Linear independence of values of G-functions

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G-functions

Definition 1 (Siegel 1929)

$F(z) = \sum_{k=0}^{\infty} A_k z^k \in \mathbb{Q}[[z]]$ is a G-function if:

(i) *it is solution of a non zero homogeneous linear differential equation with coefficients in $\mathbb{Q}(z)$*

and there exist $C, D > 0$ such that:

(ii) *for any $k \geq 0$, $|A_k| \leq C^{k+1}$.*

(iii) *for any $k \geq 0$, there exists $D_k \in \mathbb{N} \setminus \{0\}$ such that $|D_k| \leq D^{k+1}$ and $D_k A_m \in \mathbb{Z}$ for all $m \leq k$.*

$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$,

Transcendental G-functions: $-\log(1-z) = \sum_{k=1}^{\infty} \frac{z^k}{k}$ and
$\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$ $\quad (s \geq 1$ integer).

G-functions form a subring of $\overline{\mathbb{Q}}[[z]]$, stable by $\frac{d}{dz}$ et $\int_0^z$.

$$\sum_{k=0}^{\infty} \binom{3k}{2k} z^k = \frac{2\cos\left(\frac{1}{3}\arcsin(\frac{3}{2}\sqrt{3z})\right)}{\sqrt{4-27z}}, \qquad \sum_{k=0}^{\infty} \binom{4k}{2k} z^k = \frac{\sqrt{1+\sqrt{1-16z}}}{\sqrt{2-32z}},$$

$$\sum_{k=0}^{\infty} \binom{ak}{bk} z^k, \left(\binom{k+j}{j}\right)\Big) z^k = \frac{1}{\sqrt{1-6z+z^2}}.$$

$$\sum_{k=1}^{\infty} \frac{z^{2k}}{k^2 \binom{2k}{k}} = 2\arcsin\left(\frac{z}{2}\right)^2,$$

$$\sum_{k=0}^{\infty} \Big(\sum_{j=0}^{k} \binom{k}{j}^2 \binom{k+j}{j}^2 \Big) z^k, \qquad \frac{1}{\pi}\int_0^1 \sqrt{\frac{t(1-t)}{1-tz}}\, dt.$$

Hypergeometric series with rational parameters:

$${}_{p+1}F_p \left[\begin{matrix} a_1, a_2, \ldots, a_{p+1} \\ b_1, b_2, \ldots, b_p \end{matrix}; z\right] := \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_{p+1})_k}{(1)_k (b_1)_k \cdots (b_p)_k} z^k,$$

where $(\alpha)_k := \alpha(\alpha+1)\cdots(\alpha+k-1)$ for $k \geq 0$.

E-functions

$\sum_{k=0}^{\infty} \frac{A_k}{k!} z^k$ is an E-function $\iff \sum_{k=0}^{\infty} A_k z^k$ is a G-function.

$$e^z = \sum_{n=0}^{\infty} \frac{z^k}{k!},$$

$$\sum_{k=0}^{\infty} \frac{1}{k!} \Big(\sum_{j=0}^{k} \binom{k}{j} \binom{k+j}{j} \Big) z^k = e^{3z} J_0\big(i2\sqrt{2}z\big).$$

Theorem 1 (Siegel-Shidlovsky 1956)

Let $Y(z) = {}^t(F_1(z), \ldots, F_S(z))$ *be a vector of* E*-functions such that*

$$Y'(z) = A(z)Y(z), \quad A(z) \in M_S(\overline{\mathbb{Q}}(z)).$$

Let $T(z) \in \overline{\mathbb{Q}}[z] \setminus \{0\}$ *such that* $T(z)A(z) \in M_S(\overline{\mathbb{Q}}[z])$.

For any $\alpha \in \overline{\mathbb{Q}}$ *such that* $\alpha T(\alpha) \neq 0$,

$$\mathrm{degtr}_{\overline{\mathbb{Q}}}(F_1(\alpha), \ldots, F_S(\alpha)) = \mathrm{degtr}_{\overline{\mathbb{Q}}(z)}(F_1(z), \ldots, F_S(z)).$$

A Siegel-Shidlovsky type theorem is not possible for G-functions.

Diophantine results for values of specific G-functions

For any $\alpha \in \overline{\mathbb{Q}} \setminus \{0,1\}$, $\log(\alpha) \notin \overline{\mathbb{Q}}$. In particular, $\pi \notin \overline{\mathbb{Q}}$.

Chudnovsky (70's), reproved by André (1996): for any $\alpha \in \overline{\mathbb{Q}}$, $0 < |\alpha| < 1$, the two numbers

$$ {}_2F_1\left[\begin{matrix}\frac{1}{2}, \frac{1}{2} \\ 1\end{matrix}; \alpha\right], \quad {}_2F_1\left[\begin{matrix}-\frac{1}{2}, \frac{1}{2} \\ 1\end{matrix}; \alpha\right] $$

are algebraically independent over $\overline{\mathbb{Q}}$.

Wolfart's results (1988) on the values taken by Gauss' hypergeometric function ${}_2F_1$ (with rational parameters) at algebraic points.

There exist transcendental G-functions that take algebraic values on an infinite set of algebraic numbers (André, Beukers, Joyce-Zucker, Wolfart).

