Values of the Beta function: from Ramanujan's continued fraction to Hermite-Padé approximants

T. Rivoal

November 2, 2017

1 Introduction

In two recent papers [10, 11], the author presented new methods to generate sequences of rational numbers approximating the values of the Gamma function at rational points. The motivation was Diophantine, though no new arithmetical result was obtained. It is conjectured that $\Gamma(1/m)$ is transcendental for any integer $m \geq 2$, and more generally, the algebraic relations amongst the values of the Gamma function at the rational points are predicted by the Rohrlich-Lang conjecture (see [12] for a precise statement). So far, only $\Gamma(1/2) = \sqrt{\pi}, \Gamma(1/3), \Gamma(1/4)$ and $\Gamma(1/6)$ are already known to be irrationnal (and in fact they are transcendental, see [15]). In this paper, we construct good rational approximations not to the Gamma function itself but to certain values of the Beta function, which is defined by $B(x, y) := \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$. No new result will be given here concerning the arithmetic nature of the values of $B(x, y)$ for $x, y \in \mathbb{Q}$, even though our approximations fit into the now classical pattern developped for values of the zeta functions and, more generally, periods. However, in the particular case of Beta values, the Rohrlich-Lang conjecture is known to be true, by a classical result of Wolfart-Wüstholz [16]. But it is still important to find new methods in this field.

To prove the irrationality of a given number, a classical strategy is to first construct good explicit functional rational approximations to a certain function that can be specialized to our number. Hermite-Padé approximants are natural candidates to construct functional rational approximations. We recall that given $\ell \geq 2$ formal power series $F_1(z), \dots, F_\ell(z) \in \mathbb{C}[[z]]$, there exist some polynomials $P_{n,1}(z), \dots, P_{n,\ell} \in \mathbb{C}[z]$ of degree at most n such that $P_{n,1}(z)F_1(z) + \dots + P_{n,\ell}(z)F_\ell(z)$ has order at least $\ell(n+1)$ at $z = 0$; these polynomials are called the diagonal Hermite-Padé approximants $[n, \dots, n]$ of $F_1(z), \dots, F_\ell(z)$ at $z = 0$. However, in the case of the Gamma function, it is not at all clear at which point such approximations should be constructed. The obvious choice is to compute Padé approximants of the function $\Gamma(1 + x)$ at $x = 0$, where it is holomorphic. However, its n-th Taylor coefficient at $x = 0$ is a polynomial in $\gamma, \zeta(2), \dots, \zeta(n)$, and thus presumably a transcendental number; this would at best produce some sort of approximations of $\Gamma(1/m)$ in terms of these quantities, which is probably useless from a Diophantine point of view. Another possibility is to compute Padé approximants of $\Gamma(x)$ at $x = \infty$. There are some issues either. Indeed, this function is not holomorphic at infinity and it does not even have an asymptotic expansion in powers of $1/x$. We can get rid of the extra factor $x^{x+1/2}e^{-x}$ in Stirling's formula but at the cost of considering quotient of functions involving the Gamma function, such as

$$R(x,a) := \frac{\Gamma(x)^2}{\Gamma(x+a)\Gamma(x-a)} = \frac{B(x,a)}{B(x-a,a)}.$$

This is the reason for the shift from the Gamma function to the Beta function in this paper. Ramanujan found an explicit continued fraction for $R(x,a)$ (see [2, 3]), which is equivalent to the explicit construction of Padé approximants of the divergent Taylor series of $R(x,a)$ at $x = \infty$. Our main result is a generalization of this fact, and apparently it offers a new proof of Ramanujan's continued fraction expansion of $R(x,a)$.

Let us now describe our construction. For any $\ell \geq 2$, let $a, b_1, \dots, b_\ell$ be complex numbers such that $b_i - b_j \notin \mathbb{Z}$ for any $i \neq j$. We recall the definition of Pochhammer symbol: for any $x \in \mathbb{C}$, $(x)_0 = 1$ and $(x)_n = x(x+1)\cdots(x+n-1)$ for $n \geq 1$. We define the contour integral

$$I_n(x,a,\mathbf{b}) := (-1)^{\ell(n+1)+1}\frac{n!^{\ell-1}}{2i\pi}\int_{\mathcal{C}_n} \frac{\Gamma(z)\Gamma(a)}{\Gamma(z+a)} \cdot \frac{(1-z-a)_n}{\prod_{j=1}^{\ell}(z-x+b_j-n)_{n+1}}dz. \qquad (1.1)$$

The factor $n!^{\ell-1}$ is an arithmetic normalization and $\mathbf{b}$ stands for $b_1, \dots, b_\ell$. The closed direct simple path $\mathcal{C}_n$ encloses the zeros $x + b_j + k$ $(k = 0, \dots, n)$ of the polynomial $\prod_{j=1}^{\ell}(z-x-b_j-n)_{n+1}$, which are all distinct, but not the poles of $\Gamma(z)/\Gamma(z+a)$. This is possible in particular if $x - b_j \notin \mathbb{Z}_{\leq 0}$ $(j = 1, \dots, \ell)$, which we will assume from now on, and $I_n(x,a,b)$ is analytic at any such x. (In fact, $I_n(x,a,b)$ is analytic in a larger domain, whose definition depends on n.)

We shall use the integral $I_n(x,a,b)$ to construct diagonal Hermite-Padé approximations to the Taylor series at $x = \infty$ of the functions

$$B(x-b_j,a) := \frac{\Gamma(a)\Gamma(x-b_j)}{\Gamma(x+a-b_j)}, \qquad j = 1, \dots, \ell.$$

Note that $B(x-b_j,a)$ is defined at least for any x such that $x - b_j \notin \mathbb{Z}_{\leq 0}$ $(j = 1, \dots, \ell)$, but it is not meromorphic at $x = \infty$. It has there a (divergent) asymptotic expansion of the form $\sum_{n\geq 0} c_{j,n}/x^{n+a+1}$ as $x \to \infty$ in at least the half-plane $\mathcal{P}_j := \{x : \mathrm{Re}(x) > \max(\mathrm{Re}(b_j), \mathrm{Re}(b_j - a))\}$. We denote by

$$\mathcal{P} := \{x : \mathrm{Re}(x) > \max(\mathrm{Re}(b_1), \mathrm{Re}(b_1 - a), \dots, \mathrm{Re}(b_\ell), \mathrm{Re}(b_\ell - a))\}$$

the half-plane defined as the intersection of the $\mathcal{P}_j$'s, $j = 1, \dots, \ell$. For simplicity, we will consider asymptotic expansions as $x \to \infty$ in $\mathcal{P}$.

By definition, the Hermite-Padé approximants are defined for (formal) series of the form $\sum_{n\geq 0} c_{j,n}/x^{n+a+1}$. Strictly speaking, we should get rid of the factor $1/x^a$ by normalizing the functions to $\frac{B(x-b_j,a)}{B(x-a,a)}$ or to $x^a B(x-b_j,a)$ for instance. We will not explicitly choose such a normalization in Theorem 1 because the context is clear. Nonetheless, we will choose the normalized factor as $\frac{1}{B(x-a,a)} := \frac{\Gamma(x)}{\Gamma(a)\Gamma(x-a)}$ in Corollary 1. Given a formal Laurent expansion or an asymptotic expansion $\sum_{n\geq -N} d_n/x^{a+n}$, we say that it is $\mathcal{O}(1/x^{a+m+1})$ if $d_{-N} = d_{-N-1} = \ldots = d_m = 0$.

