Siegel's problem for E-functions

Tanguy Rivoal, CNRS et Université Grenoble Alpes

Joint work with Stéphane Fischler, Université Paris-Sud

Périodes, motifs et équations différentielles : entre arithmétique et géométrie

Institut Henri Poincaré, Paris, april 2022

E- and G-functions (Siegel, 1929)

Definition 1

A power series $F(z)=\sum_{n=0}^{\infty} a_n z^n/n! \in \overline{\mathbb{Q}}[[z]]$ *is an* E*-function if*

(i) $F(z)$ *is solution of a non-zero linear differential equation with coefficients in* $\overline{\mathbb{Q}}(z)$*.*

(ii) *There exists* $C>0$ *such that for any* $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *and any* $n \geq 0$, $|\sigma(a_n)| \leq C^{n+1}$*.*

(iii) *There exists a sequence of positive integers* d_n*, with* $d_n \leq C^{n+1}$*, such that* $d_n a_m$ *are algebraic integers for all* $m \leq n$*.*

Siegel's definition was more general: the two bounds $(\cdots) \leq C^{n+1}$ are replaced by: for all $\varepsilon > 0$, $(\cdots) \leq n!^{\varepsilon}$ for all $n \geq N(\varepsilon)$.

E-functions are entire functions. They form a ring stable under $\frac{d}{dz}$ and $\int_0^z$. If $F(z)$ is an E-function and $\alpha \in \overline{\mathbb{Q}}$, then $F(\alpha z)$ is an E-function.

A power series $\sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[[z]]$ is a G-function if $\sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$ is an E-function (in the sense of Definition 1).

Examples

E-**functions:** polynomials in $\overline{\mathbb{Q}}[z]$,

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad L(z) := \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{n} \right) \frac{z^n}{n!},$$

$$H(z) := \sum_{n=0}^{\infty} \left(\sum_{k=1}^{n} \frac{1}{k} \right) \frac{z^n}{n!}, \quad J_0(z) := \sum_{n=0}^{\infty} \frac{(iz/2)^{2n}}{n!^2}.$$

G-**functions:** algebraic functions over $\overline{\mathbb{Q}}(z)$ regular at 0, $\log(1-z) = -\sum_{n=1}^{\infty} z^n/n$ and (multiple) polylogarithms

$$\mathrm{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s} \qquad (s \in \mathbb{Z}),$$

$$\sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{z^{n_1}}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}} \qquad (s_1, s_2, \ldots, s_k \in \mathbb{Z}),$$

$$\frac{1}{\pi} \int_0^1 \frac{\sqrt{x(1-x)}}{1-zx} dx.$$

The intersection of both classes of series is reduced to $\overline{\mathbb{Q}}[z]$.

Why are E- and G-functions interesting?

Theorem 1 (Lindemann-Weierstrass)

If $\alpha_1, \dots, \alpha_n \in \overline{\mathbb{Q}}$ are $\mathbb{Q}$-linearly independent, then ($e^{\alpha_1 z}, \dots, e^{\alpha_n z}$ are $\overline{\mathbb{Q}}(z)$-algebraically independent and) $e^{\alpha_1}, e^{\alpha_2}, \dots, e^{\alpha_n}$ are $\overline{\mathbb{Q}}$-algebraically independent.

Consequences:

• For any $\alpha \in \overline{\mathbb{Q}} \setminus \{0\}$, $\exp(\alpha) \notin \overline{\mathbb{Q}}$.

• For any $\alpha \in \overline{\mathbb{Q}} \setminus \{0, 1\}$, $\log(\alpha) \notin \overline{\mathbb{Q}}$ for any given determination of the logarithm.

Recall that $\exp(z)$ is an E-function while $\log(1 - z)$ is a G-function: Siegel's aim was to generalize the above statements.

