetic functions, binomial coefficients, etc. Presumably, π is the only other classical constant with at least as many representations. However, contrary to π, the arithmetic nature of γ is still an open problem. Nobody knows whether it is a rational or an irrational number, an algebraic or a transcendental number, though γ is widely expected to be transcendental. Moreover, $\pi=\int_{-\infty}^{\infty}\frac{dx}{1+x^2}$ is a period; none of the above representations proves that γ is a period and in fact a folklore conjecture states that γ is not a period, though it is one of the simplest examples of an exponential period in the sense of Kontsevich-Zagier [20].

Despite very general transcendence results for various classes of *arithmetic* special functions, none of them is known to apply to γ. In the sequel, we explain that even though γ is conjecturally not a value at an algebraic point of these arithmetic special functions, certain representations hidden in (1)–(15) show that it is not very far from being one.

2. *Arithmetic special functions.*

It is not easy to define the meaning of "special functions" though there exist many books devoted to them, for instance [7]; a voluntarily imprecise definition might be "a mathematical function used for ages in some part of science and satisfying some sort of functional equation". In our Diophantine context, the arithmetic special functions we have in mind are of three types. Siegel defined E-functions and G-functions in [35]: these are series of the form $\sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$, respectively $\sum_{n=0}^{\infty} a_n z^n$, where the height of $a_n \in \overline{\mathbb{Q}}$ satisfies a certain growth condition (see [23]), and solutions of a differential equation $\sum_{j=0}^{d} p_j(z) y^{(j)}(z) = 0$ with coefficients $p_j(z) \in \overline{\mathbb{Q}}(z)$. Prototypical examples are the exponential and logarithmic functions respectively, and the intersection of both classes is reduced to polynomials functions. M-functions are power series $\sum_{n=0}^{\infty} a_n z^n$ with $a_n \in \overline{\mathbb{Q}}$, and solutions of a functional equation $\sum_{j=0}^{d} p_j(z) y(z^{\ell^j}) = 0$ for some integers $\ell \geq 2$ and $d \geq 1$, and $p_j(z) \in \overline{\mathbb{Q}}(z)$. A simple example is the series $\sum_{n=0}^{\infty} z^{2^n}$. They were first studied by Mahler [27] in a broader context. E-functions are entire and G-functions can be analytically continued to $\mathbb{C}$ minus a finite number of cuts; an M-function is either in $\overline{\mathbb{Q}}(z)$ or has the unit circle as natural boundary ([32, 9]) and moreover if it satisfies a linear differential equation, then it must be a in $\overline{\mathbb{Q}}(z)$ by [12].

The arithmetic nature of the values of E-functions at algebraic points have been studied for a long time: the Siegel-Shidlovsky Theorem [34] is the landmark result, with refinements in [3, 5, 6, 11, 28]. As a result, there exists now an algorithm which, for any given E-function $f(z)$, provides explicitly the finite list of algebraic numbers α such that $f(\alpha) \in \overline{\mathbb{Q}}$. Essentially the same things can be said of M-functions. An analogue of the Siegel-Shidlovsky Theorem was proved by Nishioka [30] and refined in [1, 2, 31]. Again, there exists now an algorithm which, for any given M-function $f(z)$, provides explicitly the list of algebraic numbers α such that $f(\alpha) \in \overline{\mathbb{Q}}$. On the other hand, the situation is not completely understood for G-functions. The best general Diophantine result is due to Chudnosvky [14], but we are far from having an algorithm to perfom the same task as for E-functions and M-functions.

Let us denote by $\mathbf{E}$, $\mathbf{G}$ and $\mathbf{M}$ the subsets of $\mathbb{C}$ of all the values taken by E-functions and (analytic continuation of) G-functions and M-functions at algebraic points, respectively. The first two sets have been introduced and studied in [23] and [24] respectively, see also [25]; in particular it was proved there that $\mathbf{E}$ and $\mathbf{G}$ are in fact subrings of $\mathbb{C}$. The set $\mathbf{M}$ does not seem to have been considered so far but it is easy to prove that it is a ring as well. As said above, none of the above Diophantine results has been applied so far to γ because this number is not known to be in either $\mathbf{E}$ or $\mathbf{G}$ or $\mathbf{M}$.