For any $j = 1, \ldots, \ell$, let us now define the polynomial of degree n in x:

$$P_{n,j}(x,a,\mathbf{b}) := (-1)^{\ell(n+1)+1} \sum_{k=0}^{n} \binom{n}{k} \frac{n!^{\ell-2}(x-b_j)_k(1-x+b_j-a)_{n-k}}{\prod_{p=1,p\neq j}^{\ell}(k-b_j+b_p-n)_{n+1}}. \tag{1.2}$$

If a is an integer, the $B(x-b_j,a)$ are rational functions and this case is of no interest to us. From now on, we assume that $a \notin \mathbb{Z}$, even though some of the results below might be true when $a \in \mathbb{Z}$.

Theorem 1. *(i) For any $x - b_j \notin \mathbb{Z}_{\leq 0}$ $(j = 1, \ldots, \ell)$, we have the identity*

$$I_n(x,a,\mathbf{b}) = \sum_{j=1}^{\ell} P_{n,j}(x,a,\mathbf{b})B(x-b_j,a) \tag{1.3}$$

and the asymptotic expansion of this function is $\mathcal{O}\big(1/x^{a+(\ell-1)(n+1)}\big)$ when $x \to \infty$ in $\mathcal{P}$.

In particular, the polynomials $P_{n,1}(x,a,\mathbf{b}), \ldots, P_{n,\ell}(x,a,\mathbf{b})$ are the diagonal Hermite-Padé approximants $[n, \ldots, n]$ of the formal Taylor series at $x = \infty$ of the (normalized) functions $B(a, x-b_1), \ldots, B(a, x-b_\ell)$.

(ii) For any x such that $\mathrm{Re}(x) > \max_j \mathrm{Re}(b_j)$ and if $(\ell-1)(n+1) > -\mathrm{Re}(a)$, we have

$$I_n(x,a,\mathbf{b}) = \frac{(1-a)_n}{n!} \int_{[0,1]^\ell} \frac{\prod_{j=1}^{\ell} u_j^{x-b_j-1}(1-u_j)^n}{(1-u_1\cdots u_\ell)^{n-a+1}} du_1 \cdots du_\ell. \tag{1.4}$$

Remarks. In (ii), the conditions on x, n, a and the b_j's are sufficient to ensure that both sides of (1.4) are defined simultaneously. They could be relaxed.

We provide more details in the case $\ell = 2$, $b_1 = 0$, $b_2 = a$ of Theorem 1. We define two polynomials in $\mathbb{Q}(a)[x]$ of degree n in x by

$$P_n(x,a) = -\sum_{k=0}^{n} \binom{n}{k} \frac{(x-a)_k(1-x)_{n-k}}{(k-a-n)_{n+1}}, \quad Q_n(x,a) = \sum_{k=0}^{n} \binom{n}{k} \frac{(x)_k(1-x-a)_{n-k}}{(k+a-n)_{n+1}},$$

which are equal to $P_{n,2}(x,a,0,a)$ and $-P_{n,1}(x,a,0,a)$ respectively.

Corollary 1. *(i) As $x \to \infty$ in the half-plane $\{x : \mathrm{Re}(x) > |\mathrm{Re}(a)|\}$, we have*

$$Q_n(x,a)R(x,a) - P_n(x,a) = \mathcal{O}\Big(\frac{1}{x^{n+1}}\Big). \tag{1.5}$$

In particular, $\frac{P_n(x,a)}{Q_n(x,a)}$ is the n-th diagonal Padé approximant $[n/n]$ of the Taylor series of Ramanujan's quotient $R(x,a)$ at $x=\infty$

(ii) The sequences $(P_n(x,a))_{n\geq 0}$ and $(Q_n(x,a))_{n\geq 0}$ satisfy the linear recurrence of order 2:

$$U_{n+2} = \frac{(2n+3)(2x-1)}{(n-a+2)(n+a+2)}U_{n+1} + \frac{(n-a+1)(n+a+1)}{(n-a+2)(n+a+2)}U_n. \tag{1.6}$$

Recurrence (1.6) can be rephrased as follows: we have the continued fraction "identity"

$$R(x,a) \approx \\ 1 - \frac{2a^2}{|\,2x-1+a^2} + \frac{(a^2-1)^2}{|\,6x-3} + \frac{(a^2-4)^2}{|\,10x-5} + \cdots + \frac{(a^2-m^2)^2}{|\,(2m+1)(2x-1)} + \cdots. \tag{1.7}$$

For any $n \geq 0$, the n-th convergent of the continued fraction is $P_n(x,a)/Q_n(x,a)$ and $\approx$ means exactly (1.5). This continued fraction is essentially due to Ramanujan [2] (see Section 3.1 for the details) so that Theorem 1 is indeed a generalization of Ramanujan's work, as claimed before.

In principle, diagonal Padé approximants may provide effective numerical approximations to the values of the underlying function, i.e. we may have the pointwise limit $\lim_{n\to+\infty}[n/n]_f(x) = f(x)$. It is natural to expect that we could replace $\approx$ by equality in (1.7) for any given x in a suitable domain. This is not always the case in the present situation. For instance, we will show in Section 3.2 that for all n and $a \notin \mathbb{Z}$, we have $\frac{P_n(1/2,a)}{Q_n(1/2,a)} = (-1)^{n+1}$, which clearly does not converge to $\frac{\Gamma(1/2)^2}{\Gamma(1/2+a)\Gamma(1/2-a)} = \cos(\pi a)$. However, this example is extremal because it is known that we have pointwise convergence in (1.7) when $\mathrm{Re}(x) > \frac{1}{2}$; see Section 3.1.

Similar non-convergence phenomenons occur in the more general setting of Theorem 1. We did not try to find out when we have pointwise convergence because the resulting numerical approximations are a priori too slow to be of any use. Instead, we shall show how to accelerate the convergence. For this, we change x to $x+rn$ (for some new fixed integer parameter $r \geq 1$) and observe that

$$B(x+rn-b_j,a) = \frac{(x-b_j)_{rn}}{(x+a-b_j)_{rn}}\,B(x-b_j,a).$$

We now set

$$\widetilde{P}_{n,j}(x,a,\mathbf{b}) = \frac{(x-b_j)_{rn}}{(x+a-b_j)_{rn}}P_{n,j}(x+rn,a,\mathbf{b}), \tag{1.8}$$

which is no longer a polynomial in x but is in $\mathbb{Q}(x,a)$.

