

# ON THE BITS COUNTING FUNCTION OF REAL NUMBERS

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ABSTRACT. Let  $B_n(x)$  denote the number of 1's occurring in the binary expansion of an irrational number  $x > 0$ . A difficult problem is to provide non-trivial lower bounds for  $B_n(x)$  for interesting number numbers like  $\sqrt{2}$ ,  $e$  or  $\pi$ : their conjectural simple normality in base 2 is equivalent to  $B_n(x) \sim n/2$ . In this article, amongst other things, we prove inequalities relating  $B_n(x+y)$ ,  $B_n(xy)$  and  $B_n(1/x)$  to  $B_n(x)$  and  $B_n(y)$  for any irrational numbers  $x, y > 0$ , which we prove to be sharp up to a multiplicative constant. As a by-product, we provide an answer to a question raised by Bailey et al. (*On the binary expansions of algebraic numbers*, J. Théor. Nombres Bordeaux **16** (2004), no. 3, 487–518) concerning the binary digits of the square of a series related to the Fibonacci sequence. We also obtain a slight refinement of the main theorem of the same article, which provides non-trivial lower bound for  $B_n(\alpha)$  for any real irrational algebraic number. We conclude the article with effective or conjectural lower bounds for  $B_n(x)$  when  $x$  is a transcendental number.

## 1. INTRODUCTION

The integer  $B(m)$  is defined to be the number of 1's in the finite binary expansion of the non negative integer  $m$ , with  $B(0) = 0$ . It is a classical fact that  $B$  is sub-additive and sub-multiplicative, i.e. that for all integers  $m, n \geq 0$ , we have  $B(m+n) \leq B(m) + B(n)$  and  $B(mn) \leq B(m)B(n)$ : see [5, 21] or Section 2, where we provide a proof for completeness' sake. These inequalities are sharp, as the examples  $10 + 1 = 11$  et  $111 \times 100100 = 111111$  (in base 2) show. The behavior of  $B(m^j)$  ( $m$  fixed and  $j \rightarrow +\infty$ ) was studied in [15, 21].

In this article, we will first prove related results when  $m$  is allowed to be an irrational number (in Theorem 1) via a study of the function

$$B_n(x) = \#\{j \leq n : x_j = 1\}$$

where  $n \geq 0$  and  $x = (x_{-p} \cdots x_{-1} x_0. x_1 x_2 x_3 \cdots)_2$  is the binary expansion of  $x \geq 0$ . The case where  $x$  is a rational number of the form  $n/2^m$  ( $n, m \in \mathbb{N}$ ) is ambiguous: in this case, there are two possible binary expansions, a finite and an infinite one. The only rational numbers of this form that we will consider in this text are the integers and we will only use their finite binary expansion; with this convention, for any positive integer  $x$  (written in its finite binary expansion), we have  $B_n(x) = B_0(x) = B(x)$ . In any occurrence of the quantity  $B_n(x)$ , it will be tacitly supposed (if not explicitly stated) that  $x$  is either an irrational number or an integer.

Theorem 1 below has some interesting consequences, which we now describe as a motivation for the article. We recall that a real number  $x$  is said to be simply normal in base 2 if

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$B_n(x) \sim n/2$  and 2-normal if any blocks of  $m$  digits appears with the frequency  $2^{-m}$ . Certain rational numbers like  $1/3 = (0.01010101\dots)_2$  and certain artificial irrational numbers are simply normal in base 2 (or even 2-normal as in the case of the base 2 Champernowne number  $\sum_{k \geq 1} k2^{-c_k}$  with  $c_k = k + \sum_{j=1}^k \lfloor \log_2 j \rfloor$ ), nobody knows if this is the case for more classical numbers.

A simple method to provide lower bound of  $B_n(x)$  for more “natural” irrational numbers  $x$  like  $e$  or  $\pi$  is the following. Assume that  $x$  has a finite irrationality exponent, i.e. that there exists  $\mu_x \in [2, +\infty[$  such that  $|x - p/q| \geq q^{-\mu_x - \varepsilon}$  for any  $\varepsilon > 0$  and  $q \gg_\varepsilon 1$ . Then, we have that

$$B_n(x) \geq \frac{\log(n)}{\log(\mu_x + \varepsilon)} + \mathcal{O}(1) \quad (1.1)$$

for  $n \gg_\varepsilon 0$ . We can apply (1.1) to  $e$  and  $\pi$  because it has been proved that  $\mu_e = 2$  (Bundschuh [7] et Davis [8]) and  $2 \leq \mu_\pi \leq 8.1$  (Hata [11]) ; surprisingly, these lower bounds for  $B_n(e)$  and  $B_n(\pi)$  are the best known to date. The proof of (1.1) runs as follows. Denoting by  $p_k 2^{-n_k}$  the truncations of the binary expansion  $\sum_{k=1}^\infty 2^{-n_k}$  of  $x$ , we have for all  $k \gg_\varepsilon 0$

$$\frac{1}{2^{(\mu_x + \varepsilon)n_k}} \leq \left| x - \frac{p_k}{2^{n_k}} \right| \leq \frac{1}{2^{n_{k+1} - 1}},$$

from which we deduce that  $n_{k+1} \leq (\mu_x + \varepsilon)n_k + 1$ . Hence  $n_k \leq c_{\varepsilon, x}(\mu_x + \varepsilon)^k$  with  $c_{\varepsilon, x} > 0$ . We now remark that  $B_n(x) = \#\{k : n_k \leq n\} \geq \#\{k : c_{\varepsilon, x}(\mu_x + \varepsilon)^k \leq n\} = \#\{k : k \leq (n - \log c_{\varepsilon, x}) / \log(\mu_x + \varepsilon)\}$  and (1.1) follows. See the final section for an elaboration of this method.

When  $\alpha$  is a real irrational algebraic number, we can obtain the same kind of bound by using Roth’s theorem [20] that  $\mu_\alpha = 2$ . Ridout’s theorem [18] (which claims that  $\mu_\alpha = 1$  when  $q$  is restricted to a power of an integer) gives a result better than Roth’s theorem but the improvement is marginal: for all  $\varepsilon > 0$ , we have  $B_n(\alpha) \geq \log(n) / \log(1 + \varepsilon) + \mathcal{O}(1)$ . However, a dramatic improvement was recently obtained by Bailey et al. [5], who proved that

$$B_n(\alpha) \geq c_\alpha n^{1/d} (1 + o(1)) \quad (1.2)$$

for all real positive algebraic numbers  $\alpha$  of degree  $d \geq 2$  over  $\mathbb{Q}$ . Denoting by  $a_d X^d + \dots + a_1 X + a_0$  the minimal polynomial of  $\alpha$  over  $\mathbb{Q}$ , they obtained  $c_\alpha = (1 + \log a_d)^{-1/d}$  when  $a_0 < 0$  and  $a_j \geq 0$  otherwise (“case S” –for special– in the rest of the article). They only got  $c_\alpha = (2a_d)^{-1/d}$  in the general case, whose proof is much more complicated <sup>(1)</sup>: it seems that the use of Ridout’s theorem instead of Roth’s theorem in their proof improves the constant to  $c_\alpha = a_d^{-1/d}$ . They used the bound (1.2) to prove that the number  $\sum_{n \geq 1} 2^{-f(n)}$  is transcendental over  $\mathbb{Q}$  provided  $f(n)$  grows faster than any power of  $n$ .