In fact, given the long list of series and integral expressions for γ, one can even argue that if it were such a value, this would be known for a long time. Hence it is plausible that γ is not in $\mathbf{E} \cup \mathbf{G} \cup \mathbf{M}$ and in this sense, the answer to the title of this note is negative. Moreover, for similar reasons it is believed that $\mathbf{E} \cap \mathbf{G} = \overline{\mathbb{Q}}$, $\mathbf{E} \cap \mathbf{M} = \overline{\mathbb{Q}}$ and $\mathbf{G} \cap \mathbf{M} = \overline{\mathbb{Q}}$. Let us also say that it is also believed no Liouville number is in $\mathbf{E} \cup \mathbf{G} \cup \mathbf{M}$, but it would certainly be surprising if γ were itself a Liouville number.

4. Extension of arithmetic sets.

We claim that γ is in some sense close to be in $\mathbf{E} \cap \mathbf{G} \cap \mathbf{M}$. Of course, this statement is paradoxical because this intersection is expected to contain only algebraic numbers, while γ is expected to be transcendental. We mean the following: we can "slightly" extend the sets $\mathbf{E}$ and $\mathbf{M}$ to certain sets $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{M}}$ respectively so that $\gamma \in \widetilde{\mathbf{E}} \cap \widetilde{\mathbf{M}}$. The extension process is the same for both classes of functions, which adds value to this observation. The process can be applied to $\mathbf{G}$ but we get $\widetilde{\mathbf{G}} = \mathbf{G}$. However, we shall prove the non-trivial fact that $\mathbf{G} \subset \widetilde{\mathbf{E}}$, so that $\widetilde{\mathbf{E}}$ can be viewed as a common extension of both $\mathbf{E}$ and $\mathbf{G}$.

We define $\widetilde{\mathbf{E}}$, respectively $\widetilde{\mathbf{M}}$, as the set of all convergent evaluations at $z \in \overline{\mathbb{Q}} \cup \{\infty\}$ of the twisted primitives

$$\int_0^z Q(x)F(x)dx \tag{16}$$

where $Q(z) \in \overline{\mathbb{Q}}(z)$ and $F(z)$ is an E-function, respectively an M-function, such that QF is holomorphic at $z = 0$. Changing the variable z to αz for any algebraic number, we can in fact assume that $z \in \{1, \infty\}$ in (16). The drawback of this voluntarily simple definition is that we do not know if $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{M}}$ carry an algebraic structure. Of course, we get two $\overline{\mathbb{Q}}$-vector spaces by considering the *a priori* larger sets of numbers $\int_\alpha^\beta Q(x)F(x)dx$ where $\alpha, \beta \in \overline{\mathbb{Q}} \cup \{\infty\}$ and F is an E-function, respectively an M-function.

We have $\mathbf{E} \subset \widetilde{\mathbf{E}}$ and $\mathbf{M} \subset \widetilde{\mathbf{M}}$ because if $F(z)$ is an E-function, respectively an M-function, then $F'(z)$ is an E-function, respectively an M-function, and $F(z) = \int_0^z F'(x)dx$. It is likely however that $\mathbf{E} \neq \widetilde{\mathbf{E}}$ and $\mathbf{M} \neq \widetilde{\mathbf{M}}$, because when $F(z)$ is an E-function or an M-function, then an integral of the form (16) is in general neither an E-function nor an M-function. Taking $F(z) = 1$ and $Q(z) = \frac{1}{1+z}$, we see that $\log(\overline{\mathbb{Q}}^*) \subset \widetilde{\mathbf{E}} \cap \widetilde{\mathbf{M}}$. Values at algebraic points of iterated primitives vanishing at 0 of E-functions and M-functions are in $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{M}}$ respectively because for any integer $p \geq 1$

$$\Big(\int_0^z\Big)^p F(x)dx = \frac{(-1)^{p-1}}{(p-1)!}\int_0^z (x-z)^{p-1}F(x)dx. \tag{17}$$

But since the class of E-functions is stable by $\frac{d}{dz}$ and $\int_0^z$, we get nothing more than $\mathbf{E}$ if F is an E-function in (17). On the other hand, F is an M-function if and only if QF is an M-function, so that $\widetilde{\mathbf{M}}$ is exactly the set of values at algebraic points of primitives of M-functions vanishing at 0.

Much more interesting is the property that $\gamma \in \widetilde{\mathbf{E}} \cap \widetilde{\mathbf{M}}$. Indeed, by (6), we have

$$\gamma = \int_0^\infty \frac{1-(1+x)e^{-x}}{x(1+x)}dx \in \widetilde{\mathbf{E}} \tag{18}$$

$$= \int_0^1 \frac{1}{x(1+x)}\sum_{n=1}^\infty x^{2^n}dx \in \widetilde{\mathbf{M}} \tag{19}$$

$$= \int_0^1 \frac{2+x}{x(1+x+x^2)}\sum_{n=1}^\infty x^{3^n}dx \in \widetilde{\mathbf{M}} \tag{20}$$

because $1-(1+z)e^{-z}$ is an E-function and $\sum_{n=1}^\infty z^{2^n}, \sum_{n=1}^\infty z^{3^n}$ are M-functions. Eq. (18) is apparently due to Dirichlet, while Eqs. (19) and (20) are due to Catalan and Ramanujan, respectively. All the series in (8) and (9) can be transformed in a straighforward manner as integrals $\int_0^1 Q(x)F(x)dx$ for some $Q(z) \in \overline{\mathbb{Q}}(z)$ and some M-function $F(z)$. Moreover, the series (11) is just a rewritting of the first series in (8).