Theorem 2. *For any x,n such that $rn > \max_j \mathrm{Re}(b_j - x)$, and $x-b_j$ are not in $\mathbb{Z}_{\leq 0}$ (for any $j=1,\ldots,\ell$), we have*

$$\sum_{j=1}^{\ell} \widetilde{P}_{n,j}(x,a,\mathbf{b})B(x-b_j,a) = \frac{(1-a)_n}{n!}\int_{[0,1]^\ell} \frac{\prod_{j=1}^{\ell} u_j^{x+rn-b_j-1}(1-u_j)^n}{(1-u_1\cdots u_\ell)^{n-a+1}}du_1\cdots du_\ell. \tag{1.9}$$

For any $\ell \geq 2, r \geq 1$, let ρ be the unique number in $(0,1)$ such that $r\rho^\ell - r + \rho\frac{\rho^{\ell-1}-1}{\rho-1} = 0$. Then

$$\lim_{n\to+\infty} \left|\sum_{j=1}^{\ell} \widetilde{P}_{n,j}(x,a,\mathbf{b})B(x-b_j,a)\right|^{1/n} = \frac{\rho^{r\ell}(1-\rho)^\ell}{1-\rho^\ell} < 1. \tag{1.10}$$

For any $\ell \geq 2, r \geq 1$, let η be the unique number in $(0,1)$ such that $\frac{(r+\eta)(1-\eta)^\ell}{\eta^\ell(r+\eta-1)} = 1$. Then, for any $j \in \{1,\ldots,\ell\}$,

$$\limsup_{n\to+\infty} \left|\widetilde{P}_{n,j}(x,a,\mathbf{b})\right|^{1/n} \leq \frac{(r+\eta)^r}{(1-\eta)^\ell(r+\eta-1)^{r-1}}, \tag{1.11}$$

When ℓ is even, the limsup is a limit and there is equality in (1.11).

Remarks. The conditions on x and n are sufficient to ensure the convergence of the integral. We can split x and n in the condition $rn > \max_j \mathrm{Re}(b_j - x)$ by assuming the stronger conditions $\mathrm{Re}(x) > 0$ and $rn \geq \max_j \mathrm{Re}(b_j)$.

When ℓ is odd, the method giving equality in (1.11) for ℓ even does not work and it is an open problem to determine the exact rate of growth of each $\widetilde{P}_{n,j}(x,a,\mathbf{b})$ as $n \to \infty$. We comment on this after the proof of Theorem 2.

Again, we extract the case $\ell = 2$, $b_1 = 0, b_2 = a$. With the notations of Corollary 1 and in accordance with (1.8), we set

$$\widetilde{P}_n(x,a) = \frac{(x-a)_{rn}}{(x)_{rn}} P_n(x+rn,a), \quad \widetilde{Q}_n(x,a) = \frac{(x)_{rn}}{(x+a)_{rn}} Q_n(x+rn,a).$$

Corollary 2. *For any $r \geq 1$, let $\rho = \frac{\sqrt{4r^2+1}-1}{2r} \in (0,1)$.*

(i) For any $x \notin \mathbb{Z}_{\leq 0}$, when $n \to +\infty$,

$$\left|\widetilde{Q}_n(x,a)R(x,a) - \widetilde{P}_n(x,a)\right| \sim \Big(\frac{\rho^{2r}(1-\rho)}{1+\rho}\Big)^{n+o(n)} \to 0. \tag{1.12}$$

(ii) For any $x \in \mathbb{C}$, when $n \to +\infty$,

$$\left|\widetilde{Q}_n(x,a)\right| \sim \Big(\frac{1+\rho}{\rho^{2r}(1-\rho)}\Big)^{n+o(n)} \to +\infty. \tag{1.13}$$

(iii) When $r = 1$, the function $R(x,a)$ can be represented as a rapidly convergent continued fraction with elements in $\mathbb{Q}[x,a]$ whose n-th convergent is $\widetilde{P}_n(x,a)/\widetilde{Q}_n(x,a)$.

The polynomials $\widetilde{P}_n(x,a)$ and $\widetilde{Q}_n(x,a)$ are solutions of a linear recurrence of finite order with polynomials coefficients in n, x, a that also depend on r (see the beginning of Section 4). We don't know how to write down explicitely the recurrence if we don't specify the value of r. We display the recurrence in the case $r = 1$ in Section 4, which is of order 2. Together with (i), this immediately translates into the convergent continued fraction alluded to in (ii), whose n-th convergent is $\widetilde{P}_n(x,a)/\widetilde{Q}_n(x,a)$. As an illustration, we write down the continued fraction for $x = 1/3$, $a = 2/3$. In this case, $(\widetilde{P}_n(x,a))_{n\geq 0}$ and $(\widetilde{Q}_n(x,a))_{n\geq 0}$ satisfy

$$U_{n+2} = \frac{s(n)}{r(n)}U_{n+1} + \frac{t(n)}{r(n)}U_n$$

with

$$r(n) = 3(3n+8)(15n+13)(3n+4)(n+2),$$
$$s(n) = 4455n^4 + 23166n^3 + 43668n^2 + 35199n + 10204,$$
$$t(n) = (3n+5)(3n-1)(3n+1)(15n+28).$$

We thus obtain the convergent continued fraction

$$-\frac{\Gamma(1/3)^3}{2\pi\sqrt{3}} = \frac{\Gamma(1/3)^2}{\Gamma(-1/3)} = 1 + \frac{104 \rvert}{\lvert s(-1)} + \frac{r(-1)t(0)\rvert}{\lvert s(0)} + \cdots + \frac{r(m-1)t(m)\rvert}{\lvert s(m)} + \cdots ,$$

whose rate of convergence is given in Corollary 2, (ii) with $r = 1$.

In [4], the authors construct simultaneous Padé approximants of type II to functions of the form $B(a_j, x)$. They do not require a large order of approximations at $x = \infty$, but instead the cancellation of the approximating forms at many chosen distinct points. This approach does not overlap with ours. We also refer to [11] for the construction of numerical approximations of the values $\Gamma(\alpha)$ for any $\alpha \in \mathbb{C}$, based on Padé approximation to series of the form $\sum_{n\geq 0} \Gamma(n+\alpha)x^n$; the approximations to Gamma values are obtained in a very indirect way, not as in this paper. We also refer to [10] for the constuction of numerical rational approximations to the values $\Gamma(a/b)^b$, a, b integers; the construction involves an integral similar to $I_n(x, a, \mathbf{b})$, but no interpretation in terms of Padé approximation is given. A continued fraction for the period $\Gamma(1/3)^3$ is also given.

2 Proof of the results

2.1 Proof of Theorem 1

Since a is not an integer, the poles of the integrand of $I_n(x, a, b)$ are simple. By the residue theorem, we thus readily obtain

$$I_n(x, a, \mathbf{b}) = \sum_{j=1}^{\ell} P_{n,j}(x, a, \mathbf{b}) B(x - b_j, a)$$

for any x such that $x - b_j \notin \mathbb{Z}_{\leq 0}$ $(j = 1, \ldots, \ell)$, where the $P_{n,j}(x, a, \mathbf{b})$ are the polynomials of degree n in x defined by (1.2) in the Introduction.