The Siegel-Shidlovskii Theorem

Theorem 2 (Siegel-Shidlovskii, 1929-1956)

Let $Y = {}^t(F_1, \ldots, F_n)$ *be a vector of E-functions (in Siegel's sense) and* $A \in M_{n\times n}(\overline{\mathbb{Q}}(z))$ *such that* $Y' = AY$*.*

Let $T \in \overline{\mathbb{Q}}[z] \setminus \{0\}$ *a common denominator of the entries of A, of minimal degree.*

Then, for all $\alpha \in \overline{\mathbb{Q}}$ *such that* $\alpha T(\alpha) \neq 0$*,*

$$\deg \operatorname{tr}_{\overline{\mathbb{Q}}(z)} \overline{\mathbb{Q}}(z)\big(F_1(z), \ldots, F_n(z)\big) = \deg \operatorname{tr}_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}\big(F_1(\alpha), \ldots, F_n(\alpha)\big).$$

We obtain (a version of) the Lindemann-Weierstrass Theorem with $F_j(z) = e^{\alpha_j z}$, $A = \mathrm{Diag}(\alpha_j)$ and $\alpha = 1$.

Siegel, 1929: The E-functions $J_0(z)$ et $J_0'(z)$ are $\overline{\mathbb{Q}}(z)$-algebraically independent and

$$\begin{pmatrix} J_0'(z) \\ J_0''(z) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -\frac{1}{z} \end{pmatrix} \begin{pmatrix} J_0(z) \\ J_0'(z) \end{pmatrix}, T(z) = z.$$

For all $\alpha \in \overline{\mathbb{Q}} \setminus \{0\}$, the numbers $J_0(\alpha)$ et $J_0'(\alpha)$ are $\overline{\mathbb{Q}}$-algebraically independent.

After the Siegel-Shidlovskii Theorem

André obtained in 2000 a new proof of the Siegel-Shidlovskii Theorem (in the restricted sense). He used the special properties of the differential equations satisfied by such E-functions.

These properties are inherited from those of the diff equations satisfied by G-functions, found in the 80's by André, Bombieri, Chudnovsky, Galochkin, Katz: *The non-zero minimal differential equation satisfied by a given G-function is fuchsian with rational exponents.*

Beukers, 2006: If $Y = {}^t(F_1, \ldots, F_n)$ is a vector of E-functions (in the restricted sense) such that $Y' = AY$ and the F_j's are linearly independent over $\overline{\mathbb{Q}}(z)$, then for any $\alpha \in \overline{\mathbb{Q}}^*$ not a singularity of A, the numbers $F_1(\alpha), \ldots, F_n(\alpha)$ are linearly independent over $\overline{\mathbb{Q}}$.

Consequence: for any non-polynomial E-function $F(z)$, there are only finitely many $\alpha \in \overline{\mathbb{Q}}$ such that $F(\alpha) \in \overline{\mathbb{Q}}$. This is not a consequence of the Siegel-Shidlovskii Theorem. An exotic evaluation: $J_0^{(4)}(\pm\sqrt{3}) = 0$.

In 2014, André extended Beukers' lifting theorem to the case of E-functions in Siegel's sense.

Chudnovsky's Theorem

Chudnovsky "completed" Siegel's program for G-functions.

Theorem 3 (Chudnovsky 1984)

Let $Y(z) = {}^t(F_1(z), \ldots, F_S(z))$ *be a vector of G-functions solution of*

$$Y'(z) = A(z)Y(z), \quad A(z) \in M_S(\overline{\mathbb{Q}}(z)).$$

Assume $F_1(z), \ldots, F_S(z)$ *to be* $\overline{\mathbb{Q}}(z)$*-algebraically independent.*

For any d, there exists $C_{Y,d} > 0$ such that, for any $\alpha \in \overline{\mathbb{Q}}$ of degree $\leq d$ with

$$0 < |\alpha| < \exp\left(-C_{Y,d} \log\left(H(\alpha)\right)^{\frac{4S}{4S+1}}\right), \tag{1}$$

there does not exist a polynomial relation of degree $\leq d$ between the values $1,\ F_1(\alpha), \ldots,\ F_S(\alpha)$ *over* $\mathbb{Q}(\alpha)$.

A condition like (1) is unavoidable: there exist transcendental G-functions that take algebraic values on a dense set of algebraic points in the disk of convergence (Wolfart).

Hypergeometric E-functions

Set $(x)_m := x(x+1)\cdots(x+m-1)$.