As a consequence of Theorem 1, we provide another proof of case S in Corollary 1 and also obtain a simpler proof of the bound  $B_n(\alpha) \gg n^{1/d}$  for a class of algebraic numbers  $\alpha$

<sup>1</sup>This result suggests an analogy with the Liouville-Thue-Siegel-Roth theorem concerning the diophantine approximation of algebraic numbers  $\alpha$  of degree  $d$  by rationals: (1.2) can be viewed as the analogue of Liouville’s theorem ( $\mu_\alpha \leq d$ ) and the expected bound that  $B_n(\alpha) \gg n$  would be the analogue of Roth’s theorem ( $\mu_\alpha = 2$ ).

that do not fall obviously into case S. We also note that a recent result of Adamczewski, Bugeaud and Luca [1, 2] implies in particular that the binary digits of algebraic real numbers can not be generated by an automaton (see [3] for definitions). These two results are the most important ones towards the conjectural normality in all bases of algebraic numbers.

We now state some of the results proved in this article.

**Theorem 1.** *Let  $x, y > 0$  be real irrational numbers, otherwise stated.*

(i) *If  $x + y$  is irrational, we have for all  $n \gg 0$ ,*

$$B_n(x + y) \leq B_n(x) + B_n(y) + 1. \quad (1.3)$$

*If  $x + y \in \mathbb{N}$ , then we replace  $B_n(x + y)$  by  $n + 1$ .*

(ii) *If  $xy$  is irrational, we have for all  $n \gg 0$ ,*

$$B_n(xy) \leq B_n(x) \cdot B_n(y) + \log_2 \lfloor x + y + 1 \rfloor. \quad (1.4)$$

*If  $y$  is an integer, we have  $B_n(xy) \leq B_n(x) \cdot B(y) + B(y)$ .*

(iii) *For all integer  $A > 0$ , we have for all  $n \gg 0$ ,*

$$B_n(x) \cdot B_n(A/x) \geq n - 1 - \lfloor \log_2(x + A/x + 1) \rfloor. \quad (1.5)$$

(iv) *The three inequalities (1.3), (1.4) and (1.5) are sharp, up to a multiplicative factor for the last two.*

**Remark.** *When  $0 < x, y < 1$ , results slightly better than (1.3) and (1.4) are indicated in [4, p. 157]:  $B_n(x + y) \leq B_n(x) + B_n(y)$  and  $B_n(xy) \leq B_n(x) \cdot B_n(y)$ . On the other hand, inequality (iii) seems to be new. The results proved in Theorem 1 are sufficient for our purposes.*

The sharpness of (1.3) (up to an additive constant) follows from the examples  $x = \sum_{k \geq 1} 2^{-2^k}$  and  $y = \sum_{k \geq 1} 2^{-3^k}$ . We shall prove the sharpness of (1.4) and of (1.5) (which are less obvious) in Section 4: the Theorem 2 of that section answers a question in [5].

We obtain lower bounds of  $B_n(\alpha)$  for certain classes of algebraic numbers  $\alpha$  of the same form as (1.2) above.

**Corollary 1.** (i) *Let  $\alpha$  be an algebraic number in class S of degree  $d \geq 2$  and dominant coefficient  $a_d$ . We have that*

$$B_n(\alpha) \geq B(a_d)^{-1/d} n^{1/d} (1 + o(1)).$$

(ii) *Let  $\beta$  be a positive irrational algebraic number, of degree  $d \geq 2$ , such that there exist two polynomials  $P, Q$  with positive integers coefficients and two positive integers  $a, b$  such that  $P(\beta) = a + bQ(\beta)^{-1}$ . Then, we have*

$$B_n(\beta) \geq (B(p)B(q))^{-1/\delta} n^{1/\delta} (1 + o(1)),$$

*where  $\delta = \deg(PQ)$  and  $p, q$  are the dominant coefficients of  $P$  and  $Q$  respectively.*

**Remarks.** The lower bound in (i) is a small improvement on Theorem 5.2 in [5] (proved with  $1 + \log(a_d)$  instead of our  $B(a_d)$ ). It is presented here in order to illustrate the usefulness of the simple bounds in Theorem 1.

The most favorable case in (ii) is when  $P(X)Q(X) - aQ(X) - b$  is the minimal polynomial of  $\beta$ , in which case  $\delta$  is the degree of  $\beta$  over  $\mathbb{Q}$ . For example, the positive real root  $\beta$  of the polynomial  $8x^3 - 2x^2 + 4x - 3$ , which is irreducible over  $\mathbb{Q}$ , satisfies the equation  $4\beta = 1 + 2(2\beta^2 + 1)^{-1}$  and therefore, by (ii), we have  $B_n(\beta) \geq n^{1/3}(1 + o(1))$ . Note that  $\beta$  does not belong to class  $S$  and Theorem 7.1 in [5] yields the weaker bound  $B_n(\beta) \geq 16^{-1/3} n^{1/3}(1 + o(1))$ .

If we allow the case  $b = 0$  in (ii), we recover (i) provided the constant term of  $P$  is  $< a$ . However, the proof of (ii) requires that  $b > 0$ .

We now state a second application of Theorem 1. At first sight, it is not completely improbable that certain powers of a transcendental number  $x$  might not be simply normal when  $x$  itself is not simply normal in base 2: indeed, the rather unpredictable “mixing” effect of carries might produce a smoothing “law of large powers”. However, this is generally not the case as shown by the following result.

**Corollary 2.** *There exists a transcendental real number  $\xi$  such that none of its powers  $\xi^j$  ( $j \in \mathbb{N}^*$ ) is simply normal in base 2.*

**Remark.** Let  $\alpha$  be a real irrational algebraic number which is not of the form  $\alpha = (a/b)^{1/d}$ ,  $a, b, d \in \mathbb{N}$ . Then, for all integer  $j \geq 1$ , the number  $\alpha^j$  is again a real irrational algebraic number and it is therefore expected that all the powers of  $\alpha$  are normal in all bases.

*Proof.* It is enough to produce transcendental numbers  $\xi$  such that, for example,  $B_n(\xi) \ll \log(n)$ : inequality (1.4) implies that  $B_n(\xi^j) \ll \log(n)^j$  for all integer  $j \geq 1$ , and thus the numbers  $\xi^j$  all fail to be simply normal in base 2. Any one of the numbers  $\xi = \sum_{n=1}^{\infty} a^{-b^n}$ , with the integers  $a \geq 2, b \geq 2$ , is suitable. Its transcendence follows, for instance, from Roth’s theorem when  $b \geq 3$  and from Mahler’s method when  $b = 2$  (see [17]).

Incidentally, Kempner [12] was the first to provide a transcendence proof of these numbers. When  $b = 2$ , an unusual and original proof, based on the counting of non-zero digits of powers of  $\xi^j$ , was recently given by Knight [13]; as noticed by Bailey et al., the lower bound (1.2) provides a proof in the same “digital” spirit.  $\square$

## 2. WARMUP

In this short section, we provide a proof of the subadditivity and submultiplicativity of the function  $B$ . We follow the presentation in [5].

