Linear independence of values of G-functions

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G-functions

Definition 1 (Siegel 1929)

A power series $F(z) = \sum_{n=0}^{\infty} A_n z^n \in \overline{\mathbb{Q}}[[z]]$ is a G-function if:

(i) *$F(z)$ is solution of an homogeneous linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$.*

(ii) *There exists $C > 0$ such that, for any $n \geq 0$, $|\sigma(A_n)| \leq C^{n+1}$, for any Galoisian conjugate $\sigma(A_n)$ of A_n.*

(iii) *For any $n \geq 0$, there exists $D_n \in \mathbb{N} \setminus \{0\}$ such that $|D_n| \leq C^{n+1}$ and $D_n A_m$ is an algebraic integer for all $m \leq n$.*

Does there exist a power series $F(z) = \sum_{n=0}^{\infty} A_n z^n \in \overline{\mathbb{Q}}[[z]]$ which is not a G-function but

(I) For any $n \geq 0$, $|A_n| \leq C^{n+1}$.

(II) Items (i) and (iii) above both hold.

Examples of G-functions

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}, \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{k+1} z^k = \frac{2}{1+\sqrt{1-4z}},$$

$$\sum_{k=0}^{\infty} \binom{3k}{2k} z^k = \frac{2\cos\left(\frac{1}{3}\arcsin(\frac{3}{2}\sqrt{3z})\right)}{\sqrt{4-27z}}, \sum_{k=0}^{\infty} \Big(\sum_{j=0}^{k} \binom{k}{j}\binom{k+j}{j} \Big) z^k = \frac{1}{\sqrt{1-6z+z^2}}.$$

Algebraic functions over $\overline{\mathbb{Q}}(z)$ and holomorphic at $z = 0$.

$$\sum_{k=1}^{\infty} \frac{z^{2k}}{k^2 \binom{2k}{k}} = 2\arcsin\left(\frac{z}{2}\right)^2, \quad \mathrm{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s}, \quad \sum_{n_1 > \cdots > n_k \geq 1} \frac{z^{n_1}}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}}$$

$$\sum_{k=0}^{\infty} \Big(\sum_{j=0}^{k} \binom{k}{j}^2 \binom{k+j}{j}^2 \Big) z^k.$$

Hypergeometric series with rational parameters:

$${}_{p+1}F_p \left[\begin{matrix} a_1, a_2, \ldots, a_{p+1} \\ b_1, b_2, \ldots, b_p \end{matrix} ; z \right] := \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_{p+1})_k}{(1)_k (b_1)_k \cdots (b_p)_k} z^k,$$

where $(\alpha)_0 := 1$ and $(\alpha)_k := \alpha(\alpha+1)\cdots(\alpha+k-1)$ for $k \geq 1$.

E-functions

E-function: $\sum_{n=0}^{\infty} \frac{A_n}{n!} z^n$ such that $\sum_{n=0}^{\infty} A_n z^n$ is a G-function.

$$\exp(z), \qquad J_0(z) := \sum_{k=0}^{\infty} (-1)^k \frac{(z/2)^{2k}}{k!^2}.$$

$$\sum_{k=0}^{\infty} \frac{1}{k!} \Big(\sum_{j=0}^{k} \binom{k}{j} \binom{k+j}{j} \Big) z^k = e^{3z} J_0(2i\sqrt{2}z).$$

A Siegel-Shidlovskii type theorem is not possible for G-functions.

Transcendental G-functions can take algebraic values on a dense set of algebraic numbers (André, Beukers, Joyce-Zucker, Wolfart):

$${}_2F_1\left[\frac{1}{12}, \frac{5}{12}; \frac{1}{2}; \frac{1323}{1331}\right] = \frac{3}{4}\sqrt[4]{11}.$$

Polynomial relations between values of G-functions are expected to follow Grothendieck's Period Conjecture, by the Bombieri-Dwork Conjecture "G-functions come from geometry".