Siegel: the "hypergeometric" series

$$ {}_pF_q\left[\begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix}; z^{q-p+1}\right] := \sum_{n=0}^{\infty} \frac{(a_1)_n\cdots(a_p)_n}{n!(b_1)_n\cdots(b_q)_n} z^{n(q-p+1)}, $$

is an E-function when $q \geq p \geq 1$, $a_j \in \mathbb{Q}$ and $b_j \in \mathbb{Q}\setminus\mathbb{Z}_{\leq 0}$ for all j.

$L(z)$ and $H(z)$ are not of ${}_pF_q(z^{q-p+1})$ type but

$$ \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{n}\Big)\frac{z^n}{n!} = e^{(3-2\sqrt{2})z}\cdot {}_1F_1\left[\begin{matrix} 1/2 \\ 1 \end{matrix}; 4\sqrt{2}z\right], $$

$$ \sum_{n=0}^{\infty}\Big(\sum_{k=1}^{n}\frac{1}{k}\Big)\frac{z^n}{n!} = ze^z\cdot {}_2F_2\left[\begin{matrix} 1,1 \\ 2,2 \end{matrix}; -z\right]. $$

Siegel's question

Question 1 (Siegel, 1949)

Is it possible to write every E-function (in Siegel's sense) as a polynomial with coefficients in $\overline{\mathbb{Q}}$ of series ${}_pF_q[a_1, \ldots, a_p; b_1, \ldots, b_q; \lambda z^{q-p+1}]$, with $q \geq p \geq 1$, $a_j, b_j \in \mathbb{Q}$ and $\lambda \in \overline{\mathbb{Q}}$?

Such a representation may not be unique. For instance

$$J_0(z) := {}_1F_2\left[\begin{matrix}1\\1,1\end{matrix}; (iz/2)^2\right] = e^{-iz} \cdot {}_1F_1\left[\begin{matrix}1/2\\1\end{matrix}; 2iz\right].$$

Gorelov, 2004: the answer is yes if the E-function (in Siegel's sense) is solution of a differential equation of order ≤ 2 with coefficients in $\overline{\mathbb{Q}}(z)$.

In 2019, Fischler and myself gave a strong reason to believe that the answer was negative in general for E-functions of differential order ≥ 4.

The answer was then shown to be negative by Fresán and Jossen in 2020, who produced an explicit counter-example.

In the rest of the talk, I will explain our 2019 result. From now on, E-functions are always understood in the restricted sense.

Rings of special values

G the ring of values taken at algebraic points by analytic continuations of G-functions. Algebraic numbers, $\Gamma(a/b)^b$ $(a, b \in \mathbb{N})$ and π are units of **G**.

H the ring generated by $\overline{\mathbb{Q}}$, $1/\pi$ and $\Gamma^{(n)}(r)$, $r \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$, $n \in \mathbb{N}$. Algebraic numbers and $\Gamma(r)$ $(r \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0})$ are units of **H**.

S the **G**-module generated by $\Gamma^{(n)}(r)$, $r \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$, $n \in \mathbb{N}$. It is a ring.

G and **H** are subrings of **S**.

Proposition 1

(i) **H** *is generated by* $\overline{\mathbb{Q}}$, $1/\pi$ *and*

$$\begin{cases} \mathrm{Li}_s(e^{2i\pi r}) & s \in \mathbb{N}^*,\ r \in \mathbb{Q},\ (s, e^{2i\pi r}) \neq (1,1) \\ \log(q) & q \in \mathbb{N}^* \\ \Gamma(r) & r \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0} \\ \gamma := -\Gamma'(1) & (Euler's\ constant) \end{cases}$$

(ii) **S** *is the* $\mathbf{G}[\gamma]$*-module generated by* $\Gamma(r)$, $r \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$.

Theorem 4 (Fischler-R., 2019)

At least one of the following statements is true:

(i) $\quad \mathbf{G} \subset \mathbf{H}$;

(ii) $\quad$ *Siegel's question has a negative answer.*

(i) is very unlikely. It contradicts a conjecture on exponential periods that generalizes Grothendieck's periods conjecture.