*Subadditivity of  $B$ .* Let  $m$  denote a positive integer written in base 2 as  $(m_k \cdots m_0)_2$  with  $m_k = 1$  and  $m_i \in \{0, 1\}$ . Let  $L_j$  denotes the minimal integer  $\ell \geq 0$  such that  $m_{j+\ell} = 0$ . The usual rule of add with carry implies that  $B(m + 2^j) = B(m) + 1 - L_j \leq B(m) + 1$ .

If  $n$  is a positive integer written as  $n = \sum_{\ell=1}^d 2^{j_\ell}$ , then by successive applications of the previous equality, we obtain

$$\begin{aligned} B(m+n) &= B\left(m + \sum_{\ell=1}^d 2^{j_\ell}\right) \leq B\left(m + \sum_{\ell=1}^{d-1} 2^{j_\ell}\right) + 1 \leq B\left(m + \sum_{\ell=1}^{d-2} 2^{j_\ell}\right) + 2 \\ &\leq \cdots \leq B\left(m + \sum_{\ell=1}^0 2^{j_\ell}\right) + d = B(m) + d = B(m) + B(n). \end{aligned}$$

*Submultiplicativity of  $B$ .* We first remark that obviously  $B(2^j m) = B(m)$ . Therefore with  $n = \sum_{\ell=1}^d 2^{j_\ell}$ , we have

$$B(mn) = B\left(m \sum_{\ell=1}^d 2^{j_\ell}\right) \leq \sum_{\ell=1}^d B(2^{j_\ell} m) = \sum_{\ell=1}^d B(m) = B(m)B(n),$$

where we have used the subadditivity of  $B$ .

### 3. PROOF OF THEOREM 1, PARTS (i), (ii) AND (iii)

For all integer  $n \geq 0$ , let  $x_n = \lfloor 2^n x \rfloor$ ,  $y_n = \lfloor 2^n y \rfloor$ ,  $s_n = \lfloor 2^n(x+y) \rfloor$  and  $p_n = \lfloor 2^n xy \rfloor$ .

(i) For all  $n \geq 0$ , we have  $x_n + y_n \leq s_n < x_n + y_n + 2$  hence  $s_n = x_n + y_n + z_n$  with  $z_n \in \{0, 1\}$ . It follows that

$$B(s_n) = B(x_n + y_n + z_n) \leq B(x_n) + B(y_n) + B(z_n).$$

Let's suppose that  $x+y$  is irrational. Then, since  $B(s_n) = B_n(x+y)$ ,  $B(x_n) = B_n(x)$ ,  $B(y_n) = B_n(y)$  and  $B(z_n) \leq 1$ , we obtain

$$B_n(x+y) \leq B_n(x) + B_n(y) + 1.$$

If  $x+y \in \mathbb{N}$ , then  $B_n(x) + B_n(y) \geq n$  because the sum of the  $n$ -th digits after the point of  $x$  and  $y$  is 1 ( $x, y$  are irrational numbers).

(ii) We now show the second inequality.

We first assume that  $x, y$  and  $xy$  are irrational. Since there exist two reals  $z_n$  and  $\sigma_n$  satisfying

$$x = \frac{x_n}{2^n} + \rho_n, \quad y = \frac{y_n}{2^n} + \sigma_n \quad \text{and} \quad 0 < \rho_n, \sigma_n < \frac{1}{2^n},$$

we have

$$0 < xy - \frac{x_n y_n}{2^{2n}} = \rho_n \frac{y_n}{2^n} + \sigma_n \frac{x_n}{2^n} + \rho_n \sigma_n < \frac{x+y+1/2^n}{2^n}.$$

Furthermore, we have the binary expansion  $x_n y_n = \sum_{\ell=0}^{2n+n_0} \varepsilon_\ell 2^\ell$  where  $n_0 \in \mathbb{Z}$  (depending only on  $x$  and  $y$ ),  $\varepsilon_\ell \in \{0, 1\}$  and  $\varepsilon_{2n+n_0} = 1$ . For all  $n \geq -n_0$ , we write

$$x_n y_n = \sum_{\ell=0}^{n-1} \varepsilon_\ell 2^\ell + \sum_{\ell=n}^{2n+n_0} \varepsilon_\ell 2^\ell = a_n + 2^n b_n$$

with  $0 \leq a_n \leq 2^n - 1$ . Thus we get

$$0 < xy - \frac{b_n}{2^n} < \frac{x+y+1/2^n}{2^n} + \frac{a_n}{2^{2n}} \leq \frac{x+y+1}{2^n}. \quad (3.1)$$

We now remark that  $2^n(p_n + 1) = 2^n(\lfloor 2^n xy \rfloor + 1) > \lfloor 2^n x \rfloor \lfloor 2^n y \rfloor = x_n y_n \geq 2^n b_n$ , which implies that  $p_n > b_n - 1$  and hence  $p_n \geq b_n$  (these are integers). Therefore, comparing the bound

$$0 < xy - \frac{p_n}{2^n} < \frac{1}{2^n}$$

and the bound (3.1), we obtain that  $p_n = b_n + v_n$  with  $v_n \in \{0, 1, \dots, \lfloor x+y+1 \rfloor\}$ .

Finally, using the fact that  $B(v_n) \leq \log_2 \lfloor x+y+1 \rfloor$  and that  $B(x_n y_n) \geq B(b_n)$  (because  $B(x_n y_n) = B(a_n + 2^n b_n) = B(a_n) + B(b_n) \geq B(b_n)$ ), we obtain that

$$\begin{aligned} B(p_n) = B(b_n + v_n) &\leq B(b_n) + B(v_n) \\ &\leq B(x_n y_n) + \log_2 \lfloor x+y+1 \rfloor \\ &\leq B(x_n) \cdot B(y_n) + \log_2 \lfloor x+y+1 \rfloor, \end{aligned}$$

which is nothing but (1.4) because  $B(p_n) = B_n(xy)$  (we use here the fact that  $xy$  is irrational).

We now suppose that  $x$  is irrational and that  $y = 2^{\ell_1} + \dots + 2^{\ell_k}$  is an integer. Then, by (i) we have

$$\begin{aligned} B_n(xy) = B_n(2^{\ell_1} x + \dots + 2^{\ell_k} x) &\leq B_n(2^{\ell_1} x) + \dots + B_n(2^{\ell_k} x) + k \\ &= k \cdot B_n(2^{\ell_1} x) + k = B_n(x) \cdot B(y) + B(y). \end{aligned}$$

This concludes the proof of (ii).

