Rational approximation to values of G-functions

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G-functions

Definition 1

A G-function is a formal power series $G(z) = \sum_{n=0}^{\infty} a_n z^n$ such that $a_n \in \overline{\mathbb{Q}}$ and there exists $C > 0$ such that:

(*i*) *For any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and any $n \geq 0$, $|\sigma(a_n)| \leq C^{n+1}$.*

(*ii*) *there exists a sequence of rational integers $d_n \neq 0$, with $|d_n| \leq C^{n+1}$, such that $d_n a_m$ is an algebraic integer for all $m \leq n$.*

(*iii*) *$G(z)$ satisfies a homogeneous linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$.*

Algebraic functions over $\overline{\mathbb{Q}}(z)$. Polylogarithms $Li_s(z) := \sum_{n=1}^{\infty} z^n/n^s$.

No general theorem about the transcendance of values of G-functions: $\log(\overline{\mathbb{Q}}^{\times}) \notin \overline{\mathbb{Q}}$ but we don't know if $\mathrm{Li}_{2s+1}(1) \notin \mathbb{Q}$ for $s \geq 2$. A result formally similar to the Siegel-Shidlovsky Theorem is impossible.

Chudnovky (70's), reproved by André (1996): for any $\alpha \in \overline{\mathbb{Q}}$, $0 < |\alpha| < 1$, the numbers ${}_2F_1[\frac{1}{2}, \frac{1}{2}; 1; \alpha]$ and ${}_2F_1[-\frac{1}{2}, \frac{1}{2}; 1; \alpha]$ are algebraically independent over $\overline{\mathbb{Q}}$.

We don't know **three** values of G-functions algebraically independent over $\overline{\mathbb{Q}}$;

Theorem 1 (Chudnovsky 1984)

Let $Y(z) = {}^t(F_1(z), \ldots, F_N(z))$ be a vector of G-functions solution of a differential system $Y'(z) = A(z)Y(z)$, where $A(z) \in M_N(\overline{\mathbb{Q}}(z))$. Assume that $F_1(z), \ldots, F_N(z)$ are $\overline{\mathbb{Q}}(z)$-algebraically independent.

Then for any d, there exists $C = C(Y, d) > 0$ such that, for any $\alpha \in \overline{\mathbb{Q}}^{\times}$ of degree $\leq d$ with

$$|\alpha| < \exp(-C \log (H(\alpha))^{\frac{4N}{4N+1}}), \tag{1}$$

there does not exist a polynomial relation between the values $F_1(\alpha), \ldots, F_N(\alpha)$ over $\mathbb{Q}(\alpha)$ of degree $\leq d$.

$H(\alpha)$ is the maximum of the modulus of the coefficients of the minimal polynomial of α over $\mathbb{Q}$.

If $\alpha = a/b \in \mathbb{Q}$, Eq. (1) reads $b > C_1|a|^{C_2}$ where $C_1, C_2 > 0$ depend on Y **and d**.

A lot of work has been devoted to improvements of Theorem 1, or alike, for classical G-functions, or to determine weaker conditions for the irrationality of the values of G-functions at rational points.

Theorem 2

Let F be a G-function in $\mathbb{Q}[[z]]$ such that $F(z) \notin \mathbb{Q}(z)$. Then there exist some positive constants C_1 and C_2, depending only on F, with the following property.

Let $a \neq 0$ and $b \geq 1$ be integers such that

$$b > C_1 |a|^{C_2}. \tag{2}$$

Then $F(a/b)$ is irrational.

In general, an irrationality measure is also obtained, showing that $F(a/b)$ is not a Liouville number.

Theorem 3 (Zudilin 1995)

Let $N \geq 2$ and $Y(z) = {}^t(F_1(z), \dots, F_N(z))$ be a vector of G-functions solution of a differential system $Y'(z) = A(z)Y(z) + B(z)$, where $A(z), B(z) \in M_N(\mathbb{C}(z))$.

If $N = 2$, assume $1, F_1(z), F_2(z)$ are $\mathbb{C}(z)$-linearly independent. If $N \geq 3$ assume $F_1(z), \dots, F_N(z)$ are $\mathbb{C}(z)$-algebraically independent.

Let $\varepsilon > 0$, $a \in \mathbb{Z}$, $a \neq 0$. Let b and q be sufficiently large positive integers, in terms of the F_j's, a and ε.

Then $F_j(a/b)$ is an irrational number and for any integer p, we have

$$\left| F_j\left(\frac{a}{b}\right) - \frac{p}{q} \right| \geq \frac{1}{q^{2+\varepsilon}}, \quad j = 1, \dots, N. \tag{3}$$

The "Roth like" irrationality exponent $2+\varepsilon$ is very strong, because prior similar results provide an exponent equal to $N+\varepsilon$.

This dramatic mprovement comes from a technique due to Chudnovsky: **graded Padé approximants**.