Diophantine results for values of specific G-functions

- $\log(\overline{\mathbb{Q}} \setminus \{0,1\}) \notin \overline{\mathbb{Q}}$.

- Chudnovsky (70's), reproved by André (1996): for any $\alpha \in \overline{\mathbb{Q}}$, $0 < |\alpha| < 1$, the two numbers

$$ {}_2F_1\left[\begin{matrix}\frac{1}{2}, \frac{1}{2} \\ 1\end{matrix}; \alpha\right], \quad {}_2F_1\left[\begin{matrix}-\frac{1}{2}, \frac{1}{2} \\ 1\end{matrix}; \alpha\right] $$

are algebraically independent over $\overline{\mathbb{Q}}$.

- Wolfart (1988) on the values of Gauss' hypergeometric function ${}_2F_1$.

In the proofs, other special functions are used. Respectively: exponential function, elliptic/modular functions, periods and Baker-Wüstholz's theory.

We don't know **three** algebraically independent (over $\overline{\mathbb{Q}}$) values of G-functions at algebraic points.

Two families of Diophantine results for G-functions

• In the first family, the G-function is evaluated at an algebraic point α close to 0. First general results by Siegel, Nurmagomedov, Galochkin, Bombieri.

Theorem 1 (Chudnovsky 1984)

Let $Y(z) = {}^t(F_1(z), \dots, F_S(z))$ be a vector of G-functions solution of

$$Y'(z) = A(z)Y(z), \quad A(z) \in M_S(\overline{\mathbb{Q}}(z)).$$

Assume that $F_1(z)$, ..., $F_S(z)$ are $\overline{\mathbb{Q}}(z)$-algebraically independent.

For any d, there exists $C_{Y,d} > 0$ such that, for any $\alpha \in \overline{\mathbb{Q}}$ of degree $\leq d$ with

$$0 < |\alpha| < \exp\big(- C_{Y,d} \log(H(\alpha))^{\frac{4S}{4S+1}} \big),$$

there does not exist a polynomial relation of degree $\leq d$ between the values $1, F_1(\alpha), \dots, F_S(\alpha)$ over $\mathbb{Q}(\alpha)$.

$H(\alpha)$ is the maximum of the modulus of the coefficients of the normalized minimal polynomial of α over $\mathbb{Q}$.

Case $d = 1$ and $\alpha = \frac{a}{b} \in \mathbb{Q}^*$: if $b > c_1 |a|^{c_2}$ then

$$\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}\big(1, F_1(a/b), \ldots, F_S(a/b)\big) = S + 1.$$

In general, $c_2 \asymp S^2$. If $F_j(z) = \mathrm{Li}_j(z)$, then $c_2 \asymp S$ (Nikishin, Hata).

- The second family of results is more recent (≈ 2000).

For any algebraic point α in the disk of convergence of F, we get estimates for the dimension of the vector space spanned by $F(\alpha)$ over a number field, where F ranges through a set of G-functions.

Theorem 2 (Marcovecchio 2006)

Let $\alpha \in \overline{\mathbb{Q}}$, $0 < |\alpha| < 1$. *As* $S \to +\infty$,

$$\dim_{\mathbb{Q}(\alpha)} \mathrm{Span}_{\mathbb{Q}(\alpha)}(1, \mathrm{Li}_1(\alpha), \ldots, \mathrm{Li}_S(\alpha)) \geq \frac{1 + o(1)}{[\mathbb{Q}(\alpha) : \mathbb{Q}] \log(2e)} \log(S).$$

R. 2001: same result under the further assumption that $\alpha \in \mathbb{R}$.

All the results in this family are for polylogarithms or Lerch function.

A new general result in the second family

$F(z) = \sum_{k=0}^{\infty} A_k z^k$ any G-function with radius of convergence R.