With the algebraic numbers, the numbers π and $\zeta(n)$ (n any integer ≥ 2) are amongst the simplest elements of $\mathbf{G}$. So far, no simple relation between γ and G-functions is known, though γ is often considered to be the regularized value of the divergent $\zeta(1)$. The "tilde" extension presented here can also be applied to G-functions but if $F(z)$ is a G-function then $\int_0^z Q(x)F(x)dx$ is still a G-function and a convergent integral $\int_0^\alpha Q(x)F(x)dx$ with $\alpha \in \overline{\mathbb{Q}} \cup \{\infty\}$ is in $\mathbf{G}$. Hence $\widetilde{\mathbf{G}} = \mathbf{G}$. However, we have the following property.

Proposition 1. *We have* $\mathbf{G} \subset \widetilde{\mathbf{E}}$.
More precisely, $\mathbf{G}$ *coincides with the set of all convergent integrals* $\int_0^\infty F(x)dx$ *of* E*-functions* F*. Equivalently,* $\mathbf{G}$ *coincides with the set of all finite limits of* E*-functions at infinity (along some direction).*

Proof. From the stability of the class of E-function by $\frac{d}{dz}$ and $\int_0^z$, we deduce that the set of convergent integrals $\int_0^\infty F(x)dx$ of E-functions and the set of finite limits of E-functions along some direction as $z \to \infty$ are the same.

Theorem 2(iii) in [24] implies that if an E-function has a a finite limit as $z \to \infty$ along some direction, then this limit must be in $\mathbf{G}$.

Conversely, let $\beta \in \mathbf{G}$. By Theorem 1 in [23], there exists a G-function $G(z) = \sum_{n=0}^\infty a_n z^n$ of radius of convergence ≥ 2 (say) such that $G(1) = \beta$. Let $F(z) = \sum_{n=0}^\infty \frac{a_n}{n!} z^n$ be the associated E-function. Then for any z such that $\mathrm{Re}(z) > \frac{1}{2}$, we have

$$\frac{1}{z}G\Big(\frac{1}{z}\Big) = \int_0^{+\infty} e^{-xz}F(x)dx.$$

Hence, $\beta = \int_0^{+\infty} e^{-x}F(x)dx$. Now, $e^{-z}F(z)$ is an E-function, so that $\beta \in \widetilde{\mathbf{E}}$. □

This proposition shows how difficult it would be to obtain a Siegel-Shidlovsky like result for the elements of $\widetilde{\mathbf{E}}$. A generalization of Nishioka's Theorem to elements of $\widetilde{\mathbf{M}}$ would probably be difficult to obtain as well. On the other hand, it is not formally necessary to prove such general theorems if one is only interested in γ, as *ad hoc* Diophantine arguments might be simpler to obtain for the integrals (18), (19) or (20).

The number $\log(4/\pi)$ is sometimes named "the alternating Euler's constant" because [37]

$$\log(4/\pi) = \sum_{n=1}^\infty (-1)^{n-1}\Big(\frac{1}{n} - \log\big(1 + \frac{1}{n}\big)\Big).$$

(Compare with (1).) The identity [15]

$$\log(4/\pi) = \int_0^1 \frac{x^2 - x - 1}{1 - x^2} \sum_{n=0}^\infty \frac{(1 - x^{2^n})x^{2^n - 1}}{1 + x^{2^n}} dx$$

proves that $\log(4/\pi) \in \widetilde{\mathbf{M}}$. There exist other representations similar to certain for γ, for instance [33, 37, 4]:

$$\begin{aligned}\log(4/\pi) &= \int_0^1 \frac{1-x}{1+x}\Big(\frac{1}{\log(x)}+\frac{1}{1-x}\Big)dx\\ &= \int_0^1\int_0^1 \frac{x-1}{(1+xy)\log(xy)}dxdy\\ &= \sum_{n=1}^{\infty}\frac{N_0(n)-N_1(n)}{2n(2n+1)},\end{aligned}$$

but it does not seem to be known whether $\log(4/\pi) \in \widetilde{\mathbf{E}}$ or not.