(iii) For all integer  $n \geq 0$ , letting  $x_n = \lfloor 2^n x \rfloor$  and  $y_n = \lfloor 2^n A/x \rfloor$ , we have that

$$0 < u_n = x - \frac{x_n}{2^n} < \frac{1}{2^n} \quad \text{et} \quad 0 < v_n = \frac{A}{x} - \frac{y_n}{2^n} < \frac{1}{2^n}.$$

(The inequalities for  $v_n$  hold because  $A > 0$ .) Hence,

$$A = x \cdot \frac{A}{x} = \left( \frac{x_n}{2^n} + u_n \right) \cdot \left( \frac{y_n}{2^n} + v_n \right) = \frac{x_n y_n}{2^{2n}} + v_n \frac{x_n}{2^n} + u_n \frac{y_n}{2^n} + u_n v_n$$

and

$$0 < v_n \frac{x_n}{2^n} + u_n \frac{y_n}{2^n} + u_n v_n < \frac{x + A/x + 1}{2^n}.$$

For  $n \gg_{A,x} 1$ , we have  $1 - (x + A/x + 1)/2^n > 0$  and thus

$$\begin{aligned} \left\{ \frac{x_n y_n}{2^{2n}} \right\} &= \left\{ A - v_n \frac{x_n}{2^n} - u_n \frac{y_n}{2^n} - u_n v_n \right\} \\ &= \left\{ 1 - v_n \frac{x_n}{2^n} - u_n \frac{y_n}{2^n} - u_n v_n \right\} > 1 - \frac{x + A/x + 1}{2^n} > 1 - \frac{1}{2^{n - \lfloor 1 + \log_2(x + A/x + 1) \rfloor}} > 0. \end{aligned}$$

(The fact that  $A$  is an integer is used to invoke the 1-periodicity of the fractional part function  $\{\cdot\}$ .) These inequalities imply that we have at least  $B(x_n y_n) \geq n - 1 - \lfloor \log_2(x + A/x + 1) \rfloor$ .

Since  $x$  and  $A/x$  are irrational, we have  $B(x_n) = B(x)$  et  $B(y_n) = B(A/x)$ . The submultiplicativity of  $B$  finally yields that

$$B_n(x) \cdot B_n(A/x) \geq n - 1 - \lfloor \log_2(x + A/x + 1) \rfloor.$$

#### 4. PROOF OF THEOREM 1, PART (iv)

Borel proved that almost all real numbers are normal in all bases ([16, p. 98]) and thus the set of pairs of real numbers which are normal in base 2 is of full measure. Hence, for almost all pairs of irrational numbers  $(x, y)$ , inequality (1.4) is of the trivial form  $2n \leq n^2$ : therefore, we could expect to prove a stronger inequality for all pairs of real numbers. However, this is not the case; the following examples show that there exist pairs  $(x, y)$  for which (1.4) is optimal (up to a multiplicative constant) without further hypothesis on  $x$  and  $y$ . Let

$$\mathcal{K} = \sum_{n=1}^{\infty} \frac{1}{2^{2^n}} \quad \text{and} \quad \mathcal{F} = \sum_{n=2}^{\infty} \frac{1}{2^{F_n}},$$

where  $(F_n)_{n \geq 0}$  is the Fibonacci sequence defined by  $F_1 = F_2 = 1$  and  $F_{n+2} = F_{n+1} + F_n$  for  $n \geq 1$ . Let  $\phi = (1 + \sqrt{5})/2$ .

**Theorem 2.** *As  $n \rightarrow +\infty$ , we have*

$$B_n(\mathcal{K}) \sim \frac{\log(n)}{\log(2)} \quad \text{and} \quad B_n(\mathcal{K}^2) \sim \frac{\log^2(n)}{2 \log^2(2)},$$

as well as

$$B_n(\mathcal{F}) \sim \frac{\log(n)}{\log(\phi)} \quad \text{and} \quad B_n(\mathcal{F}^2) \sim \frac{\log^2(n)}{2 \log^2(\phi)}.$$

**Remarks.** *The powers of  $\mathcal{K}$  are also considered in [14], where it is proved that their binary expansions are automatic.*

*The result for  $\mathcal{F}^2$  answers the second question in Section “Open problems”, p. 27 of [5]. More generally, numerical computations suggest that, for all  $j \in \mathbb{N}^*$ , we have*

$$B_n(\mathcal{K}^j) \sim \frac{\log^j(n)}{j! \log^j(2)} \quad \text{and} \quad B_n(\mathcal{F}^j) \sim \frac{\log^j(n)}{j! \log^j(\phi)},$$

as  $n \rightarrow +\infty$ .

*Proof of Theorem 2.* The statements for  $B_n(\mathcal{K})$  and  $B_n(\mathcal{F})$  are clear. We now consider the cases of  $\mathcal{K}^2$ . We have

$$\mathcal{K}^2 = \left( \sum_{j=k \geq 1} + 2 \sum_{j>k \geq 1} \right) \frac{1}{2^{2^j+2^k}} = \sum_{n=2}^{\infty} \frac{1}{2^{2^n}} + 2 \sum_{m=1}^{\infty} \frac{e_m}{2^m},$$

where  $e_m$  is the number of ways of writing  $m$  as  $m = 2^j + 2^k$  with  $j > k \geq 1$ . Obviously, by uniqueness of the binary expansion, we have  $e_m \in \{0, 1\}$ . Since the sets  $P = \{2^j, j \geq 1\}$  and  $Q = \{2^j + 2^k - 1, j > k \geq 1\}$  have empty intersection, we have

$$\mathcal{K}^2 = \sum_{n \in P \sqcup Q} \frac{1}{2^n}.$$

Let  $P \sqcup Q = R = \{R_1, R_2, \dots\}$ : by definition,  $B_n(\mathcal{K}^2) = \#\{k : R_k \leq n\}$ . For all  $\ell \geq 1$ , there exist exactly  $\ell - 1$  elements in  $Q \cap \{2^\ell + 1, \dots, 2^{\ell+1}\}$ , which are of the form  $2^\ell + 2^k - 1$  with  $1 \leq k < \ell$ . Hence, the set  $Q \cap \{1, 2, \dots, 2^{n+1}\}$  has  $\sum_{\ell=1}^n (\ell - 1) = n(n-1)/2$  elements. Furthermore,  $P \cap \{1, 2, \dots, 2^{n+1}\} = n$  hence  $R \cap \{1, 2, \dots, 2^{n+1}\}$  has  $k_n = n(n+1)/2$  elements, whose largest is  $2^{n+1}$ , i.e.  $R_{k_n} = 2^{n+1}$ .

Let now  $j \geq 1$  be any integer: there exists an integer  $n \geq 0$  such that  $j \in \{k_n, k_n + 1, \dots, k_{n+1}\}$ . We then have

$$\frac{2(n+1)}{(n+2)} = \frac{\log_2(R_{k_n})^2}{k_{n+1}} \leq \frac{\log_2(R_j)^2}{j} \leq \frac{\log_2(R_{k_{n+1}})^2}{k_n} = \frac{2(n+2)^2}{n(n+1)},$$

which implies that  $\lim_{j \rightarrow +\infty} \log(R_j)^2/j = 2 \log^2(2)$  and hence that

$$\#\{k : R_k \leq n\} = \#\{k : \log^2(R_k) \leq \log^2(n)\} \sim \frac{\log^2(n)}{2 \log^2(2)},$$

as desired.