To prove the other assertions, we will first find another expression for $I_n(x,a,\mathbf{b})$. We define the integral

$$K(N) := (-1)^{\ell(n+1)+1}\frac{n!^{\ell-1}}{2i\pi}\int_{\widetilde{\mathcal{C}}_N}\frac{\Gamma(z)\Gamma(a)}{\Gamma(z+a)}\cdot\frac{(1-z-a)_n}{\prod_{j=1}^{\ell}(z-x+b_j-n)_{n+1}}dz$$

over the circle $\widetilde{\mathcal{C}}_N$ of center 0 and radius half an integer $N+\eta$ (where $\eta\in(0,1)$ depends on a and the b_j's, but not N) and is such that the circle does not pass on the poles of the integrand. By standard estimates based on Stirling's formula (and the complements formula when z is closed to the negative axis), we get the estimate

$$\left|\frac{\Gamma(z)\Gamma(a)}{\Gamma(z+a)}\cdot\frac{(1-z-a)_n}{\prod_{j=1}^{\ell}(z-x+b_j-n)_{n+1}}\right| \ll \frac{1}{|N|^{(n+1)(\ell-1)+\mathrm{Re}(a)+1}}$$

for $z\in\widetilde{\mathcal{C}}_N$ as $N\to+\infty$. Therefore, $\lim_{N\to+\infty}K(N)=0$ provided that $(\ell-1)(n+1)>-\mathrm{Re}(a)$. Hence, under this condition on n, the sum of the residues over all the poles of the integrand is equal to 0. In other words,

$$I_n(x,a,\mathbf{b}) = (-1)^{\ell(n+1)}n!^{\ell-1}\sum_{k=0}^{\infty}\mathrm{Residue}\left(\frac{\Gamma(z)\Gamma(a)}{\Gamma(z+a)}\cdot\frac{(1-z-a)_n}{\prod_{j=1}^{\ell}(z-x+b_j-n)_{n+1}}, z=-k\right).$$

The poles involved here are those of $\Gamma(z)$ at the negative integers $-k$, which are simple with residue $(-1)^k/k!$.

After some simplifications, we get that, when $(\ell-1)(n+1)>-\mathrm{Re}(a)$,

$$\begin{aligned}I_n(x,a,\mathbf{b}) &= (-1)^{\ell(n+1)}n!^{\ell-1}(1-a)_n\sum_{k=0}^{\infty}\frac{(n-a+1)_k}{k!\prod_{j=1}^{\ell}(k+x-b_j)_{n+1}}\\ &= \frac{n!^{\ell-1}(1-a)_n}{\prod_{j=1}^{\ell}(x-b_j)_{n+1}}\cdot {}_{\ell+1}F_{\ell}\left[\begin{matrix}n-a+1, x-b_1,\dots,x-b_\ell\\ n+x-b_1+1,\dots,n+x-b_\ell+1\end{matrix};1\right].\end{aligned}$$

(The last equality is just the definition of an hypergeometric series.)

We now use a classical integral represention of such series to obtain

$$I_n(x,a,\mathbf{b}) = \frac{(1-a)_n}{n!}\int_{[0,1]^\ell}\frac{\prod_{j=1}^{\ell}u_j^{x-b_j-1}(1-u_j)^n}{(1-u_1\cdots u_\ell)^{n-a+1}}du_1\cdots du_\ell$$

for any $x\in\mathcal{P}$, provided $(\ell-1)(n+1)>-\mathrm{Re}(a)$. Under the same condition on n, we can relax the condition $x\in\mathcal{P}$ to $\mathrm{Re}(x)>\max_j\mathrm{Re}(b_j)$ by analytic continuation of both sides of the identity.

Since each $B(x-b_j,a)$ has an asymptotic expansion $\sum_{m\geq 0}c_{j,m}/x^{m+a}$ as $x\to\infty$ in $\mathcal{P}$, $I_n(x,a,\mathbf{b})$ also has an asymptotic expansion of the form $\sum_{m\geq -n}C_m/x^{m+a}$. We will now prove that $C_{-n}=C_{-n+1}=\cdots=C_{(\ell-1)(n+1)-1}=0$. Let us assume for the moment that

x is a real number and set $A = \mathrm{Re}(a)$, $B_j = \mathrm{Re}(b_j)$. For any $u_1, \dots, u_\ell \in [0,1]$, we have $(1-u_1\cdots u_\ell)^\ell \geq (1-u_1)\cdots(1-u_\ell)$ so that

$$\left|\int_{[0,1]^\ell} \frac{\prod_{j=1}^{\ell} u_j^{x-b_j-1}(1-u_j)^n}{(1-u_1\cdots u_\ell)^{n-a+1}} du_1\cdots du_\ell\right| \leq \int_{[0,1]} \prod_{j=1}^{\ell} u_j^{x-B_j-1}(1-u_j)^{n-\frac{n-A+1}{\ell}} du_j$$
$$= \prod_{j=1}^{\ell} \frac{\Gamma(x-B_j)\Gamma(n-\frac{n-A+1}{\ell}+1)}{\Gamma(x-B_j+n-\frac{n-A+1}{\ell}+1)}.$$

Therefore,

$$|I_n(x,a,\mathbf{b})| \leq \left|\frac{(1-a)_n}{n!}\prod_{j=1}^{\ell}\frac{\Gamma(x-B_j)\Gamma(n-\frac{n-A+1}{\ell}+1)}{\Gamma(x-B_j+n-\frac{n-A+1}{\ell}+1)}\right|. \tag{2.1}$$

As $x \to +\infty$, $x>0$, Stirling's formula shows that for any complex numbers α, β, we have

$$\left|\frac{\Gamma(x+\alpha)}{\Gamma(x+\beta)}\right| \ll \frac{1}{x^{\mathrm{Re}(\beta-\alpha)}}.$$

Applying this estimate to the right-hand side of (2.1), we see that it is $\mathcal{O}(1/x^{(\ell-1)(n+1)+A})$ as $x \to +\infty$. Now, if we denote by M the smallest integer such that $C_M \neq 0$ in the asymptotic expansion of $I_n(x,a,b)$ in $\mathcal{P}$, we have

$$\left|\frac{1}{x^{M+a}}\right| \ll \frac{1}{x^{(\ell-1)(n+1)+A}}$$

as $x \to +\infty$, $x>0$, where the implicit constant is independent of x. Since $|x^a| = x^A$ when $x>0$, this forces $M \geq (\ell-1)(n+1)$ as expected.

We have thus proved that the asymptotic expansion of

$$\sum_{j=1}^{\ell} P_{n,j}(x,a,\mathbf{b})B(x-b_j,a)$$

is $\mathcal{O}(1/x^{(\ell-1)(n+1)+a})$ as $x\to\infty$ in $\mathcal{P}$. We refer to [9, Section 2] for the explanation of why this implies that the polynomials $P_{n,1}(x,a,\mathbf{b}),\dots,P_{n,\ell}(x,a,\mathbf{b})$ are the diagonal Hermite-Padé approximants $[n,\dots,n]$ of the formal Taylor series at $x=\infty$ of the (normalized) functions $B(a,x-b_1),\dots,B(a,x-b_\ell)$.

2.2 Proof of Corollary 1

The first assertion (i) is an immediate consequence of Theorem 1. To prove (ii), we run Zeilberger's algorithm on $P_n(x,a)$ and $Q_n(x,a)$: we observe that they do satisfy the same recurrence (1.6).

2.3 Proof of Theorem 2

Eq. (1.9) is an immediate consequence of Eqs. (1.3) and (1.4) with $x+rn$ instead of x.