However, (3) is not of the strength of Roth's theorem, because b depends on ε.

A hybrid measure

Theorem 4 (Fischler-R, 2016)

Let F be a G-function in $\mathbb{Q}[[z]]$ with $F(z) \notin \mathbb{Q}(z)$, and $t \geq 0$.

There exist some effective constants c_1, c_2, c_3, $c_4 > 0$, depending only on F (and t as well for c_3), such that the following property holds.

Let $a \neq 0$ and $b, B \geq 1$ be integers such that

$$b > c_1 |a|^{c_2} \quad \text{and} \quad B \leq b^t. \tag{4}$$

Then for any $n \in \mathbb{Z}$ and any $m \geq c_3 \frac{\log(b)}{\log(|a|+1)}$ we have

$$\left| F\left(\frac{a}{b}\right) - \frac{n}{B \cdot b^m} \right| \geq \frac{1}{B \cdot b^m \cdot (|a|+1)^{c_4 m}}. \tag{5}$$

Earlier results: Beukers (1979), Bugeaud, Bennett for $(1-z)^\alpha$ (with $\alpha \in \mathbb{Q} \setminus \mathbb{Z}$, $B = 1$), and myself for $\log(1-z)$.

Method : **non-diagonal Padé approximants**, first used by Beukers for $(1-z)^\alpha$.

Application 1

The lower bound (5) implies an effective irrationality measure of $F(a/b)$.

Let A and $B \geq 1$ be any integers, $t = \frac{\log(B)}{\log(b)}$ and $m = \lfloor c_3 \frac{\log(b)}{\log(|a|+1)} \rfloor + 1$. From the proof of the theorem, we get $c_3 = \frac{4}{3}t$ if B is large enough in terms of F.

Then, with $n = A \cdot b^m$, Eq. (5) implies that, provided $b > c_1 |a|^{c_2}$,

$$\left| F\left(\frac{a}{b}\right) - \frac{A}{B} \right| \geq \frac{\kappa}{B^\mu} \tag{6}$$

for $\kappa, \mu > 0$ depending effectively on a, b and F.

The constant μ is worse than Zudilin's, at least when b is large with respect to a.

But (6) applies to a larger class of G-functions because we only need to assume that $F(z) \notin \mathbb{Q}(z)$.

Application 2

The lower bound (5) also implies a measure of the distance of $F(a/b)$ to rational numbers of a special type.

Given $\varepsilon > 0$ and assuming that $t = 0$, $b > (|a| + 1)^{2c_4/\varepsilon}$, we obtain the

Corollary 1 (F-R 2016)

Let F be a G-function in $\mathbb{Q}[[z]]$ with $F(z) \notin \mathbb{Q}(z)$, $\varepsilon > 0$, and $a \in \mathbb{Z}$, $a \neq 0$. Let b and m be positive integers, sufficiently large in terms of F, ε, a.

Then $F(a/b) \notin \mathbb{Q}$ and for any integer n, we have

$$\left| F\left(\frac{a}{b}\right) - \frac{n}{b^m} \right| \geq \frac{1}{b^{m(1+\varepsilon)}}.$$

This is a "fake" Ridout Theorem for G-values. However, using Zudilin's Theorem, we only get $2 + \varepsilon$.

Open problem: does a similar result hold when F is assumed to be an E-function? Nothing is known in this direction, not even for $\exp(z)$.

Let b, t be integers with $b \geq 2$ and $t \geq 1$, and let $\xi \in \mathbb{R} \setminus \mathbb{Q}$.

Let $0.a_1 a_2 a_3 \ldots$ be the expansion in base b of the fractional part of ξ.

For any $n \geq 1$, let $\mathcal{N}_b(\xi, t, n)$ denote the number of times the pattern $a_n a_{n+1} \ldots a_{n+t-1}$ is repeated starting from a_n.

If $t = 1$, $\mathcal{N}_b(\xi, t, n)$ is the number of consecutive equal digits in the expansion of ξ, starting from a_n.

Theorem 5 (F-R 2016)

Let F be a G-function in $\mathbb{Q}[[z]]$ with $F(z) \notin \mathbb{Q}(z)$, $\varepsilon > 0$, and $a \in \mathbb{Z}$, $a \neq 0$. Let $b \geq 2$.

Then for any integer s sufficiently large in terms of F, ε, and a, we have for any $t \geq 1$:

$$\limsup_{n \to \infty} \frac{1}{n} \mathcal{N}_b\big(F(a/b^s), t, n\big) \leq \varepsilon / t.$$

A similar bound for this upper limit, with $1+\varepsilon$ instead of ε, follows from (and under the assumptions of) Zudilin's Theorem 3.

When ξ is an irrational algebraic number, Ridout's theorem yields $\lim_n \frac{1}{n}\mathcal{N}_b(\xi,t,n)=0$ for any b and any t.

