Arithmetic theory of E-operators

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E-functions

We fix an embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}$.

Definition 1

An E-function is a formal power series $E(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$ such that $a_n \in \overline{\mathbb{Q}}$ and there exists $C > 0$ such that:

(*i*) *the maximum of the moduli of the conjugates of a_n is $\leq C^{n+1}$ for any n.*

(*ii*) *there exists a sequence of rational integers d_n, with $|d_n| \leq C^{n+1}$, such that $d_n a_m$ is an algebraic integer for all $m \leq n$.*

(*iii*) *$E(z)$ satisfies a homogeneous linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$.*

A G-function $\sum_{n=0}^{\infty} a_n z^n$ is defined similarly.

Three sets of numbers related to E and G-functions

Definition 2

(*i*) *The set* **E** *is the set of all the values taken at algebraic points by E-functions.*

It is a ring. Its group of units contains $\overline{\mathbb{Q}}^*$ *and* $\exp(\overline{\mathbb{Q}})$.

(*ii*) *The set* **G** *is the set of all the values taken at algebraic points by (analytic continuation of) G-functions.*

It is a ring. Its group of units contains $\overline{\mathbb{Q}}^*$ *and the Beta values* $B(\mathbb{Q}, \mathbb{Q})$.

(*iii*) *The set* **S** *is the module generated over* **G** *by all the values of derivatives of the Gamma function at rational points.*

It is also the module generated over $\mathbf{G}[\gamma]$ *by all the values of* Γ *at rational points, where* γ *is Euler's constant.*

It is a ring.

E-operators

Definition 3 (André, 2000)

A differential operator $L \in \overline{\mathbb{Q}}[x, \frac{d}{dx}]$ *is an* E*-operator if the operator* $M \in \overline{\mathbb{Q}}[z, \frac{d}{dz}]$ *obtained from* L *by formally changing*

$$x \to -\frac{d}{dz}, \qquad \frac{d}{dx} \to z \qquad (\textbf{Fourier-Laplace transform of } L)$$

is a G*-operator, i.e.* $My(z) = 0$ *has at least one* G*-function solution for which it is minimal.*

Motivation: Given an E-function $E(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$, there exists an E-operator L, of order μ say, such that $LE(x) = 0$. Moreover, let

$$g(z) = \int_0^{\infty} E(x) e^{-xz} dx = \sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}} \qquad (\textbf{Laplace transform of } E).$$

Then

$$\left(\left(\frac{d}{dz} \right)^{\mu} \circ M \right) g(z) = 0.$$

Basis of solutions of L at $z = 0$

Theorem 1 (André 2000)

(*i*) *An E-operator has at most 0 and ∞ as singularities: 0 is always a regular singularity, while ∞ is an irregular one in general.*

(*ii*) *An E-operator L of order μ has a basis of solutions at $z = 0$ of the form*

$$(E_1(z), \dots, E_\mu(z)) \cdot z^M$$

where M is an upper triangular $\mu \times \mu$ matrix with coefficients in $\mathbb{Q}$ and the $E_j(z)$ are E-functions.

Any local solution $F(z)$ of $Ly(z) = 0$ at $z = 0$ is of the form

$$F(z) = \sum_{j=1}^{\mu} \Big(\sum_{s \in S_j} \sum_{k \in K_j} \phi_{j,s,k} z^s \log(z)^k \Big) E_j(z) \tag{1}$$

where $S_j \subset \mathbb{Q}$, $K_j \subset \mathbb{N}$ are finite and $\phi_{j,s,k} \in \mathbb{C}$.

Interesting case for us: $\phi_{j,s,k} \in \overline{\mathbb{Q}}$.

Connection constants at finite distance

Let $F(z)$ be a local solution of L at $z = 0$, of the form given in (**??**).

Any point $\alpha \in \overline{\mathbb{Q}} \setminus \{0\}$ is a regular point of L.

There exists a basis of local solutions $F_1(z), \ldots, F_\mu(z) \in \overline{\mathbb{Q}}[[z - \alpha]]$, holomorphic around $z = \alpha$, such that

$$F(z) = \omega_1 F_1(z) + \cdots + \omega_\mu F_\mu(z) \tag{2}$$

where $\omega_1, \ldots, \omega_\mu$ are **connection constants**.

Theorem 2 (Fischler-R, 2014)

If $\phi_{j,s,k} \in \overline{\mathbb{Q}}$ *in* (**??**), *then* $\omega_1, \ldots, \omega_\mu$ *belong to* $\mathbf{E}[\log \alpha]$, *and even to* $\mathbf{E}$ *if* $F(z)$ *is an* E*-function.*

Proof: Differentiate $\mu - 1$ times (**??**) to construct a $\mu \times \mu$ linear system with the ω_j's as unknown. Solve it at $z = \alpha$ using the wronskian built on the F_j's (Cramer's rule). Use in particular the fact that, by André's result on singularities of E-operators, the wronskian $= cz^\rho e^{\beta z}$ with $c \in \overline{\mathbb{Q}}^*$, $\rho \in \mathbb{Q}$ and $\beta \in \overline{\mathbb{Q}}$.