It is standard that

$$\lim_{n\to+\infty}\left|\int_{[0,1]^\ell}\frac{\prod_{j=1}^{\ell}u_j^{x+rn-b_j-1}(1-u_j)^n}{(1-u_1\cdots u_\ell)^{n-a+1}}du_1\cdots du_\ell\right|^{1/n}=\sup_{(u_1,\ldots,u_\ell)\in[0,1]^\ell}\frac{\prod_{j=1}^{\ell}u_j^r(1-u_j)}{1-u_1\cdots u_\ell}.$$

This supremum is achieved on the diagonal $u_1=\ldots=u_\ell=\rho$ where ρ is the unique number in $(0,1)$ such that $r\rho^\ell-r+\rho\frac{\rho^{\ell-1}-1}{\rho-1}=0$. (Note that for fixed ℓ, we have $\rho=1-\frac{\ell-1}{\ell r}+\mathcal{O}(\frac{1}{r^2})$ as $r\to+\infty$.) See [1] for more details in a similar situation. We thus have

$$\lim_{n\to+\infty}\left|\int_{[0,1]^\ell}\frac{\prod_{j=1}^{\ell}u_j^{x-b_j-1}(1-u_j)^n}{(1-u_1\cdots u_\ell)^{n-a+1}}du_1\cdots du_\ell\right|^{1/n}=\frac{\rho^{r\ell}(1-\rho)^\ell}{1-\rho^\ell}=\frac{\rho^{r\ell}(1-\rho)^{\ell-1}}{1+\rho+\ldots+\rho^{\ell-1}}<1. \tag{2.2}$$

For fixed ℓ, we have $\frac{\rho^{r\ell}(1-\rho)^\ell}{1-\rho^\ell}\sim\frac{\ell^\ell}{(\ell+1)^{\ell+1}}\cdot\frac{1}{e^\ell r^\ell}$ as $r\to+\infty$.

We shall now indicate how to bound $\widetilde{P}_{n,j}(x,a,\mathbf{b})$ as $n\to+\infty$. We recall that

$$\widetilde{P}_{n,j}(x,a,\mathbf{b})=\frac{(x)_{rn}}{(x+a)_{rn}}\sum_{k=0}^{n}\binom{n}{k}\frac{n!^{\ell-2}(x+rn-b_j)_k(1-x-rn+b_j-a)_{n-k}}{\prod_{p=1,p\neq j}^{\ell}(k-b_j+b_p-n)_{n+1}}, \tag{2.3}$$

which is defined for any $x\in\mathbb{C}$.

We first assume that ℓ is even. By Stirling's formula, we obtain that

$$\widetilde{P}_{n,j}(x,a,\mathbf{b})=c_n(x,a,\mathbf{b})\sum_{k=0}^{n}\binom{n}{k}^\ell\frac{(rn)_k((r-1)n+k)_{n-k}}{n!} \tag{2.4}$$

where $|c_n(x,a,\mathbf{b})|^{1/n}\to1$ when $n\to+\infty$. To prove (2.4), it is important that the sum has positive terms (the fact that ℓ is even is implicitly used here) and moreover this fact even enables us to estimate precisely the sum by the discrete Laplace method. We refer to [1] or [5] where the details are given in similar situations. We first change k to tn for $t\in[0,1]$ and we determine the asymptotic behavior of $\binom{n}{k}^\ell\frac{(rn)_k((r-1)n+k)_{n-k}}{n!}$ as n to $+\infty$. By Stirling's formula,

$$\binom{n}{k}^\ell\frac{(rn)_k((r-1)n+k)_{n-k}}{n!}=d_n(t)\left(\frac{(r+t)^{r+t}}{t^{\ell t}(1-t)^{\ell(1-t)}(r+t-1)^{r+t-1}}\right)^n$$

where $d_n(t)^{1/n}\to1$. Taking logarithmic derivatives to simplify the computations, one proves that

$$\varphi(t):=\frac{(r+t)^{r+t}}{t^{\ell t}(1-t)^{\ell(1-t)}(r+t-1)^{r+t-1}}$$

has a unique maximum in $(0,1)$ achieved at $t=\eta$, where η is the unique number in $(0,1)$ such that

$$\frac{(\eta+r)(1-\eta)^\ell}{\eta^\ell(\eta+r-1)}=1.$$

(For fixed ℓ, $\eta=\frac{1}{2}+\frac{1}{4\ell r}+\mathcal{O}(\frac{1}{r^2})$ when $r\to+\infty$.) We conclude that, when ℓ is even,

$$\lim_{n\to+\infty}\left|\widetilde{P}_{n,j}(x,a,\mathbf{b})\right|^{1/n}=\varphi(\eta)=\frac{(r+\eta)^r}{(1-\eta)^\ell(r+\eta-1)^{r-1}}>1. \tag{2.5}$$

For fixed ℓ, we have $\varphi(\eta)\sim 2^\ell r$ as $r\to+\infty$.

If ℓ is odd, it is more complicated to obtain the exact behavior of $|\widetilde{P}_{n,j}(x,a,\mathbf{b})|$. We comment on the issues after the proof. However, an upper bound is easily obtained. Indeed, from (2.3), we have

$$|\widetilde{P}_{n,j}(x,a,\mathbf{b})|\le\left|\frac{(x)_{rn}}{(x+a)_{rn}}\right|\sum_{k=0}^n\binom{n}{k}\frac{n!^{\ell-2}|(x+rn-b_j)_k(1-x-rn+b_j-a)_{n-k}|}{|\prod_{p=1,p\neq j}^\ell(k-b_j+b_p-n)_{n+1}|},$$

from which we deduce that

$$|\widetilde{P}_{n,j}(x,a,\mathbf{b})|\le\widetilde{c}_n(x,a,\mathbf{b})\sum_{k=0}^n\binom{n}{k}^\ell\frac{(rn)_k((r-1)n+k)_{n-k}}{n!}$$

where $|\widetilde{c}_n(x,a,\mathbf{b})|^{1/n}\to 1$. Hence, we deduce as above that $\limsup_{n\to+\infty}|\widetilde{P}_{n,j}(x,a,\mathbf{b})|^{1/n}\le\varphi(\eta)$. This completes the proof of Theorem 2.

We now explain the difficulties when ℓ is odd. A computation similar to the one giving (2.4) would formally lead to

$$\widetilde{P}_{n,j}(x,a,\mathbf{b})\overset{?}{=}\widehat{c}_n(x,a,\mathbf{b})\sum_{k=0}^n(-1)^{n-k}\binom{n}{k}^\ell\frac{(rn)_k((r-1)n+k)_{n-k}}{n!} \tag{2.6}$$

where $|\widehat{c}_n(x,a,\mathbf{b})|^{1/n}\to 1$. However, it is not clear to us that (2.6) is true because the sum on the right-hand side is alternating. Even if (2.6) holds, getting the exact asymtotic behavior of this sum does not seem to be easy. It might be possible to estimate precisely $\widetilde{P}_{n,j}(x,a,\mathbf{b})$ by applying the saddle point method after transformation of (2.3) into the contour integral:

$$\begin{aligned}\widetilde{P}_{n,j}(x,a,\mathbf{b})&=n!^{\ell-1}\frac{\Gamma(rn+x-b_j+a)}{\Gamma(rn+x-b_j)}\frac{(x)_{rn}}{(x+a)_{rn}}\\&\quad\times\frac{1}{2i\pi}\int_{\mathcal{C}}\frac{\Gamma(x+rn-b_j+t)\Gamma(t)}{\Gamma((r-1)n+t+a-b_j+x)\Gamma(t-n)\prod_{p=1,p\neq j}^\ell(t-b_j+b_p-n)_{n+1}}dt.\end{aligned}$$

Here, $\mathcal{C}$ is a closed contour that surrounds the points $0,1,\dots,n$ but no other poles of the integrand. The usual procedure would then be to change t to tn, apply Stirling's formula (like we did above), estimate the critical points and deform the contour through these points to be in position to apply the saddle point method. However, doing so involves a lot of technicalities and we did not try to transform this procedure into a formal proof because this is not essential for potential applications. Indeed, even though there is apparently no interesting Diophantine applications here (see Section 5.3), we point out that an upper bound for $|\widetilde{P}_{n,j}(x,a,\mathbf{b})|$ is enough when one applies Nesterenko's linear independence criterion (see [1] for the statement of the criterion), because in our situation we know the exact rate of decrease of the linear form in Beta values.