For any integers $n \geq 1$ and $s \geq 0$, define the twisted iterated antiderivatives

$$F_n^{[s]}(z) := \sum_{k=0}^{\infty} \frac{A_k}{(k+n)^s} z^{k+n}.$$

$\mathbb{K}$ any number field containing the A_k's.

For any $S \geq 0$ and any $\alpha \in \mathbb{K}$ such that $0 < |\alpha| < R$, set

$$\Phi_{\alpha,S} := \mathrm{Span}_{\mathbb{K}}\big(F_n^{[s]}(\alpha), n \geq 1, 0 \leq s \leq S\big).$$

Theorem 3 (Fischler-R. 2020)

Assume F is not a polynomial. There exist $\mu, \ell_0, C(F) > 0$ effective such that, for any $\alpha \in \mathbb{K}$, $0 < |\alpha| < R$,

$$\frac{1+o(1)}{[\mathbb{K}:\mathbb{Q}]C(F)} \log(S) \leq \dim_{\mathbb{K}}(\Phi_{\alpha,S}) \leq \ell_0 S + \mu.$$

$\Phi_{\alpha,S} = \mathrm{Span}_{\mathbb{K}}\big(F_n^{[s]}(\alpha), 1 \leq n \leq \ell_0, 0 \leq s \leq S\big).$

Examples with $\ell_0 = 1$

Corollary 1

Fix some $a_1, \ldots, a_{p+1} \in \mathbb{Q}$ and $b_1, \ldots, b_p \in \mathbb{Q}$ such that

$$a_i \notin \mathbb{Z} \setminus \{1\} \quad \text{and} \quad b_j \notin -\mathbb{N} \quad \text{for any } i, j.$$

For any $\alpha \in \overline{\mathbb{Q}}$ such that $0 < |\alpha| < 1$, infinitely many of the hypergeometric values

$$\sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_{p+1})_k}{(1)_k (b_1)_k \cdots (b_p)_k} \frac{\alpha^k}{(k+1)^s}, \quad s \geq 0,$$

are linearly independent over $\mathbb{Q}(\alpha)$.

For any integers $1 \leq b \leq a$ and any $\alpha \in \overline{\mathbb{Q}} \cap \mathbb{R}$ such that $0 < |\alpha| < \frac{b^b (a-b)^{a-b}}{a^a}$, infinitely many of the numbers

$$\sum_{k=1}^{\infty} \binom{ak}{bk} \frac{\alpha^k}{k^s}, \quad s \geq 0,$$

are irrational.

Beyond the disk of convergence

In 2021, we extended our theorem to the outside of the disk of convergence.

The G-function F can be analytically continued to $\mathbb{C}$ minus cuts $[\xi_j, \infty)$ at the finite singularities $\xi_1, \ldots, \xi_p$ of F. We choose the directions of the cuts such that the straightlines defined by each cut contains 0. This enables to continue F and each $F_n^{[s]}$ to a domain $\mathcal{D}_F$ which is star-shaped at the origin.

Theorem 3 holds for all $\alpha \in \overline{\mathbb{Q}}^* \cap \mathcal{D}_F$ and α not a singularity of F.

The result on hypergeometric series holds for all $\alpha \in \overline{\mathbb{Q}}^*$, $\alpha \notin [1, +\infty)$.

In particular, the polylogarithms $\mathrm{Li}_s(z)$ can be extended to $\mathbb{C} \setminus [1, +\infty)$ by the identity (with the principal determination of $\log(z)$)

$$\mathrm{Li}_s(z) = (-1)^{s+1}\mathrm{Li}_s(1/z) - \frac{(2i\pi)^s}{s!} B_s\big(\log(z)/2i\pi\big).$$

Theorem 4 (Fischler-R. 2021)

For any $\alpha \in \overline{\mathbb{Q}}^*$, $\alpha \notin [1, +\infty)$,

$$\dim_{\mathbb{Q}(\alpha)} \mathrm{Span}_{\mathbb{Q}(\alpha)}(1, \mathrm{Li}_1(\alpha), \ldots, \mathrm{Li}_S(\alpha)) \gg \log(S).$$

Upper bound for $\dim_{\mathbb{K}}(\Phi_{\alpha,S})$ in Theorem 3

Let $\theta := z\frac{d}{dz}$.