Finally, let us mention that arithmetic special functions could also include "q-analogues". It is proved in [22] that

$$\gamma = \lim_{q\to 1^+}\left(\sum_{n=1}^{\infty}\Big(\frac{1-q}{1-q^n}\Big)+\frac{(q-1)\log(q-1)}{\log(q)}+\frac{1-q}{2}\right)$$

which is a q-analogue of the definition of γ stated in (1). But no evaluation of a q-series or a q-integral in terms of γ seems to be known.

5. Other extensions.

The form of the integral in (16) also suggests to define a subset $\widetilde{\mathbf{EGM}}$ of $\mathbb{C}$ as follows: it is the set of all convergent integrals

$$\int_0^{\alpha} e(z)g(z)m(z)dz$$

where $\alpha \in \{1,\infty\}$, and $e(z)$ is E-fonction, $g(z)$ is a G-function $g(z)$ and $m(z)$ is an M-function. Since $\frac{1-(1+z)e^{-z}}{z}$ is an E-function, $\frac{1}{1+z}$ is a G-function (and an M-function as well), and $\sum_{n=1}^{\infty} z^{2^n-1}$ is an M-function, (6) shows in many different ways that $\gamma \in \widetilde{\mathbf{EGM}}$. However, this set is quite big because it contains periods, exponential periods, $\mathbf{E}$, $\mathbf{G}$, $\mathbf{M}$, $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{M}}$. This likely makes $\widetilde{\mathbf{EGM}}$ very difficult to study from a Diophantine point of view.

To get other constants than γ without extending too much $\mathbf{E}$, $\mathbf{G}$ or $\mathbf{M}$, we consider the twisted primitives

$$\int_0^z Q(x)F(x)dx$$

where $Q(z)$ is now an algebraic function over $\overline{\mathbb{Q}}(z)$ and $F(z)$ is either an E-function or an M-function such that QF is integrable at $z=0$. When $F(z)$ is an E-function, respectively an M-function, any convergent integral $\int_0^{\alpha} Q(x)F(x)dx$, with $\alpha \in \{1,\infty\}$, defines an element of $\widetilde{\widetilde{\mathbf{E}}}$ and $\widetilde{\widetilde{\mathbf{M}}}$. Again, the "double tilde" extension is not interesting for G-functions and $\widetilde{\widetilde{\mathbf{G}}} = \mathbf{G}$, but obviously we still have $\mathbf{G} \subset \widetilde{\widetilde{\mathbf{E}}}$.

For instance, for any rational number $s>0$, we have

$$\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx \in \widetilde{\widetilde{\mathbf{E}}}$$

and [10]

$$\Psi(s) := \frac{\Gamma'(s)}{\Gamma(s)} = \int_0^1 \left(\frac{x^{s-1}}{1-x} - \frac{2x^{2s-2}}{1-x^2}\right) \sum_{n=0}^\infty x^{2^n} dx \in \widetilde{\widetilde{\mathbf{M}}}.$$

However, it does not seem to be known whether $\Gamma(s) \in \widetilde{\widetilde{\mathbf{M}}}$ and $\Psi(s) \in \widetilde{\widetilde{\mathbf{E}}}$.

We don't know if $\widetilde{\mathbf{EGM}}$, $\widetilde{\widetilde{\mathbf{E}}}$ and $\widetilde{\widetilde{\mathbf{M}}}$ carry any interesting algebraic structure. We get three $\overline{\mathbb{Q}}$-vector spaces by considering the set of convergent integrals $\int_\alpha^\beta e(z)g(z)m(z)dz$, where $\alpha, \beta \in \overline{\mathbb{Q}} \cup \{\infty\}$, e, g, m are an E-fonction, a G-function and an M-function respectively, as well as the two sets of convergent integrals $\int_\alpha^\beta Q(x)F(x)dx$ where $\alpha, \beta \in \overline{\mathbb{Q}} \cup \{\infty\}$, Q is algebraic over $\overline{\mathbb{Q}}(z)$ and F an E-function, respectively an M-function.

6. A generalization of Vacca's second identity.

Vacca [39] proved in a geometric way the identity

$$\gamma = \sum_{n=1}^\infty n\Big(\sum_{k=2^n}^{2^{n+1}-1} \frac{(-1)^k}{k}\Big),$$

which is a variant of the first identity displayed in (8). Analytic proofs were quickly found by Glaisher [17] and Hardy [18], the latter showing that Vacca's identity is also a variant of Catalan's integral [13] $\gamma = \int_0^1 \frac{1}{x(1+x)} \sum_{n=1}^\infty x^{2^n} dx$, displayed in (6).