2.4 Proof of Corollary 2

(i) Since we let $n \to +\infty$, the assumption $rn > \max(\mathrm{Re}(-x), \mathrm{Re}(a-x))$ and $x \notin \mathbb{Z}_{\leq 0}$ simply becomes $x \notin \mathbb{Z}_{\leq 0}$. From the proof of Theorem 2, we see that when $\ell = 2$, $\rho = \frac{\sqrt{4r^2+1}-1}{2r}$ and (2.2) gives (1.12).

(ii) We now prove the assertion for $\widetilde{Q}_n(x,a)$, which is defined for any $x \in \mathbb{C}$. Since ℓ is even, by Theorem 2, we have

$$\lim_{n\to+\infty} \left|\widetilde{Q}_n(x)\right|^{1/n} = \frac{(r+\eta)^r}{(1-\eta)^2(r+\eta-1)^{r-1}},$$

where η is the unique number in $(0,1)$ such that $\frac{(\eta+r)(1-\eta)^2}{\eta^2(r+\eta-1)} = 1$. Solving this equation, we get $\eta = \frac{1}{2}\sqrt{4r^2+1} + \frac{1}{2} - r$ and using the relation $\eta = r\rho + \frac{1}{2} - r$, we see that

$$\frac{(r+\eta)^r}{(1-\eta)^2(r+\eta-1)^{r-1}} = \frac{1+\rho}{\rho^{2r}(1-\rho)},$$

which completes the proof of (1.13).

3 Remarks on the continued fraction (1.7)

3.1 Ramanujan's continued fraction

Let x, α, β be complex numbers and set

$$Q := \frac{\Gamma(\frac{x+\alpha-\beta+1}{2})\Gamma(\frac{x-\alpha+\beta+1}{2})}{\Gamma(\frac{x+\alpha+\beta+1}{2})\Gamma(\frac{x-\alpha-\beta+1}{2})}.$$

In [2], Ramanujan stated that if either α or β is an integer, or if $\mathrm{Re}(x) > 0$, then

$$\frac{1-Q}{1+Q} = \frac{\alpha\beta|}{|x} + \frac{(\alpha^2-1)(\beta^2-1)|}{|3x} + \frac{(\alpha^2-4)(\beta^2-4)|}{|5x} + \cdots + \frac{(\alpha^2-m^2)(\beta^2-m^2)|}{|(2m+1)x} + \cdots \quad (3.1)$$

See [3] for a proof. If we set $\alpha = \beta = 2a$ and change x to $2x-1$, we see that $Q = \frac{\Gamma(x)^2}{\Gamma(x-a)\Gamma(x+a)}$ and that (3.1) is nothing but the continued fraction (1.7), in a disguised form. In particular, the continued fraction in (1.7) converges to $\frac{\Gamma(x)^2}{\Gamma(x-a)\Gamma(x+a)}$ as soon as $\mathrm{Re}(x) > \frac{1}{2}$. The case $x = \frac{1}{2}$ studied in the next section shows that we cannot expect convergence to a larger domain.

Ramanujan's continued fraction can be proved from the general case $\ell = 2$ in Theorem 1. We obtain Padé approximants to the function $\frac{\Gamma(x-b_1)\Gamma(x+a-b_2)}{\Gamma(x+a-b_1)\Gamma(x-b_2)}$, which is easily transformed into Ramanujan's Q quotient.

3.2 Summation of $P_n(\frac{1}{2}, a)$ and $Q_n(\frac{1}{2}, a)$

In this section, we give the proof that $P_n(1/2, a) = (-1)^{n+1} Q_n(1/2, a)$. We recall that

$$P_n(x, a) = -\sum_{k=0}^{n} \binom{n}{k} \frac{(x-a)_k (1-x)_{n-k}}{(k-a-n)_{n+1}}, \quad Q_n(x, a) = \sum_{k=0}^{n} \binom{n}{k} \frac{(x)_k (1-x-a)_{n-k}}{(k+a-n)_{n+1}}.$$

defined in the Introduction in the context of Corollary 1. In hypergeometric notation, we have:

$$P_n(x, a) = -\frac{(1-x)_n}{(-a-n)_{n+1}} {}_3F_2 \left[\begin{matrix} -a-n, x-a, -n \\ x-n, -1-a \end{matrix} ; 1 \right]$$

and

$$Q_n(x, a) = \frac{(1-x-a)_n}{(a-n)_{n+1}} {}_3F_2 \left[\begin{matrix} a-n, -n, x \\ a+1, a-n+x \end{matrix} ; 1 \right].$$

In general, these hypergeometric series cannot be summed. However, if $x = 1/2$, both are well-poised and can be summed by Dixon's formula ([14, p. 52, eq. (2.3.3.5)]). We then get

$$P_n(1/2, a) = -\frac{2^{n+a} \Gamma(\frac{a+n+1}{2})^2 \Gamma(-a-n) \cos(\frac{\pi}{2}(a+n))}{\sqrt{\pi} \Gamma(\frac{1+a-n}{2}) \Gamma(\frac{n-a}{2}+1) \cos(\pi n)},$$

$$Q_n(1/2, a) = \frac{\sqrt{\pi} 2^{n-a} \Gamma(\frac{a+n+1}{2}) \Gamma(a-n) \cos(\pi a)}{\Gamma(\frac{1+a-n}{2})^2 \Gamma(\frac{a+n}{2}+1) \cos(\pi(a-n))}.$$

After some simplifications (involving the complements formula), we obtain that for all $n \in \mathbb{N}$ and all $a \notin \mathbb{Z}$, we have

$$\frac{P_n(1/2, a)}{Q_n(1/2, a)} = -\frac{2 \sin(\frac{\pi}{2}(n+1+a)) \sin(\frac{\pi}{2}(2+a-n)) \sin(\frac{\pi}{2}(1+2a-2n))}{\sin(\pi(n+1+a)) \sin(\frac{\pi}{2}(1+2n)) \sin(\frac{\pi}{2}(1+2a))} = (-1)^{n+1}$$

as expected.

4 The linear recurrence associated to Corollary 2

Zeilberger's algorithm can produce recurrences for $\widetilde{P}_n(x,a)$ and $\widetilde{Q}_n(x,a)$ for any specified value of r but not when r is a variable. The reason is that the algorithm considers that, for instance, $\binom{2n}{2k}$ is hypergeometric (and thus can find the recurrence for $\sum_k \binom{2n}{2k}$) but not $\binom{rn}{rk}$. At least, since $\widetilde{P}_n(x,a)$ and $\widetilde{Q}_n(x,a)$ can be expressed as values of ${}_3F_2(z)$ function at $z=1$ and with parameters of the form $u(x,a)n+v(x,a)$, it is known that they satisfy linear recurrences of order at most 3 with polynomials coefficients in $\mathbb{Q}[n,x,a]$ (that depend on r). Presumably, the order is always 2 but we can't prove it in general.

