Linear independence of values of $G$-functions

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Definition 1 (Siegel 1929)

\( F(z) = \sum_{k=0}^{\infty} A_k z^k \in \mathbb{Q}[[z]] \) is a \( G \)-function if:

(i) it is solution of a non zero homogeneous linear differential equation with coefficients in \( \mathbb{Q}(z) \)

and there exist \( C, D > 0 \) such that:

(ii) for any \( k \geq 0 \), \( |A_k| \leq C^{k+1} \).

(iii) for any \( k \geq 0 \), there exists \( D_k \in \mathbb{N} \setminus \{0\} \) such that \( |D_k| \leq D^{k+1} \) and \( D_k A_m \in \mathbb{Z} \) for all \( m \leq k \).

\[
\sum_{k=0}^{\infty} z^k = \frac{1}{1-z},
\]

Transcendental \( G \)-functions: \( -\log(1-z) = \sum_{k=1}^{\infty} \frac{z^k}{k} \) and \( \text{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s} \) \( (s \geq 1 \text{ integer}) \).

\( G \)-functions form a subring of \( \overline{\mathbb{Q}}[[z]] \), stable by \( \frac{d}{dz} \) et \( \int_0^z \).
\[
\sum_{k=0}^{\infty} \binom{3k}{2k} z^k = 2 \cos \left( \frac{1}{3} \arcsin \left( \frac{3}{2} \sqrt{3z} \right) \right), \quad \sum_{k=0}^{\infty} \binom{4k}{2k} z^k = \frac{\sqrt{1 + \sqrt{1 - 16z}}}{\sqrt{2 - 32z}},
\]

\[
\sum_{k=0}^{\infty} \binom{ak}{bk} z^k \binom{k+j}{j} z^k = \frac{1}{\sqrt{1 - 6z + z^2}}.
\]

\[
\sum_{k=1}^{\infty} \frac{z^{2k}}{k^2 \binom{2k}{k}} = 2 \arcsin \left( \frac{z}{2} \right)^2,
\]

\[
\sum_{k=0}^{\infty} \left( \sum_{j=0}^{k} \binom{k}{j}^2 \binom{k+j}{j}^2 \right) z^k, \quad \frac{1}{\pi} \int_0^1 \sqrt{\frac{t(1-t)}{1-tz}} dt.
\]

Hypergeometric series with rational parameters:

\[
p+1 F_p \left[ \begin{array}{c} a_1, a_2, \ldots, a_{p+1} \\ b_1, b_2, \ldots, b_p \end{array} ; z \right] := \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_{p+1})_k}{(1)_k (b_1)_k \cdots (b_p)_k} z^k,
\]

where \((\alpha)_k := \alpha(\alpha + 1) \cdots (\alpha + k - 1)\) for \(k \geq 0\).
**E-functions**

\[ \sum_{k=0}^{\infty} \frac{A_k}{k!} z^k \text{ is an } E\text{-function } \iff \sum_{k=0}^{\infty} A_k z^k \text{ is a } G\text{-function.} \]

\[ e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \]

\[ \sum_{k=0}^{\infty} \frac{1}{k!} \left( \sum_{j=0}^{k} \binom{k}{j} \binom{k+j}{j} \right) z^k = e^{3z} J_0(i2\sqrt{2}z). \]

**Theorem 1 (Siegel-Shidlovsky 1956)**

Let \( Y(z) = (F_1(z), \ldots, F_S(z)) \) be a vector of \( E \)-functions such that

\[ Y'(z) = A(z) Y(z), \quad A(z) \in M_S(\bar{\mathbb{Q}}(z)). \]

Let \( T(z) \in \bar{\mathbb{Q}}[z] \setminus \{0\} \) such that \( T(z)A(z) \in M_S(\bar{\mathbb{Q}}[z]) \).

For any \( \alpha \in \bar{\mathbb{Q}} \) such that \( \alpha T(\alpha) \neq 0 \),

\[ \degtr_{\overline{\mathbb{Q}}}(F_1(\alpha), \ldots, F_S(\alpha)) = \degtr_{\overline{\mathbb{Q}}(z)}(F_1(z), \ldots, F_S(z)). \]

A Siegel-Shidlovsky type theorem is not possible for \( G \)-functions.
Diophantine results for values of specific $G$-functions

For any $\alpha \in \overline{\mathbb{Q}} \setminus \{0, 1\}$, $\log(\alpha) \notin \overline{\mathbb{Q}}$. In particular, $\pi \notin \overline{\mathbb{Q}}$.

Chudnovsky (70’s), reproved by André (1996): for any $\alpha \in \overline{\mathbb{Q}}$, $0 < |\alpha| < 1$, the two numbers

$$2F_1\left[\frac{1}{2}, \frac{1}{2}; \alpha\right], \quad 2F_1\left[-\frac{1}{2}, \frac{1}{2}; \alpha\right]$$

are algebraically independent over $\overline{\mathbb{Q}}$.

Wolfart’s results (1988) on the values taken by Gauss’ hypergeometric function $2F_1$ (with rational parameters) at algebraic points.

There exist transcendental $G$-functions that take algebraic values on an infinite set of algebraic numbers (André, Beukers, Joyce-Zucker, Wolfart).

Polynomial relations between values of $G$-functions follow from Grothendieck’s “Periods Conjecture” provided “$G$-functions come from geometry”.