Later on, Vacca proved in [40] a second identity, stated in (12), namely

$$\gamma + \zeta(2) = \sum_{n=1}^\infty \Big(\frac{1}{\lfloor\sqrt{n}\rfloor^2} - \frac{1}{n}\Big). \tag{21}$$

Because the paper [40] might be difficult to find, for the reader's convenience, we present a proof of the following generalization of (21).

Proposition 2. *For any integer $s \geq 2$, we have*

$$\gamma = \frac{1}{s-1} \sum_{n=1}^\infty \Big(\frac{1}{\lfloor n^{1/s}\rfloor^s} - \frac{1}{n} - \sum_{\ell=2}^s \frac{\binom{s}{\ell}}{n^\ell}\Big). \tag{22}$$

Proof. Indeed, we have

$$\sum_{n=1}^{N}\Big(\frac{1}{\lfloor n^{1/s}\rfloor^s}-\frac{1}{n}\Big)$$

$$=\sum_{j=1}^{\lfloor N^{1/s}\rfloor}\sum_{n=j^s}^{(j+1)^s-1}\frac{1}{j^s}-\sum_{n=1}^{N}\frac{1}{n}=\sum_{j=1}^{\lfloor N^{1/s}\rfloor}\frac{(j+1)^s-j^s}{j^s}-\sum_{n=1}^{N}\frac{1}{n}$$

$$=s\sum_{j=1}^{\lfloor N^{1/s}\rfloor}\frac{1}{j}-\sum_{n=1}^{N}\frac{1}{n}+\sum_{\ell=2}^{s}\binom{s}{\ell}\sum_{j=1}^{\lfloor N^{1/s}\rfloor}\frac{1}{j^\ell}$$

$$=(s-1)\gamma+s\log(\lfloor N^{1/s}\rfloor)-\log(N)+\mathcal{O}\big(\frac{1}{N^{1/s}}\big)+\sum_{\ell=2}^{s}\binom{s}{\ell}\sum_{j=1}^{\lfloor N^{1/s}\rfloor}\frac{1}{j^\ell},\ N\to\infty$$

$$=(s-1)\gamma+\sum_{\ell=2}^{s}\binom{s}{\ell}\zeta(\ell)+o(1),\ N\to\infty$$

where we used $\sum_{1\le j\le x}\frac{1}{j}=\gamma+\log(x)+\mathcal{O}(\frac{1}{x})$ as $x\to+\infty$. Eq. (22) follows. □

The convergence is slow because $\frac{1}{\lfloor n^{1/s}\rfloor^s}-\frac{1}{n}=\mathcal{O}(\frac{1}{n^{1+1/s}})$ and the exponent can not be improved.

A similar argument enables one to sum the series

$$\sum_{n=1}^{\infty}\Big(\frac{n^{t-1}}{\lfloor n^{1/s}\rfloor^{st}}-\frac{1}{n}\Big)$$

for any integers $s\ge 2$ and $t\ge 1$. The formula is slightly more complicated to write down explicitly. We use the summation formula

$$\sum_{n=j^s}^{(j+1)^s-1}n^{t-1}=\frac{1}{t}\Big(P_t\big((j+1)^s\big)-P_t(j^s)\Big)=:\sum_{m=0}^{ts-1}\alpha_{m,s,t}j^m$$

where $P_k(X)$ is the k-th Bernoulli polynomial, and the $\alpha_{m,s,t}\in\mathbb{Q}$ could be explicited in terms of the Bernoulli numbers and binomial coefficients. We then get

$$\sum_{n=1}^{\infty}\Big(\frac{n^{t-1}}{\lfloor n^{1/s}\rfloor^{st}}-\frac{1}{n}\Big)=(s-1)\gamma+\sum_{m=0}^{st-2}\alpha_{m,s,t}\zeta(st-m).$$

The case $s=t=2$ reads

$$\sum_{n=1}^{\infty}n\Big(\frac{1}{\lfloor\sqrt{n}\rfloor^4}-\frac{1}{n^2}\Big)=\gamma+3\zeta(2)+\zeta(3)$$

or

$$\gamma=\sum_{n=1}^{\infty}n\Big(\frac{1}{\lfloor\sqrt{n}\rfloor^4}-\frac{1}{n^2}-\frac{3}{n^3}-\frac{1}{n^4}\Big).$$

None of the identities in this section seems to express γ as a value of an arithmetic special function.

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T. Rivoal, Institut Fourier, CNRS et Université Grenoble Alpes, CS 40700, 38058 Grenoble cedex 9, France