For instance, when $r=1$, Zeilberger's algorithm readily computes the linear recurrence of order 2 satisfied by $\widetilde{P}_n(x,a)$ and $\widetilde{Q}_n(x,a)$. Both sequences satisfy

$$p_n(x,a)U_{n+2} = q_n(x,a)U_{n+1} + r_n(x,a)U_n$$

with

$$p_n(x,a) = (x+n+1)(n+2+a)(n+2-a)(x+1+n+a)(a^2-5n^2-6nx-2x^2-4n-2x-1)$$
$$q_n(x,a) = (x+n)(n+a+1)(n+1-a)(x+n-a)(a^2-5n^2-6nx-2x^2-14n-8x-10)$$

$$\begin{aligned}r_n(x,a) = 55n^6 + (176x+319)n^5 + (-31a^2+234x^2+852x+744)n^4 + \\ (-56a^2x+160x^3-130a^2+924x^2+1580x+891)n^3 + \\ (9a^4-36a^2x^2+56x^4-180a^2x+496x^3-204a^2+1300x^2+1394x+579)n^2 + \\ (8a^4x-8a^2x^3+8x^5+19a^4-84a^2x^2+128x^4-188a^2x+480x^3-143a^2+766x^2+582x+196)n - \\ a^6+2a^4x^2+8a^4x-12a^2x^3+12x^5+12a^4-46a^2x^2+66x^4-64a^2x+144x^3-39a^2+158x^2+92x+28.\end{aligned}$$

From this, we could write down the general continued fraction that converges to $\frac{\Gamma(x)^2}{\Gamma(x+a)\Gamma(x-a)}$ but this is not illuminating. Note that the characteristic equation of this recurrence is $X^2-11X-1=0$, whose solutions are $\frac{11-5\sqrt{5}}{2}$ and $\frac{11+5\sqrt{5}}{2}$. By the Poincaré-Perron Theorem, the first solution governs the asymptotic behavior of $\widetilde{Q}_n(x,a)R(x,a)-\widetilde{P}_n(x,a)$, and the second the behavior of $\widetilde{P}_n(x,a)$ and $\widetilde{Q}_n(x,a)$. Of course, this is in accordance with the rates of convergence in Corollary 2.

5 Concluding remarks

5.1 Some hypergeometry

As a side result of the proof of Theorem 1, we have proved that

$$\frac{n!^{\ell-1}(1-a)_n}{\prod_{j=1}^{\ell}(x-b_j)_{n+1}} \cdot {}_{\ell+1}F_\ell\left[\begin{matrix} n+1-a, x-b_1, \ldots, x-b_\ell \\ n+x-b_1+1, \ldots, n+x-b_\ell+1 \end{matrix}; 1\right]$$
$$= \sum_{j=1}^{\ell} P_{n,j}(x)B(x-b_j,a). \quad (5.1)$$

For $n = 0$, this gives the summation formula

$$ {}_{\ell+1}F_\ell \left[\begin{matrix} 1-a, x-b_1, \dots, x-b_\ell \\ x-b_1+1, \dots, x-b_\ell+1 \end{matrix}; 1 \right] = \sum_{j=1}^{\ell} \frac{\prod_{j=1}^{\ell}(x-b_j)}{\prod_{p=1,p\neq j}^{\ell}(b_j-b_p)} \cdot B(x-b_j, a). $$

This formula is not difficult to prove directly. Indeed, by definition,

$$ {}_{\ell+1}F_\ell \left[\begin{matrix} 1-a, x-b_1, \dots, x-b_\ell \\ x-b_1+1, \dots, x-b_\ell+1 \end{matrix}; 1 \right] = \sum_{k=0}^{\infty} \frac{(1-a)_k}{k!} \cdot \frac{\prod_{j=1}^{\ell}(x-b_j)}{\prod_{j=1}^{\ell}(x-b_j+k)}. \tag{5.2} $$

Since $b_i - b_j \notin \mathbb{Z}$, the zeros of $k \mapsto \prod_{j=1}^{\ell}(x-b_j+k)$ are simple and by decomposition in partial fractions (in k), we see the series on the right-hand side of (5.2) is equal to

$$ \sum_{j=1}^{\ell} \frac{\prod_{p=1,p\neq j}^{\ell}(x-b_p)}{\prod_{p=1,p\neq j}^{\ell}(b_j-b_p)} \cdot {}_2F_1 \left[\begin{matrix} 1-a, x-b_j \\ x-b_j+1 \end{matrix}; 1 \right]. $$

We conclude by Gauss' summation formula ([14, p. 28, eq. (1.7.6)]) and analytic continuation. The more general Eq. (5.1) can be proved along the same lines. In fact, we could have started the paper from this hypergeometric series instead of the integral $I_n(x, a, \mathbf{b})$, but we have chosen this approach because this kind of integral (of Barnes type) is classical in Padé approximation theory and Diophantine approximation: see [6, 13, 17] for instance.

5.2 Another approach

The formulas stated in the previous section suggest an alternative approach to the construction of good numerical approximations to the values of $B(x-b_j, a)$. Consider the hypergeometric series

$$ \begin{aligned} J_n(x,a,\mathbf{b}) &:= \frac{n!^\ell}{\Gamma(sn+1-a)} \sum_{k=0}^{\infty} \frac{\Gamma(k+1-a)}{k!} \cdot \frac{(k-sn+1)_{sn}}{\prod_{j=1}^{\ell}(k+x-b_j)_{n+1}} \\ &= \frac{n!^\ell}{\prod_{j=1}^{\ell}(sn+x-b_j)_{n+1}} \cdot {}_{\ell+1}F_\ell \left[\begin{matrix} sn+1-a, sn+x-b_1, \dots, sn+x-b_\ell \\ (s+1)n+x-b_1+1, \dots, (s+1)n+x-b_\ell+1 \end{matrix}; 1 \right] \end{aligned} $$

where s is an integer such that $(\ell - s)n + \ell > 1 - \mathrm{Re}(a)$, so that the series is convergent. If $s = \ell$, this assumption is fulfilled if we assume that $\mathrm{Re}(a) > -1$ because $\ell \geq 2$. The factor $\frac{n!^\ell}{(sn)!}$ is an arithmetic normalization, we could have chosen $n!^{\ell-s}$ as well. At first sight, $J_n(x, a, \mathbf{b})$ seems to be rather different from the series for $I_n(x, a, \mathbf{b})$: at the numerator, we have $(k-sn+1)_{sn}$ instead of $(k-a)_{n+1}$. The factor $(k-sn+1)_{sn}$ means that the first sn terms of the series are 0 and that it should be "small" (see Section 5.3).

We define

$$ Q_{n,j}(x,a,\mathbf{b}) = \sum_{p=0}^{n} \frac{(-1)^{n-p} n!^\ell (-x+b_j-p-sn)_{sn}(x-b_j)_p}{p!(n-p)!(1-a)_{sn}(x+a-b_j)_p \prod_{q=1,q\neq j}^{\ell}(b_j-b_q-p)_{n+1}} \in \mathbb{Q}(x, a, b_1, \dots, b_\ell). $$

Theorem 3. *We have*

$$J_n(x,a,\mathbf{b}) = \sum_{j=1}^{\ell} Q_{n,j}(x,a,\mathbf{b})B(x-b_j,a)$$
$$= \int_{[0,1]^\ell} \frac{\prod_{j=1}^{\ell} u_j^{sn+x-b_j}(1-u_j)^n}{(1-u_1\cdots u_\ell)^{sn+1-a}} du_1\cdots du_\ell.$$

From the integral expression, we remark that in fact $J_n(x,a,\mathbf{b}) = \frac{n!}{(1-a)_n} I_n(x+sn+1, a-(s-1)n, \mathbf{b})$. Hence, both approaches essentially provide the same informations. We now present a direct proof of Theorem 3.