For  $\mathcal{F}^2$ , we adapt the previous proof by using the fact that every positive integer  $N$  can be written uniquely in the form  $N = F_{n_1} + F_{n_2} + \dots + F_{n_k}$ , where the set  $\{1 < n_1 < n_2 < \dots < n_k\}$  does not contain two consecutive integers. This expansion is known as Zeckendorf decomposition [22],  $Z$ -decomposition in short. We have

$$\begin{aligned} \mathcal{F}^2 &= \left( \sum_{j=k \geq 2} + 2 \sum_{\substack{j=k+1 \\ k \geq 2}} + 2 \sum_{\substack{j \geq k+2 \\ k \geq 2}} \right) \frac{1}{2^{F_j + F_k}} \\ &= \sum_{n=2}^{\infty} \frac{1}{2^{F_n + F_n}} + \sum_{n=2}^{\infty} \frac{1}{2^{F_{n+1} + F_{n-1}}} + \sum_{\substack{j \geq k+2 \\ k \geq 2}} \frac{1}{2^{F_j + F_k - 1}} \\ &= \sum_{n=2}^4 \frac{1}{2^{2F_n}} + \sum_{n=5}^{\infty} \frac{1}{2^{2F_n}} + \sum_{n=4}^{\infty} \frac{1}{2^{F_{n-1}}} + \sum_{\substack{j \geq k+2 \\ k \geq 2}} \frac{1}{2^{F_j + F_k - 1}}. \end{aligned} \quad (4.1)$$

We will now show that the terms of the three infinite sums in (4.1) are pairwise distinct and thus that these three series provides the binary expansion of  $\mathcal{F}^2$ , up to the first finite sum, which implies a possible finite number of harmless carries.

For this, we note that the  $Z$ -decomposition of  $2F_n + 1$  is  $F_{n+1} + F_{n-2} + F_2$  for  $n \geq 5$  and that  $F_m$  ( $m \geq 4$ ) and  $F_j + F_k$  ( $j \geq k+2, k \geq 2$ ) are already  $Z$ -decompositions. We deduce that the equations  $2F_n = F_m - 1$  ( $n \geq 5, m \geq 4$ ),  $2F_n = F_j + F_k - 1$  ( $n \geq 5, j \geq k+2, k \geq 2$ )



and  $F_m = F_j + F_k$  ( $m \geq 4, j \geq k + 2, k \geq 2$ ) have no solution, which proves the above claim.

Thus, we have

$$\mathcal{F}^2 = \sum_{n=2}^4 \frac{1}{2^{2F_n}} + \sum_{m \in A_1 \sqcup A_2 \sqcup A_3} \frac{1}{2^m}$$

where the sets  $A_1 = \{2F_n, n \geq 5\}$ ,  $A_2 = \{F_m - 1, m \geq 4\}$  and  $A_3 = \{F_j + F_k - 1, j \geq k + 2, k \geq 2\}$  have pairwise empty intersections. The rest of the proof is now similar to that of  $\mathcal{K}^2$ . We note that for  $\ell \geq 4$ , the set

$$A_3 \cap \{F_\ell + 1, F_\ell + 2, \dots, F_{\ell+1}\} = \{F_\ell + F_3 - 1, F_\ell + F_4 - 1, \dots, F_\ell + F_{\ell-2} - 1\}$$

has  $\ell - 4$  elements while  $A_1 \cap \{F_\ell + 1, F_\ell + 2, \dots, F_{\ell+1}\}$  and  $A_2 \cap \{F_\ell + 1, F_\ell + 2, \dots, F_{\ell+1}\}$  have (at most) one element. It follows that  $B_n(\mathcal{F}^2) \sim \log^2(n)/2 \log^2(\phi)$ .  $\square$

We now consider the optimality of (1.5). It will be enough to produce an irrational number  $\mathcal{E} > 0$  such that  $B_n(\mathcal{E}) \asymp \sqrt{n}$  and  $B_n(1/\mathcal{E}) \asymp \sqrt{n}$ : such a number was mentioned in [5] (for another purpose) but no proof of the growth of  $B_n(\mathcal{E})$  was provided. We therefore give one below. Starting from the binary expansion  $n = (x_p \cdots x_1 x_0)_2$  of an integer  $n$ , we define the sets  $S = \{n \in \mathbb{N} : \forall j \geq 0, x_{2j} = 0\}$  and  $T = \{n \in \mathbb{N} : \forall j \geq 0, x_{2j+1} = 0\}$  (which satisfy  $S \cap T = \{0\}$  et  $2T = S$ ) and then the real numbers

$$\mathcal{E} = \prod_{n=0}^{\infty} \left(1 + \frac{1}{2^{2^{2^n}}}\right) = \sum_{s \in S} \frac{1}{2^s} \quad \text{and} \quad \widehat{\mathcal{E}} = \prod_{n=0}^{\infty} \left(1 + \frac{1}{2^{2^{2^{n+1}}}}\right) = \sum_{t \in T} \frac{1}{2^t}.$$

Since any integer  $n$  can be written exactly once under the form  $n = s + t$  with  $(s, t) \in S \times T$ , we have

$$\mathcal{E} \widehat{\mathcal{E}} = \sum_{(s,t) \in S \times T} \frac{1}{2^{s+t}} = \sum_{n=0}^{\infty} \frac{1}{2^n} = 2.$$

The inverse of  $\mathcal{E}$  is thus simply  $\widehat{\mathcal{E}}/2$  and it remains to check the property (which is probably well-known, as it is given in [5] without proof) that  $B_n(\mathcal{E}) \asymp \sqrt{n}$  et  $B_n(\widehat{\mathcal{E}}) \asymp \sqrt{n}$ . Only the upper bounds are useful for our purposes.

**Proposition 1.** *As  $n \rightarrow +\infty$ , we have that*

$$\liminf_{n \rightarrow +\infty} \frac{B_n(\mathcal{E})}{\sqrt{n}} \geq 2\sqrt{2/3}, \quad \limsup_{n \rightarrow +\infty} \frac{B_n(\mathcal{E})}{\sqrt{n}} \leq \sqrt{2}$$

and

$$\liminf_{n \rightarrow +\infty} \frac{B_n(\widehat{\mathcal{E}})}{\sqrt{n}} \geq 2/\sqrt{3}, \quad \limsup_{n \rightarrow +\infty} \frac{B_n(\widehat{\mathcal{E}})}{\sqrt{n}} \leq 2.$$

*In particular, for all  $\varepsilon > 0$  and all  $n \gg_\varepsilon 0$ , we have  $B_n(\mathcal{E}) \cdot B_n(\widehat{\mathcal{E}}/2) \leq (2\sqrt{2} + \varepsilon)n$ .*