Basis of solutions of L at $z = \infty$

The situation is more complicated because of divergent series and of Stokes' phenomenon.

Let $\theta \in [0, 2\pi)$ not in some explicit finite set which contains the anti-Stokes directions. We have a generalized asymptotic expansion

$$E(z) \sim \sum_{\rho \in \Sigma} e^{\rho z} \sum_{\alpha \in S} z^{\alpha} \sum_{i \in T} \log(z)^i \sum_{n=0}^{\infty} \frac{c_{\theta,\rho,\alpha,i}(n)}{z^n} \tag{3}$$

as $|z| \to \infty$ in a large angular sector bisected by $\{z : \arg(z) = \theta\}$.

The sets $\Sigma \subset \overline{\mathbb{Q}}$, $S \subset \mathbb{Q}$ and $T \subset \mathbb{N}$ are finite, and $c_{\theta,\rho,\alpha,i}(n) \in \mathbb{C}$.

We have found a new explicit construction of (**??**) by deforming the integral

$$E(x) = \frac{1}{2i\pi} \int_L g(z) e^{zx} dz \qquad (L \text{ ``vertical''}).$$

The series $\sum_{n=0}^{\infty} c_{\theta,\rho,\alpha,i}(n) z^{-n}$ in (**??**) are divergent, but

$$\sum_{n=0}^{\infty} \frac{1}{n!} c_{\theta,\rho,\alpha,i}(n) z^{n}$$

are finite linear combinations of G-functions.

André (2000): Construction of a special basis $H_1(z), \ldots, H_\mu(z)$ of formal solutions at infinity of the E-operator L that annihilates $E(z)$. The H_k's involve series like in (**??**) but with coefficients in $c_k\overline{\mathbb{Q}}$ for some c_k.

The asymptotic expansion (**??**) of $E(z)$ in a large sector bisected by $\{z : \arg(z) = \theta\}$ can be rewritten with this basis as

$$\omega_{\theta,1} H_1(z) + \cdots + \omega_{\theta,\mu} H_\mu(z) \tag{4}$$

with **Stokes' constants** $\omega_{\theta,k}$.

When θ "crosses" one of the anti-Stokes directions, the values of the $\omega_{\theta,k}$ may change . This is the Stokes phenomenon.

Stokes' constants at infinity

Setting:

$$E(z) \sim \omega_{\theta,1} H_1(z) + \cdots + \omega_{\theta,\mu} H_\mu(z)$$
$$\sim \sum_{\rho\in\Sigma} e^{\rho z} \sum_{\alpha\in S} z^\alpha \sum_{i\in T} \log(z)^i \sum_{n=0}^{\infty} \frac{c_{\theta,\rho,\alpha,i}(n)}{z^n}.$$

Theorem 3 (Fischler-R, 2014)

Let $\theta \in [0, 2\pi)$ be a direction not in some explicit finite set. Then:

(*i*) *The Stokes constants $\omega_{\theta,k}$ belong to* **S**.

(*ii*) *All the coefficients $c_{\theta,\rho,\alpha,i}(n)$ belong to* **S**.

(*iii*) *Let $F(z)$ be a local solution at $z = 0$ of L, with $\phi_{j,s,k} \in \overline{\mathbb{Q}}$ in* (**??**). *Then Assertions (i) and (ii) hold with $F(z)$ instead of $E(z)$.*

Applications to E-approximations

Definition 4

Sequences (P_n) and (Q_n) of algebraic numbers are said to form E-approximations of $\alpha \in \mathbb{C}$ if

$$\lim_{n \to +\infty} \frac{P_n}{Q_n} = \alpha$$

and

$$\sum_{n=0}^{\infty} P_n z^n = a(z) \cdot E\big(b(z)\big), \quad \sum_{n=0}^{\infty} Q_n z^n = c(z) \cdot F\big(d(z)\big)$$

where E and F are E-functions, and a, b, c, d are algebraic functions in $\overline{\mathbb{Q}}[[z]]$ with $b(0) = d(0) = 0$.

Diophantine motivation: Many sequences of algebraic approximations of classical numbers are E-approximations. For instance diagonal Padé approximants to $\exp(z)$ evaluated at z algebraic, and in particular the convergents to e.

The set of E-approximable numbers

Given two subsets X and Y of $\mathbb{C}$, let

$$X \cdot Y = \{xy \mid x \in X, y \in Y\}, \quad \frac{X}{Y} = \left\{\frac{x}{y} \,\middle|\, x \in X, y \in Y \setminus \{0\}\right\}.$$

Theorem 4 (Fischler-R, 2014)

The set of numbers having E-approximations contains

$$\frac{\mathbf{E} \cup \Gamma(\mathbb{Q})}{\mathbf{E} \cup \Gamma(\mathbb{Q})} \cup \mathrm{Frac}\,\mathbf{G} \tag{5}$$

and it is contained in

$$\frac{\mathbf{E} \cup (\Gamma(\mathbb{Q}) \cdot \mathbf{G})}{\mathbf{E} \cup (\Gamma(\mathbb{Q}) \cdot \mathbf{G})} \cup \Big(\Gamma(\mathbb{Q}) \cdot \exp(\overline{\mathbb{Q}}) \cdot \mathrm{Frac}\,\mathbf{G}\Big). \tag{6}$$

Proof of (**??**): Explicit constructions.