We don’t know yet three $\overline{\mathbb{Q}}$-algebraically independent values of $G$-functions at algebraic points.

Apéry (1978): $\text{Li}_3(1) = \zeta(3) \notin \mathbb{Q}$. 

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Two families of Diophantine results for general $G$-functions

In the first family, the $G$-function is evaluated at an algebraic point $\alpha$ close to 0: Siegel, Nurmagomedov, Galochkin, Bombieri.

**Theorem 2 (Chudnovsky 1984)**

Let $Y(z) = (F_1(z), \ldots, F_S(z))$ be a vector of $G$-functions such that

$$Y'(z) = A(z) Y(z), \quad A(z) \in M_S(\overline{\mathbb{Q}}(z)).$$

Assume that $F_1(z), \ldots, F_S(z)$ are $\overline{\mathbb{Q}}(z)$-algebraically independent.

Then, for any $d \geq 1$, there exists $C_{Y,d} > 0$ such that, for any $\alpha \in \overline{\mathbb{Q}}$ of degree $d$ with

$$0 < |\alpha| < \exp \left( - C_{Y,d} \log \left( H(\alpha) \frac{4S}{4S+1} \right) \right),$$

there does not exist a polynomial relation of degree $\leq d$ between the values $F_1(\alpha), \ldots, F_S(\alpha)$ over $\mathbb{Q}(\alpha)$.

$H(\alpha)$ is the maximum of the modulus of the coefficients of the normalized minimal polynomial of $\alpha$ over $\mathbb{Q}$. 
\( d = 1 \) and \( \alpha = \frac{a}{b} \in \mathbb{Q}^* \): if \( |b| > |c_0a|^{S^2} \) then

\[
\dim_{\mathbb{Q}} \text{Span}_{\mathbb{Q}}(1, F_1(a/b), \ldots, F_S(a/b)) = S + 1.
\]

\( F_j(z) = \text{Li}_j(z) \) and \( \alpha = 1/b: \ |b| > e^{S^2} \) \hspace{1em} \text{(Nikishin, Hata)}.

The second family of results is more recent (\( \approx 2000 \)).

For any algebraic point \( \alpha \) in the disk of convergence: estimates for the dimension of a vector space spanned by values \( F(\alpha) \), where \( F \) ranges through a set of \( G \)-functions.

**Theorem 3 (Marcovecchio 2006)**

*Let \( \alpha \in \overline{\mathbb{Q}}, \ 0 < |\alpha| < 1. \) Then,*

\[
\dim_{\mathbb{Q}(\alpha)} \text{Span}_{\mathbb{Q}(\alpha)}(1, \text{Li}_1(\alpha), \ldots, \text{Li}_S(\alpha)) \geq \frac{1 + o(1)}{[\mathbb{Q}(\alpha) : \mathbb{Q}] \log(2e) \log(S)} \log(S).
\]

R. 2001: same result under the further assumption that \( \alpha \in \mathbb{R} \).

All the results in this family concerned polylogarithms or Lerch function.
A new general result in the second family

\[ F(z) = \sum_{k=0}^{\infty} A_k z^k \in \mathbb{Q}[[z]] \text{ a } G\text{-function with radius of convergence 1.} \]

\( n \geq 1 \) and \( s \geq 0 \) integers:

\[ F_n^s(z) := \sum_{k=0}^{\infty} \frac{A_k}{(k+n)^s} z^{k+n} \in \mathbb{Q}[[z]]. \]

\( S \geq 0 \) and \( \alpha \in \mathbb{Q} \) such that \( 0 < |\alpha| < 1 \):

\[ \Phi_{\alpha,S} := \text{Span}_\mathbb{Q}(F_n^s(\alpha), 0 \leq s \leq S, n \geq 1). \]

**Theorem 4 (Fischler-R. 2017)**

*Assume \( F \notin \mathbb{Q}[z] \). There exist \( \mu, \ell_0, C > 0 \) effective such that, for any \( \alpha \in \mathbb{Q}, 0 < |\alpha| < 1 \),

\[ C \cdot \log(S) \leq \dim_\mathbb{Q}(\Phi_{\alpha,S}) \leq \ell_0 \cdot S + \mu. \]

If \( F \in \mathbb{Q}[z] \), then \( \Phi_{\alpha,S} \subset \mathbb{Q} \).
Corollary 1

\[ a_i \notin \mathbb{Z} \setminus \{1\} \quad \text{and} \quad b_j \notin \mathbb{N} \quad \text{for any } i, j. \]

For any \( \alpha \in \mathbb{Q} \) such that \( 0 < |\alpha| < 1 \): infinitely many of the hypergeometric values

\[
\sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k \cdots (a_{p+1})_k}{(1)_k(b_1)_k \cdots (b_p)_k} \frac{\alpha^k}{(k+1)^s}, \quad s \in \mathbb{N},
\]

are \( \mathbb{Q} \)-linearly independent.

For any integers \( 1 \leq b \leq a \) and \( \alpha \in \mathbb{Q} \) such that \( 0 < |\alpha| < \frac{b^b(a-b)^{a-b}}{a^a} \), infinitely many of the numbers

\[
\sum_{k=1}^{\infty} \binom{ak}{bk} \frac{\alpha^k}{k^s}, \quad s \in \mathbb{N},
\]

are irrational.