**Remark.** *These inequalities are probably not sharp.*

*Proof of proposition 1.* Let us write the elements of the set  $S = \{0, 2, 8, 10, 32, 34, \dots\}$  as  $\{S_0 < S_1 < S_2 < \dots\}$  and those of  $T = \{0, 1, 4, 5, 16, 17, \dots\}$  as  $\{T_0 < T_1 < T_2 < \dots\}$ . Clearly,

$$\liminf_{n \rightarrow +\infty} \frac{B_n(\mathcal{E}, n)}{\sqrt{n}} = \limsup_{k \rightarrow +\infty} \frac{S_k}{k^2} \quad \text{and} \quad \limsup_{n \rightarrow +\infty} \frac{B_n(\mathcal{E}, n)}{\sqrt{n}} = \liminf_{k \rightarrow +\infty} \frac{S_k}{k^2}$$

and similar expressions for  $\widehat{\mathcal{E}}$  and  $T_k$ . It can be easily proved that

$$S_k = \sum_{j=0}^{\infty} \left( \left\lfloor \frac{k}{2^j} \right\rfloor \bmod 2 \right) 2^{2j+1}$$

from which it follows that  $S_{2^n} = 2^{2n+1}$  and  $S_{2^{n-1}} = \frac{2}{3}(4^n - 1)$ . For all  $k \in \{2^n, \dots, 2^{n+1} - 1\}$ , we thus have

$$\frac{S_{2^n}}{(2^{n-1} - 1)^2} \leq \frac{S_k}{k^2} \leq \frac{S_{2^{n+1}-1}}{2^{2n}}.$$

Hence,

$$\liminf_{k \rightarrow +\infty} \frac{S_k}{k^2} \geq \lim_{n \rightarrow +\infty} \frac{S_{2^n}}{(2^{n+1} - 1)^2} = \frac{1}{2} \quad \text{and} \quad \limsup_{k \rightarrow +\infty} \frac{S_k}{k^2} \leq \lim_{n \rightarrow +\infty} \frac{S_{2^{n+1}-1}}{(2^n)^2} = \frac{8}{3}.$$

Since  $T = S/2$ , we also deduce that

$$\liminf_{k \rightarrow +\infty} \frac{T_k}{k^2} \geq \frac{1}{4} \quad \text{and} \quad \limsup_{k \rightarrow +\infty} \frac{T_k}{k^2} \leq \frac{4}{3}.$$

This completes the proof.  $\square$

The transcendence of  $\mathcal{E}$  and  $\widehat{\mathcal{E}}$  can be proved thanks to Mahler's method: the proof can be found in an article of Blanchard and Mendès-France [6]. It is in fact easy to produce other pairs of numbers which are inverse of each others and whose binary expansions are explicitly known: given any given subset  $A$  of  $\mathbb{N}$ , we can use the identity

$$\prod_{n \in A} (1 + z^{2^n}) \times \prod_{n \in \mathbb{N} \setminus A} (1 + z^{2^n}) = \prod_{n \in \mathbb{N}} (1 + z^{2^n}) = \frac{1}{1 - z}.$$

This suggests looking at the inverse of real numbers with few 1 in base 2: if  $\psi : \mathbb{N} \rightarrow \mathbb{N}$  is strictly increasing, of inverse  $\psi^{[-1]}$ , let  $\mathcal{D}_\psi = \sum_{n=0}^{\infty} 2^{-\psi(n)}$ : we have

$$B_n(1/\mathcal{D}_\psi) \geq \frac{n}{\psi^{[-1]}(n)}(1 + o(1)).$$

A related problem, due to Mendès-France, is to prove (or disprove) the existence of an irrational number simply normal in base  $b$  whose inverse is not simply normal or numbers with this property with respect to normality in base  $b$  or absolute normality. Clearly, the answer is positive in the case of simple normality in base 2 if we allow rational numbers (for example:  $3 = (11)_2$  and  $1/3 = (0,01010101010\dots)_2$ ). It would be equally interesting to prove (or disprove) the existence a real number  $x$  such that both  $x$  and  $1/x$  are simply normal in base  $b$  (or  $b$ -normal or normal in all bases).

## 5. PROOF OF COROLLARY 1

(i) Since  $\alpha$  is of degree  $d \geq 2$ , the number  $|a_0|/\alpha$  is irrational, as are the numbers  $a_j\alpha^{j-1}$  for those  $j \in \{2, \dots, d\}$  such that  $a_j \neq 0$ . Since

$$|a_0| \alpha^{-1} = a_1 + a_2\alpha + \dots + a_d\alpha^{d-1},$$

the various inequalities in Theorem 1 imply that

$$\begin{aligned} B_n(|a_0| \alpha^{-1}) &\leq d + B(a_1) + B_n(a_2\alpha) + \dots + B_n(a_d\alpha^{d-1}) \\ &\leq d + B(a_1) + B(a_2)(B_n(\alpha) + 1) + \dots + B(a_d)(B_n(\alpha^{d-1}) + 1) \\ &\leq B(a_d)B_n(\alpha)^{d-1}(1 + o(1)). \end{aligned}$$

The last inequality holds because  $B_n(\alpha) \rightarrow +\infty$  as  $n \rightarrow +\infty$ . Furthermore, we have that

$$B_n(|a_0| \alpha^{-1}) \geq \frac{n}{B_n(\alpha)} - c_0,$$

with  $c_0 > 0$  independent of  $n$ . Therefore

$$B(a_d)B_n(\alpha)^d(1 + o(1)) + c_0B_n(\alpha) \geq n$$

and we finally obtain that  $B_n(\alpha) \geq B(a_d)^{-1/d}n^{1/d}(1 + o(1))$  as desired.

(ii) Since  $a$  and  $b$  are positive integers, we have

$$B_n(P(\beta)) = B_n(a + b/Q(\beta)) \geq B_n(b/Q(\beta)) - c_1 \geq \frac{n}{B_n(Q(\beta))} - c_2$$

where  $c_1 > 0, c_2 > 0$  do not depend on  $n$ . (In the right most inequality, we use the fact that  $b > 0$ .) Hence

$$(B_n(P(\beta)) + c_2)B_n(Q(\beta)) \geq n$$

and essentially the same argument as above yields that

$$B(p)B_n(\beta)^{\deg(P)}B(q)B_n(\beta)^{\deg(Q)} \geq n(1 + o(1)).$$

The assertion follows.

## 6. BINARY EXPANSION OF ROOTS OF ANALYTICAL FUNCTIONS

One might wonder how to extend the lower bounds in Corollary 1 to the case of roots of analytical functions. This could provide non trivial informations for numbers like  $1 - e$  and  $\pi$  which are solutions of the equations  $\log(1 - z) = 1$  et  $\sin(z) = 0$ . Such results exist but are far from satisfactory, therefore, we quote them without details.

Let  $F$  be a power series of radius of convergence  $R > 0$ ,

$$F(z) = \sum_{n=1}^{\infty} a_n z^{b_n},$$

with  $a_n = p_n/q_n \in \mathbb{Q}^+$  and  $(b_n)_{n \geq 1}$  a strictly increasing sequence of integer. Let  $d_n = \text{lcm}(q_0, q_1, \dots, q_n)$ ,  $\omega(n) = \log_2(d_n)$  and

$$\left| \sum_{n=k+1}^{\infty} a_n z^{b_n} \right| \leq 2^{-\Omega_z(k)},$$

where the function  $k \mapsto \Omega_z(k)$  is assumed to be strictly increasing, with inverse  $k \mapsto \Omega_z^{[-1]}(k)$ .