Proof. Since $s \le n$, we have the partial fraction expansion (in k)

$$\frac{(k-sn)_{sn}}{\prod_{j=1}^{\ell}(k+x-b_j)_{n+1}} = \sum_{j=1}^{\ell}\sum_{p=0}^{n} \frac{(-1)^{n-p}(-x+b_j-p-sn)_{sn}}{p!(n-p)!\prod_{q=1,q\neq j}^{\ell}(b_j-b_q-p)_{n+1}} \frac{1}{k+x-b_j+p}.$$

Hence

$$J_n(x,a,\mathbf{b}) = \frac{1}{\Gamma(sn+1-a)} \sum_{p=0}^{n} \frac{(-1)^{n-p}n!^\ell(-x+b_j-p-sn)_{sn}}{p!(n-p)!\prod_{q=1,q\neq j}^{\ell}(b_j-b_q-p)_{n+1}} \sum_{k=0}^{\infty} \frac{\Gamma(k+1-a)}{k!(k+x-b_j+p)}.$$

Now, by Gauss' summation formula

$$\frac{1}{\Gamma(sn+1-a)} \sum_{k=0}^{\infty} \frac{\Gamma(k+1-a)}{k!(k+x-b_j+p)} = \frac{\Gamma(p+x-b_j)\Gamma(a)}{\Gamma(x-b+a-p)(1-a)_{sn}}$$
$$= \frac{(x-b_j)_p}{(x+a-b)_p(1-a)_{sn}} B(x-b_j,a),$$

so that

$$J_n(x,a,\mathbf{b}) =$$
$$\sum_{j=1}^{\ell}\left(\sum_{p=0}^{n}(-1)^{n-p}\frac{n!^\ell(-x+b_j-p-sn)_{sn}(x-b_j)_p}{p!(n-p)!(1-a)_{sn}(x+a-b_j)_p\prod_{q=1,q\neq j}^{\ell}(b_j-b_q-p)_{n+1}}\right)B(x-b_j,a).$$

The integral expression is a consequence of the fact that $J_n(x,a,b)$ is hypergeometric. □

5.3 Diophantine questions

Hypergeometric series or integrals such that those studied in this paper are very common, in particular in the study of the arithmetic nature of zeta values. See for instance [1, 7, 8]. In Theorem 2, it is not difficult to prove that

$$\left|\sum_{j=1}^{\ell} \widetilde{P}_{n,j}(x,a,\mathbf{b})B(x-b_j,a)\right| \ll \frac{1}{r^{\ell n+o(n)}}$$

for any r, where the implicit constant does not depend on ℓ. Similarly, in Theorem 3, we have

$$\Big|\sum_{j=1}^{\ell} Q_{n,j}(x,a,\mathbf{b})B(x-b_j,a)\Big| \ll \frac{1}{(es)^{(\ell-s)n+o(n)}}$$

for any $s<\ell$. When we take x, a and the b_j's as rational numbers, the $\widetilde{P}_{n,j}(x,a,\mathbf{b})$ and $Q_{n,j}(x,a,\mathbf{b})$ are rational numbers. Unfortunately, it seems that the common denominator of the $\widetilde{P}_{n,j}(x,a,\mathbf{b})$'s, respectively the $Q_{n,j}(x,a,\mathbf{b})$'s, are too large to apply Nesterenko's linear independence criterion successfuly. Even in the simple case $x=1$, $a=\frac{1}{2}$, $b_j=-\frac{j}{\ell}$, it seems that we can't obtain a non-trivial lower bound for the dimension of the vector space generated over $\mathbb{Q}$ by $B(\frac{1}{2},\frac{1}{\ell}),\ldots,B(\frac{1}{2},\frac{\ell-1}{\ell})$. We do not give any details because even if we could obtain a non-trivial lower bound for the above dimension, the result would not be new. Indeed, the results of [16] on the linear independence of Beta values enables one to get the exact value of this dimension, as predicted by the Rohrlich-Lang conjecture.

Bibliography

[1] K. M. Ball, T. Rivoal, *Irrationalité d'une infinité de valeurs de la fonction zêta aux entiers impairs*, Invent. Math. **146**.1 (2001), 193–207.

[2] B.C. Berndt, *Ramanujan's Notebooks, Part II*, Springer-Verlag, 1989.

[3] L. Jacobsen, *Domains of validity for some of Ramanujan's continued fraction formulas*, J. Math. Anal. Appl. **143** (1989), 412–437.

[4] A. A. Kandayan, V. N. Sorokin, *Multipoint Hermite-Padé approximations for beta functions*, Math. Notes **87**.2 (2010), 204–217; english translation from Mat. Zametki **87**.2 (2010), 217–232, in russian.

[5] R. J. McIntosh, *An Asymptotic Formula for Binomial Sums*, J. Number Theory **58** (1996), 158–172.

[6] Yu. V. Nesterenko, *On the irrationality exponent of the number* $\ln 2$, Math. Notes **88**.4 (2010), 530–543; translation from Mat. Zametki **88**.4 (2010), 549–564 (russian).

[7] E. Nikishin, *On the irrationality of the values of the functions* $F(x,s)$, Math. Sbornik **37**.3 (1979), 381–388.

[8] T. Rivoal, *La fonction zêta de Riemann prend une infinité de valeurs irrationnelles aux entiers impairs*, C. R. Acad. Sci. Paris, Série I Math. **331**.4 (2000), 267–270.

[9] T. Rivoal, *Simultaneous polynomial approximations of the Lerch function*, Canadian J. Math. **61**.6 (2009), 1341–1356.

[10] T. Rivoal, *Approximations rationnelles des valeurs de la fonction Gamma aux rationnels : le cas des puissances*, Acta Arith. **142**.4 (2010), 347–365.

[11] T. Rivoal, *Approximations rationnelles des valeurs de la fonction Gamma aux rationnels*, J. Number Theory **130**.4 (2010), 944–955.

[12] T. Rivoal, J. Roques, *Hadamard products of algebraic functions*, J. Number Theory **145** (2014), 579–603.

[13] T. Schneider, *Introduction aux nombres transcendants*, traduit de l'allemand par P. Eymard, Gauthier-Villars, Paris, 1959.

[14] L. J. Slater, *Generalized Hypergeometric Functions*, Cambridge University Press.

[15] M. Waldschmidt, *Transcendance de périodes : état des connaissances*, Proceedings of the Tunisian Mathematical Society. **11** (2007), 89–116.

[16] J. Wolfart, G. Wüstholz, *Der Überlagerungsradius gewisser algebraischer Kurven und die Werte der Betafunktion an rationalen Stellen*, Math. Ann. **273**.1 (1985), 1–15.

[17] W. Zudilin, *Arithmetic of linear forms involving odd zeta values*, J. Théor. Nombres Bordx. **16**.1 (2004), 251–291.

T. Rivoal, Institut Fourier, CNRS et Université Grenoble 1, 100 rue des maths, BP 74, 38402 St Martin d'Hères Cedex, France

This a preprint of the Work accepted for publication in Proceedings of the Steklov Institute of Mathematics (2018).