Proof of (**??**): Saddle point method, singularity analysis, and Theorems **??** and **??** because E-approximable numbers appear either as connection constants or as Stokes' constants.

E-approximations of Gamma values

Let

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!(n+\alpha)}, \quad \alpha \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$$

and define $P_n(\alpha)$ by

$$\frac{1}{(1-z)^{\alpha+1}} E_\alpha\left(-\frac{z}{1-z}\right) = \sum_{n=0}^{\infty} P_n(\alpha) z^n \in \mathbb{Q}[[z]].$$

Then,

$$P_n(\alpha) = \sum_{k=0}^{n} \binom{n+\alpha}{k+\alpha} \frac{(-1)^k}{k!(k+\alpha)} \longrightarrow \Gamma(\alpha) \qquad \text{if } \alpha < 1.$$

The number $\Gamma(\alpha)$ appears as a Stokes constant in the expansion

$$E_\alpha(-z) \sim \frac{\Gamma(\alpha)}{z^\alpha} - e^{-z} \sum_{n=0}^{\infty} (-1)^n \frac{(1-\alpha)_n}{z^{n+1}}.$$

What about Euler's constant $\gamma = -\Gamma'(1)$?

We conjecture that γ does not have E-approximations. However, let

$$E(z) = \sum_{n=1}^{\infty} \frac{z^n}{n!n}$$

and define the sequence (P_n) by

$$-\frac{1}{1-z} E\left(-\frac{z}{1-z}\right) + \frac{\log(1-z)}{1-z} = \sum_{n=0}^{\infty} P_n z^n \in \mathbb{Q}[[z]].$$

Then

$$P_n = \sum_{k=1}^{n} (-1)^k \binom{n}{k} \frac{1}{k} \left(1 - \frac{1}{k!}\right) \longrightarrow \gamma.$$

Again, γ appears as a Stokes' constant in the asymptotic expansion

$$E(-z) \sim -\gamma - \log(z) - e^{-z} \sum_{n=0}^{\infty} (-1)^n \frac{n!}{z^{n+1}}.$$

Linear recurrences

$$
\begin{aligned}
(n+3)(n+3+\alpha)P_{n+3}(\alpha) \\
-(3n^2+4n\alpha+14n+\alpha^2+9\alpha+17)P_{n+2}(\alpha) \\
+(3n+5+2\alpha)(n+2+\alpha)P_{n+1}(\alpha) \\
-(n+2+\alpha)(n+1+\alpha)P_n(\alpha) = 0
\end{aligned}
$$

with $P_0(\alpha) = \frac{1}{\alpha}$, $P_1(\alpha) = \frac{1+\alpha+\alpha^2}{\alpha(\alpha+1)}$ and $P_2(\alpha) = \frac{4+5\alpha+6\alpha^2+4\alpha^3+\alpha^4}{2\alpha(\alpha+1)(\alpha+2)}$.

$$
\begin{aligned}
(n+3)^2P_{n+3} - (3n^2+14n+17)P_{n+2} \\
+(3n+5)(n+2)P_{n+1} - (n+2)(n+1)P_n = 0
\end{aligned}
$$

with $P_0 = 0$, $P_1 = 0$ and $P_2 = \frac{1}{4}$.

Connection constants for G-functions

Let $G(z)$ be a G-function solution of the minimal differential equation $My(z) = 0$ of order μ.

By a deep theorem due to André, Chudnovskii and Katz, at any point $\alpha \in \overline{\mathbb{Q}} \cup \{\infty\}$, there is a basis of solutions $G_1(z), \dots, G_\mu(z)$ of $My(z) = 0$, locally holomorphic in a slit neighbourhood of α, which are (essentially) G-functions of $z - \alpha$ or $1/z$.

Locally around $\alpha \in \overline{\mathbb{Q}} \cup \{\infty\}$, we have

$$G(z) = \omega_1 G_1(z) + \cdots + \omega_\mu G_\mu(z).$$

Theorem 5 (Fischler-R, 2012)

The connection constants $\omega_1, \dots, \omega_\mu$ belong to $\mathbf{G}$.

The set of G-approximable numbers

Definition 5

Sequences (P_n) and (Q_n) of algebraic numbers are said to form G-approximations of $\alpha \in \mathbb{C}$ if

$$\lim_{n \to +\infty} \frac{P_n}{Q_n} = \alpha$$

and the generating functions

$$\sum_{n=0}^{\infty} P_n z^n, \qquad \sum_{n=0}^{\infty} Q_n z^n$$

are both G-functions.

Theorem 6 (Fischler-R, 2012)

The set of numbers having G-approximations is $\mathrm{Frac}\,\mathbf{G}$.