**Theorem 3.** *For any rational  $\rho > 0$ , let  $\xi_\rho$  denotes the unique real number  $\xi \in [0, R[$  such that  $F(\xi) = \rho$ . Then, with  $k = \Omega_{\xi_\rho}^{[-1]}(n)$ , we have*

$$B_n(\xi_\rho) \geq B(d_k a_k)^{-1/k} \cdot (n - \omega(k))^{1/k} (1 + o(n)).$$

The proof is a straightforward adaptation of that of Theorem 5.2 in [5]. Nothing interesting occurs if  $b_n$  grows slower than  $c^n$ . With  $a_n = 1$  and  $b_n = 2^{n^2}$ , we have  $B_n(\xi_\rho) \gg \exp(c_\rho \sqrt{\log(n)})$ . When  $a_n = 1$ , the faster  $b_n$  grows, the closer  $B_n(\xi_\rho)$  is to  $n$ .

## 7. FURTHER LINKS BETWEEN DIOPHANTINE APPROXIMATION AND NORMALITY

We conclude this article with some reflections that could lead, in certain cases, to refinements of (1.1):

$$B_n(x) \geq \frac{\log(n)}{\log(\mu_x + \varepsilon)} + \mathcal{O}(1)$$

for  $n \gg_\varepsilon 0$  for irrational numbers  $x$  with finite irrational exposant  $\mu_x$  (i.e. non Liouville numbers). We remark that the proof of (1.1) only partially uses the strength of the irrationality measure  $|x - p/q| \geq q^{-\mu_x - \varepsilon}$  (valid for all rational  $p/q$  with  $q \gg_\varepsilon 1$ ). Indeed, we used this measure only when  $q$  is a power of 2. Therefore, it is enough to use an irrationality measure of the following form: given an irrational number  $x$  and an integer  $b \geq 2$ , there exists  $\nu_{x,b} \geq 1$  such that for all  $m \gg_\varepsilon 0$

$$\left| x - \frac{p}{b^m} \right| \geq \frac{1}{b^{(\nu_{x,b} + \varepsilon)m}}. \quad (7.1)$$

We deduce a small improvement of (1.1):

$$B_n(x) \geq \frac{\log(n)}{\log(\nu_{x,2} + \varepsilon)} + \mathcal{O}(1).$$

For example, Ridout's theorem yields  $\nu_{\alpha,b} = 1$  for all irrational algebraic numbers  $\alpha$  and all  $b \geq 2$ , where Roth's theorem only yields  $\nu_{\alpha,b} \leq 2$ . The situation is worse for numbers like  $e$  or  $\pi$  because, for all  $b \geq 2$ , we only know the trivial upper bounds  $\nu_{e,b} \leq \mu_e = 2$  and  $\nu_{\pi,b} \leq \mu_\pi \leq 8.1$ : we can not improve on the lower bounds for  $B_n(e)$  and  $B_n(\pi)$  provided by (1.1) when  $b = 2$ . On a more positive note, the author recently proved in [19] that for the numbers  $x = \log(1 - 1/b)$ , we have  $\nu_{x,|b|} \rightarrow 1$  as  $b \rightarrow \pm\infty$  but this is clearly not as good as Ridout's theorem for algebraic numbers.

In the most favorable case where  $\nu_{x,2} = 1$ , the presence of  $\varepsilon$  rules out the possibility of getting something better than  $B_n(x) \gg \log(n)$ . However, as we now show, an improvement

would occur if, in (7.1), we could replace  $q^{1+\varepsilon}$  by a more explicit function of  $q$ . It is not difficult to prove that, given a positive-valued and non-increasing function  $\psi$  defined on the powers  $(b^m)_{m \geq 0}$ , for almost all  $x$ , the equation

$$\left| x - \frac{p}{b^m} \right| < \frac{\psi(b^m)}{b^m} \quad (7.2)$$

admits finitely many solutions  $(p, m) \in \mathbb{Z} \times \mathbb{N}$  if the series  $\sum_{m \geq 0} \psi(b^m)$  converges (this result is a particular case of more general ones: see [10, Chapter 2]). The proof runs as follows: we restrict  $x \in [0, 1[$  and consider the set

$$E_m = \left[ 0, 1 \left[ \bigcap_{p=0}^{b^m-1} \right] \frac{p - \psi(b^m)}{b^m}, \frac{p + \psi(b^m)}{b^m} \right].$$

The set  $E$  of those  $x \in [0, 1[$  such that (7.2) admits infinitely many solutions  $(p, m)$  is  $E = \bigcap_{n \geq 0} \bigcup_{m \geq n} E_m$  and to conclude it is enough to prove that the Lebesgue measure  $\lambda(E)$  of  $E$  is 0. But

$$\lambda(E) \leq \sum_{m=n}^{\infty} \lambda(E_m) \leq 2 \sum_{m=n}^{\infty} \psi(b^m) \rightarrow 0$$

as  $n \rightarrow +\infty$  by hypothesis.

The converse is also true but more difficult to prove: this is a consequence of a theorem of Duffin-Schaeffer [9, p. 245], see also [10, Theorem 2.9]. Hence, with  $\psi(q) = 1/(\log(q) \log \log(q))$ , Equation (7.2) admits infinitely many solutions and, as pointed out by Harman, this implies that, for almost all  $x$ , there exist infinitely many  $n$  such that  $B_{n+m}(x) = B_n(x) + m$  for  $m = \lfloor \lambda \log_2(n) \rfloor$  when  $\lambda = 1$  and only finitely many  $n$  when  $\lambda > 1$ .

On the other hand, for all  $\rho > 1$ , Equation (7.2) admits almost surely only finitely many solutions with  $\psi(q) = 1/\log(q)^\rho$  and we can also draw some conclusions on  $B_n(x)$  from this when  $b = 2$ .

**Proposition 2.** *Let  $x$  be an irrational number. We suppose that there exists a constant  $\rho > 1$  (depending at most on  $x$ ) such that, for all  $p \in \mathbb{Z}$  and  $m \in \mathbb{N}$  with  $m \gg 0$ , we have*

$$\left| x - \frac{p}{2^m} \right| \geq \frac{1}{2^m \log^\rho(2^m)}. \quad (7.3)$$

Then, we have

$$B_n(x) \geq \frac{n}{\rho \log(n)} (1 + o(1)). \quad (7.4)$$

**Remark.** *Since for almost all  $x$ , estimate (7.3) holds, we also have that, for almost all  $x$ , (7.4) holds. But we already know that, for almost all  $x$ , the much stronger estimate  $B_n(x) \sim n/2$  holds. Furthermore, it seems difficult to deduce something better than (7.4) from (7.3) without further information on  $x$ . Thus, general methods based on diophantine inequalities such as (7.3) are probably not strong enough to prove the normality of natural constants. However, estimates of the form (7.3) might be simpler to prove than normality.*

*Proof.* We denote by  $p_k 2^{-n_k}$  the truncations of the binary expansion  $\sum_{k=1}^{\infty} 2^{-n_k}$  of  $x$ . For all  $k \gg 0$ , we have

$$\frac{1}{2^{n_k} \log^{\rho}(2^{n_k})} \leq \left| x - \frac{p_k}{2^{n_k}} \right| \leq \frac{1}{2^{n_{k+1}-1}},$$

which implies that  $n_{k+1} \leq n_k + \rho \log(n_k) + \rho \log \log(2) + 1$  for  $k \geq k_0$ . Clearly, we have  $n_k \leq u_k$  where  $u_k$  is defined by  $u_{k_0} = n_{k_0}$  and  $u_{k+1} = u_k + \rho \log(u_k) + \rho \log \log(2) + 1$  and it will be sufficient to bound  $u_k$ . We remark that in absence of further informations on  $x$ , bounding  $u_k$  is in fact the best we can do here since we can not exclude that  $n_{k+1} \geq n_k + \rho \log(n_k) + \mathcal{O}(1)$ .