Proposition 1

There exists ℓ_0, ℓ, μ that depend only on F such that, for any $s, n \geq 1$, there exist $\kappa_{j,t,s,n} \in \mathbb{K}$ and $K_{j,s,n}(z) \in \mathbb{K}[z]$ of degree $\leq n + s(\ell - 1)$, such that

$$F_n^{[s]}(z) = \sum_{t=1}^{s} \sum_{j=1}^{\ell_0} \kappa_{j,t,s,n} F_j^{[t]}(z) + \sum_{j=0}^{\mu-1} K_{j,s,n}(z)(\theta^j F)(z). \qquad (1)$$

With $z = \alpha \in \mathbb{K}$ in (1):

$$\dim_{\mathbb{K}} \mathrm{Span}_{\mathbb{K}}\left(F_n^{[s]}(\alpha), n \geq 1, 0 \leq s \leq S\right) \leq \ell_0 S + \mu.$$

Lower bound for $\dim_{\mathbb{K}}(\Phi_{\alpha,S})$ in Theorem 3

We want to apply a Nesterenko-type linear independence criterion.

For $\xi \in \overline{\mathbb{Q}}$, let $\overline{|\xi|}$ = the maximum modulus of the Galois conjugates of ξ.

Theorem 5 (Fischler-R. 2017)

Consider N numbers $\vartheta_1, \ldots, \vartheta_N \in \mathbb{C}$. Assume there exist $(p_{j,n})_{n\geq 0}$, $j = 1, \ldots, N$ such that

(i) *For all j, $p_{j,n} \in \mathcal{O}_{\mathbb{K}}$ and $\overline{|p_{j,n}|} \leq \beta^{n(1+o(1))}$ for some $\beta > 1$;*

(ii) *We have*

$$\sum_{j=1}^{N} p_{j,n}\vartheta_j = \omega^n n^{\kappa} (\log n)^{\lambda} \Big(\sum_{t=1}^{T} c_t \zeta_t^n + o(1) \Big)$$

where $\omega \in (0,1)$, $\kappa, \lambda \in \mathbb{C}$, $T \geq 1$, $c_1, \ldots, c_T \in \mathbb{C} \setminus \{0\}$, $\zeta_1, \ldots, \zeta_T \in \mathbb{C}$ are pairwise distinct and such that $|\zeta_t| = 1$.

Then

$$\dim_{\mathbb{K}} \mathrm{Span}_{\mathbb{K}}(\vartheta_1, \ldots, \vartheta_N) \geq \frac{1}{[\mathbb{K} : \mathbb{Q}]} \Big(1 - \frac{\log(\omega)}{\log(\beta)} \Big).$$

Construction of an auxiliary function

Let $r, S \geq 0$ be integers such that $r \leq S$. Later: $r = [S/\log(S)^2]$ and S "large".

For $n \geq 0$ and $|z| > 1/R$, set

$$T_{S,r,n}(z) = n!^{S-r} \sum_{k=0}^{\infty} \frac{k(k-1)\cdots(k-rn+1)}{(k+1)^S(k+2)^S\cdots(k+n)^S} A_k \, z^{-k}$$

Lemma 1

There exist $C_{j,s,n}(X), \widetilde{C}_{j,n}(X) \in \mathbb{K}[X]$ *of degree* $\leq n+1+S(\ell-1)$ *such that*

$$T_{S,r,n}(z) := \sum_{j=1}^{\ell_0} \sum_{s=1}^{S} C_{j,s,n}(z) F_j^{[s]}(1/z) + \sum_{j=0}^{\mu-1} \widetilde{C}_{j,n}(z) z^{-S(\ell-1)} (\theta^j F)(1/z),$$