Polynomial relations between values of G-functions follow from Grothendieck's "Periods Conjecture" provided "G-functions come from geometry".

We don't know yet **three** $\overline{\mathbb{Q}}$-algebraically independent values of G-functions at algebraic points.

Apéry (1978): $\mathrm{Li}_3(1) = \zeta(3) \notin \mathbb{Q}$.

Two families of Diophantine results for general G-functions

In the first family, the G-function is evaluated at an algebraic point α **close to** 0: Siegel, Nurmagomedov, Galochkin, Bombieri.

Theorem 2 (Chudnovsky 1984)

Let $Y(z) = {}^t(F_1(z), \dots, F_S(z))$ be a vector of G-functions such that

$$Y'(z) = A(z)Y(z), \quad A(z) \in M_S(\overline{\mathbb{Q}}(z)).$$

Assume that $F_1(z), \dots, F_S(z)$ are $\overline{\mathbb{Q}}(z)$-algebraically independent.

Then, for any $d \geq 1$, there exists $C_{Y,d} > 0$ such that, for any $\alpha \in \overline{\mathbb{Q}}$ of degree d with

$$0 < |\alpha| < \exp\big(-C_{Y,d} \log\big(H(\alpha)\big)^{\frac{4S}{4S+1}}\big),$$

there does not exist a polynomial relation of degree $\leq d$ between the values $F_1(\alpha), \dots, F_S(\alpha)$ over $\mathbb{Q}(\alpha)$.

$H(\alpha)$ is the maximum of the modulus of the coefficients of the normalized minimal polynomial of α over $\mathbb{Q}$.

$d = 1$ and $\alpha = \frac{a}{b} \in \mathbb{Q}^*$: if $|b| > |c_0 a|^{S^2}$ then

$$\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}\big(1, F_1(a/b), \ldots, F_S(a/b)\big) = S + 1.$$

$F_j(z) = \mathrm{Li}_j(z)$ and $\alpha = 1/b$: $|b| > e^{S^2}$ (Nikishin, Hata).

The second family of results is more recent (≈ 2000).

For **any** algebraic point α in the disk of convergence: estimates for the dimension of a vector space spanned by values $F(\alpha)$, where F ranges through a set of G-functions.

Theorem 3 (Marcovecchio 2006)

Let $\alpha \in \overline{\mathbb{Q}}$, $0 < |\alpha| < 1$. Then,

$$\dim_{\mathbb{Q}(\alpha)} \mathrm{Span}_{\mathbb{Q}(\alpha)}(1, \mathrm{Li}_1(\alpha), \ldots, \mathrm{Li}_S(\alpha)) \geq \frac{1 + o(1)}{[\mathbb{Q}(\alpha) : \mathbb{Q}] \log(2e)} \log(S).$$

R. 2001: same result under the further assumption that $\alpha \in \mathbb{R}$.

All the results in this family concerned polylogarithms or Lerch function.

A new general result in the second family

$F(z) = \sum_{k=0}^{\infty} A_k z^k \in \mathbb{Q}[[z]]$ a G-function with radius of convergence 1.

$n \geq 1$ and $s \geq 0$ integers:

$$F_n^{[s]}(z) := \sum_{k=0}^{\infty} \frac{A_k}{(k+n)^s} z^{k+n} \in \mathbb{Q}[[z]].$$

$S \geq 0$ and $\alpha \in \mathbb{Q}$ such that $0 < |\alpha| < 1$:

$$\Phi_{\alpha,S} := \mathrm{Span}_{\mathbb{Q}}\big(F_n^{[s]}(\alpha), 0 \leq s \leq S, n \geq 1\big).$$

Theorem 4 (Fischler-R. 2017)

Assume $F \notin \mathbb{Q}[z]$. There exist $\mu, \ell_0, C > 0$ effective such that, for any $\alpha \in \mathbb{Q}$, $0 < |\alpha| < 1$,

$$C \cdot \log(S) \leq \dim_{\mathbb{Q}}(\Phi_{\alpha,S}) \leq \ell_0 \cdot S + \mu.$$

If $F \in \mathbb{Q}[z]$, then $\Phi_{\alpha,S} \subset \mathbb{Q}$.

Corollary 1

$$a_i \notin \mathbb{Z} \setminus \{1\} \quad \text{and} \quad b_j \notin -\mathbb{N} \quad \text{for any } i, j.$$

For any $\alpha \in \mathbb{Q}$ such that $0 < |\alpha| < 1$: infinitely many of the hypergeometric values

$$\sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_{p+1})_k}{(1)_k (b_1)_k \cdots (b_p)_k} \frac{\alpha^k}{(k+1)^s}, \quad s \in \mathbb{N},$$

are $\mathbb{Q}$-linearly independent.

For any integers $1 \leq b \leq a$ and $\alpha \in \mathbb{Q}$ such that $0 < |\alpha| < \frac{b^b (a-b)^{a-b}}{a^a}$, infinitely many of the numbers

$$\sum_{k=1}^{\infty} \binom{ak}{bk} \frac{\alpha^k}{k^s}, \quad s \in \mathbb{N},$$

are irrational.