We will prove below the following fact. For any  $a \geq 1$  and  $b > 0$ , let  $v_k$  be defined by  $v_1 = a$  and  $v_{k+1} = v_k + \log(v_k) + b$  (the conditions on  $a$  and  $b$  ensure that  $v_k$  is defined for all  $k \geq 1$ ). Then there exists a constant  $d = d(a, b)$  such that

$$v_k \leq k \log(k + d) + k \log \log(k + d) + bk. \quad (7.5)$$

Applying (7.5) to  $v_n = u_{n+k_0}/\rho$  (with a suitable  $b$ ), we obtain that  $u_k \leq \rho k \log(k) + g(k)$  with  $g(k) = \mathcal{O}(k \log \log(k))$  for  $k \geq 1$ . Hence, we have

$$\begin{aligned} B_n(x) &= \#\{k \geq 1 : n_k \leq n\} \\ &\geq \#\{k \geq 1 : u_k \leq n\} \\ &\geq \#\{k \geq 1 : \rho k \log(k) + g(k) \leq n\} \sim R(n) \end{aligned}$$

where  $R$  is the inverse function of  $k \mapsto \rho k \log(k) + g(k)$ . Since  $R(k) \sim k/(\rho \log(k))$  as  $k \rightarrow +\infty$ , equation (7.4) follows.

It remains to prove (7.5). We choose  $d$  such that firstly

$$\log(d + 1) + b + \log \log(d + 1) \geq a \quad (7.6)$$

and secondly, for all  $k \geq 1$ ,

$$b + \log \log(d + k) \leq \log(d + k) \quad (7.7)$$

(obviously, this is possible if  $d$  is large enough with respect to  $b$ ). The inequality  $\log(1+x) \leq x$  for all  $x > -1$  and (7.7) imply that, for all  $k \geq 1$ , we have

$$\log \left( 1 + \frac{b + \log \log(k + d)}{\log(k + d)} \right) \leq 1. \quad (7.8)$$

By *calculus*, one proves that

$$k \log(k + d) + \log(k) + 1 \leq (k + 1) \log(k + d + 1). \quad (7.9)$$

Inequalities (7.8) and (7.9) will be implicitly used below. We now proceed by induction. We first remark that (7.6) implies that (7.5) is true for  $k = 1$ . We suppose that (7.5) is

true for  $k$ . Then, we have

$$\begin{aligned}
 v_{k+1} &\leq k \log(k+d) + k \log \log(k+d) + bk \\
 &\quad + \log(k \log(k+d)) + \log \left( 1 + \frac{b + \log \log(k+d)}{\log(k+d)} \right) + b \\
 &\leq k \log(k+d) + \log(k) + 1 + (k+1) \log \log(k+d) + b(k+1) \\
 &\leq (k+1) \log(k+d+1) + (k+1) \log \log(k+d+1) + b(k+1),
 \end{aligned}$$

which completes the proof of the induction.  $\square$

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#### BIBLIOGRAPHY

- [1] B. Adamczewski, Y. Bugeaud, *On the complexity of algebraic numbers I. Expansions in integer bases*, to appear in *Annals of Math.*
- [2] B. Adamczewski, Y. Bugeaud, F. Luca, *Sur la complexité des nombres algébriques*, *C. R. Acad. Sci. Paris* **339** (2004), 11–14.
- [3] J.-P. Allouche, J. Shallit, *Automatic sequences. Theory, applications, generalizations*, Cambridge University Press, Cambridge, 2003.
- [4] D. H. Bailey, J. M. Borwein, *Mathematics by Experiments: Plausible Reasoning in the 21st century*, A. K. Peters, 2004.
- [5] D. H. Bailey, J. M. Borwein, R. E. Crandall, C. Pomerance, *On the binary expansions of algebraic numbers*, *J. Théor. Nombres Bordeaux* **16** (2004), no. 3, 487–518.
- [6] A. Blanchard, M. Mendès-France, *Symétrie et transcendance*, *Bull. Sci. Math. (2)* **106** (1982), no. 3, 325–335.
- [7] P. Bundschuh, *Irrationalitätsmaße für  $e^a$ ,  $a \neq 0$  rational oder Liouville-Zahl*, *Math. Ann.* **192** (1971), 229–242.
- [8] C. S. Davis, *Rational approximations to  $e$* , *J. Aust. Math. Soc.* **25** (1978), 497–502.
- [9] R. J. Duffin, A. C. Schaeffer, *Khinchine’s problem in metric diophantine approximation*, *Duke Math. J.* **41**, 241–255.
- [10] G. Harman, *Metric Number Theory*, London Mathematical Society Monographs **18**, The Clarendon Press, Oxford University Press, New York, 1998.
- [11] M. Hata, *Rational approximations to  $\pi$  and some other numbers*, *Acta Arith.* **63.3** (1993), 335–349.
- [12] A. J. Kempner, *On transcendental numbers*, *Trans. AMS* **17** (1916), 476–482.
- [13] M. J. Knight, *An ‘ocean of zeroes’ proof that a certain non-Liouville number is transcendental*, *Amer. Math. Monthly* **98** (1991), 947–949.
- [14] S. Lehr, J. Shallit, J. Tromp, *On the vector space of the automatic reals*, *Theoret. Comput. Sci.* **163** (1996), no. 1-2, 193–210.
- [15] B. Lindström, *On the binary digits of a power*, *J. Number Theory* **65** (1997), no. 2, 321–324
- [16] I. Niven, *Irrational numbers*, Third printing, The Carus Mathematical Monographs, Number 11, 1967.
- [17] K. Nishioka, *Mahler functions and transcendence*, *L. N. M.* **1631**, Springer, 1996.
- [18] D. Ridout, *Rational approximations to algebraic numbers*, *Mathematika* **4** (1957), 125–131.
- [19] T. Rivoal, *Convergents and irrationality measures of logarithms*, 18 pages, to appear in *Rev. Mat. Iberoamericana*.
- [20] K. F. Roth, *Rational approximations to algebraic numbers*, *Mathematika* **2** (1955), 1–20 ; corrigendum 168.

- [21] K. B. Stolarsky, *The binary digits of a power*, Proc. Amer. Math. Soc. **71** (1978), no. 1, 1–5.
- [22] E. Zeckendorf, *Représentation des nombres naturels par une somme des nombres de Fibonacci ou de nombres de Lucas*, Bull. Soc. Roy. Sci. Liège **41** (1972), 179–182.

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