This is a non-trivial generalization of the trivial identity

$$\sum_{k=1}^{\infty} \frac{z^{-k-n}}{(k+n)^s} = \mathrm{Li}_s(1/z) - \sum_{k=1}^{n} \frac{z^{-k}}{k^s}.$$

There exist $C_1(F)$ and $C_2(F) > 0$ such that:

Lemma 2

For any $z \in \overline{\mathbb{Q}}$*,*

$$\limsup_{n\to+\infty} \left(\max_{j,s} \overline{\left|C_{j,s,n}(z)\right|}\right)^{1/n} \leq C_1(F)^S r^r 2^{S+r+1} \max(1, \overline{|z|})$$

and

$$\limsup_{n\to+\infty} \left(\max_{j} \overline{\left|\widetilde{C}_{j,n}(z)\right|}\right)^{1/n} \leq C_1(F)^S r^r 2^{S+r+1} \max(1, \overline{|z|}).$$

Lemma 3

For $z \in \mathbb{K}$ *and* $q \in \mathbb{N}^*$ *such that* $qz \in \mathcal{O}_{\mathbb{K}}$*, there exists* $\Delta_n \in \mathbb{N}^*$ *such that*

$$\Delta_n C_{j,s,n}(z) \in \mathcal{O}_{\mathbb{K}}, \quad \Delta_n \widetilde{C}_{j,n}(z) \in \mathcal{O}_{\mathbb{K}},$$

$$\lim_{n\to+\infty} \Delta_n^{1/n} = q C_2(F)^S e^S.$$

Lemma 4

Let $\alpha \in \mathbb{C}$ be such that $0 < |\alpha| < R$. Assume that $r = [\frac{S}{(\log S)^2}]$ and that S is large enough.

There exist $a, \kappa, \lambda \in \mathbb{R}$, $Q \geq 1$, $c_1, \ldots, c_Q \in \mathbb{C} \setminus \{0\}$, pairwise distinct $\zeta_1, \ldots, \zeta_Q \in \mathbb{C}$ with $|\zeta_q| = 1$ such that

$$T_{S,r,n}(1/\alpha) = \omega^n n^\kappa \log(n)^\lambda \Big(\sum_{q=1}^{Q} c_q \zeta_q^n + o(1) \Big) \text{ as } n \to \infty,$$

and

$$0 < \omega \leq \frac{1}{r^{S-r}} \approx \frac{1}{S^S}.$$

The involved quantities depend on the singularities of F.

We now use the three lemmas in Theorem 5 and get the lower bound in Theorem 3 with

$$C(F) = \log(2e^2 C_1(F) C_2(F)).$$

Analytic continuation of $T_{S,r,n}(1/z)$

For $|z| < R$:

$$T_{S,r,n}(1/z) = \frac{z^{rn}}{n!^r} \int_{[0,1]^S} F^{(rn)}(t_1 \cdots t_S z) \prod_{j=1}^{S} t_j^{rn}(1-t_j)^n dt_j.$$

1) We obtain the analytic continuation of $T_{S,r,n}(1/z)$ for $|z| \geq R$ and z in $\mathcal{D}_F$, the star-shaped domain at 0 to which F is continued.
2) $T_{S,r,n}(1/z)$ is "small" in $\mathcal{D}_F$:

$$\limsup_{n\to+\infty} |T_{S,r,n}(1/z)|^{1/n} \leq \frac{1}{r^{S-r}}.$$

3) For $0 < z < R$ and the Taylor coefficients of F are non-negative, we have

$$\lim_{n\to+\infty} |T_{S,r,n}(1/z)|^{1/n} \approx \frac{1}{r^{S-r}}.$$

4) For $|z| < R$, we use a different expression for $T_{S,r,n}(1/z)$ and the saddle point method.
5) For $z \in \mathcal{D}_F$, we construct suitable (complicated) Padé type problems and conclude with a Shidlovskii type lemma.