But this is not effective: given b, t and $\varepsilon>0$, no explicit value of $M=M(b,\xi,t,\varepsilon)$ is known such that $\mathcal{N}_b(\xi,t,n)\le\varepsilon n$ for any $n\ge M$.

If $\xi=F(a/b^s)\in\overline{\mathbb{Q}}$ then Theorem 5 provides such an explicit value provided b^s is large enough. For a given ξ, Theorem 5 applies only if ε is not too small: this is not an effective version of Ridout's theorem for ξ.

Conjecturally, we have $\lim_n \frac{1}{n}\mathcal{N}_b(\xi,t,n)=0$ whenever ξ is a transcendental value of a G-function.

The only such ξ for which the upper bound $\limsup_n \frac{1}{n}\mathcal{N}_b(\xi,1,n)<1$ was known are values of the logarithm.

Idea of the proof of Theorem 4

Let $F(z)$ in $\mathbb{Q}[[z]]$. Given three integers $\mathbf{p} \geq \mathbf{q} \geq h \geq 0$, we can find $P(z)$ and $Q(z)$ in $\mathbb{Z}[z]$, of degree $\leq p$ and $\leq q$ resp., and such that

$$R(z) := Q(z)F(z) - P(z) = O(z^{p+h+1}).$$

Let $P = b^p P(a/b) \in \mathbb{Z}$ and $Q = b^q Q(a/b) \in \mathbb{Z}$. Then

$$\left|F\left(\frac{a}{b}\right) - \frac{u}{v}\right| \geq \left|\frac{u}{v} - \frac{P}{b^{p-q}Q}\right| - \left|F\left(\frac{a}{b}\right) - \frac{P}{b^{p-q}Q}\right| \geq \frac{1}{2}\left|\frac{u}{v} - \frac{P}{b^{p-q}Q}\right| \qquad (7)$$

1) provided $R(a/b)/Q$ is smaller than $\frac{1}{2}\left|\frac{u}{v} - \frac{P}{b^{p-q}Q}\right|$.

2) If the LHS is not 0 and v is any integer,

$$\left|\frac{u}{v} - \frac{P}{b^{p-q}Q}\right| \geq \frac{1}{vb^{p-q}Q}$$

3) If $v = b^m$, we have a "better" lower bound:

$$\left|\frac{u}{v} - \frac{P}{b^{p-q}Q}\right| \geq \frac{1}{b^{\max(p-q,m)}Q}.$$

We then take $p = q + m$, which justifies to use **non-diagonal** Padé approximants. This is crucial to improve on Zudilin's measure.

4) We cannot achieve 1) in general without assumptions on $F(z)$. If it is a G-function, we use Siegel's lemma to construct and control $P(z)$ and $Q(z)$. We have a good upper bound for $R(z)$ but to deduce an hybrid measure from (7), we also need to prove that $R(a/b) \neq 0$.

5) We cannot work only with $F(z)$. We need a vector of G-functions $(F, F_2, \dots, F_N)$ solution of a differential system. We then construct simultaneous type II Padé approximants

$$R_j(z) := Q(z)F_j(z) - P_j(z) = O(z^{p+h+1}), \quad j = 1, \dots, N$$

where $\deg(P_j) \leq p$, $\deg(Q) \leq q$ and $p \geq q \geq Nh$.

6) Using the action of the differential system on the $R_j(z)$, we construct some other "small" approximations $\widetilde{R}_j(z)$ to which Shidlovsky's lemma can be applied (in a form given in André's book). We then get $\widetilde{R}_j(a/b) \neq 0$.

7) We can try to do the same thing if $F(z)$ is an E-function. Everything works fine, except that observation 3) is useless.

Proof of Theorem 5

Let $\xi = F(a/b^s)$, $q_n = b^{n-1}(b^t - 1)$, and

$$p_n = (b^t - 1)\lfloor b^{n-1}\xi \rfloor + a_n b^{t-1} + a_{n-1}b^{t-2} + \ldots + a_{n+t-1}.$$

Then the b-ary expansion of

$$\frac{p_n}{q_n} = \frac{\lfloor b^{n-1}\xi \rfloor}{b^{n-1}} + \frac{a_n b^{t-1} + a_{n-1}b^{t-2} + \ldots + a_{n+t-1}}{b^{n-1}(b^t - 1)}$$

has the same $n + t\mathcal{N}_b(\xi, t, n) - 1$ first digits as the b-ary expansion of ξ.

Therefore

$$\left| \xi - \frac{p_n}{q_n} \right| \leq \frac{b-1}{b^{n+t\mathcal{N}_b(\xi,t,n)}}.$$

Now Theorem 4 with b^s for b, $B = b^t - 1$ and $m = \lfloor \frac{n-1}{s} \rfloor$ yields

$$\left| \xi - \frac{p_n}{q_n} \right| \geq \frac{1}{b^{\lfloor \frac{n-1}{s} \rfloor s(1+\varepsilon)}}.$$

The comparison of both inequalities concludes the proof.