If there exist $s \in \mathbb{N}^*$ and $\alpha \in \overline{\mathbb{Q}}$ such that $\mathrm{Li}_s(\alpha) \in \mathbf{G}$ is not in $\mathbf{H}$, then the E-function

$$\sum_{n=2}^{\infty} \left(\sum_{k=1}^{n-1} \frac{\alpha^k}{k^s} \right) \frac{z^n}{n!}$$

is a counter-example, of differential order (at most) $s+3$.

I will outline the proof of Theorem 4 when in Siegel's question we further assume that $p = q$.

The proof of the general case is based on the case $p = q$ together with more complicated arguments.

Asymptotic expansions in large sectors

Definition 2

Let $\theta \in \mathbb{R}$. We write

$$f(z) \sim \sum_{\rho\in\mathbb{C}} e^{\rho z} \sum_{\alpha\in\mathbb{C}} z^{\alpha} \sum_{i\in\mathbb{N}} \log(z)^i \sum_{n=0}^{\infty} c_{\rho,\alpha,i,n}(\theta)/z^n$$

where the sums on ρ, α, i are finite, and say (in this talk) that the RHS is the asymptotic expansion of f at ∞ in a large sector bisected by the direction θ, when there exist $\varepsilon, R, B, C > 0$ and certain functions $f_\rho(z)$ holomorphic in the sector

$$U := \{z \in \mathbb{C},\ |z| \geq R,\ \theta - \pi/2 - \varepsilon \leq \arg(z) \leq \theta + \pi/2 + \varepsilon\},$$

such that $f(z) = \sum_\rho e^{\rho z} f_\rho(z)$ and

$$\left| f_\rho(z) - \sum_{\alpha\in\mathbb{C}} z^{\alpha} \sum_{i\in\mathbb{N}} \log(z)^i \sum_{n=0}^{N-1} c_{\rho,\alpha,i,n}(\theta)/z^n \right| \leq C^N N! |z|^{B-N}, \quad z \in U,\ N \geq 1.$$

If such an expansion of $f(z)$ exists in a large sector, it is unique in this sector.

Asymptotic expansions of E-functions

Theorem 5

(i) (André, 2000) *Let $f(z)$ be an E-function. There exists a finite set A such that, for any $\theta \in (-\pi, \pi) \setminus A$,*

$$f(z) \sim \sum_{\rho \in \overline{\mathbb{Q}}} e^{\rho z} \sum_{\alpha \in \mathbb{Q}} z^{\alpha} \sum_{i \in \mathbb{N}} \log(z)^{i} \sum_{n=0}^{\infty} \frac{c_{\rho,\alpha,i,n}(\theta)}{z^{n}},$$

in a large sector bisected by the direction θ, where (Fischler-R., 2016) *the coefficients*

$$c_{\rho,\alpha,i,n}(\theta) \in \mathbf{S}.$$

(ii) (Fischler-R., 2019) *Let $\xi \in \mathbf{G}$. There exists an E-function $F(z)$ and a finite set S such that for any $\theta \in (-\pi, \pi) \setminus S$, ξ is one of the $c_{\rho,\alpha,i,n}(\theta)$ of the expansion of $F(z)$ in a large sector bisected by θ.*

Asymptotic expansions of ${}_pF_p$ hypergeometric series

Theorem 6

Let $\theta \in (-\pi, \pi) \setminus \{0\}$, and $f(z)$ be a hypergeometric series ${}_pF_p(z)$ with rational parameters. Then,

$$f(z) \sim \sum_{\rho \in \{0,1\}} e^{\rho z} \sum_{\alpha \in \mathbb{Q}} z^{\alpha} \sum_{i \in \mathbb{N}} \log(z)^i \sum_{n=0}^{\infty} \frac{c_{\rho,\alpha,i,n}(\theta)}{z^n}$$

in a large sector bisected by the direction θ where (Fischler-R., 2019) *the coefficients*

$$c_{\rho,\alpha,i,n}(\theta) \in \mathbf{H}.$$

It is a consequence of Barnes and Wright's classical results, with refinements coming from the theory of Meijer's G-function.

Proof of Theorem 4 in the case $p = q$

Let $\xi \in \mathbf{G}$.

By Theorem 5(*ii*), there exist an E-function $F(z)$ and a finite set S such that for any $\theta \in (-\pi, \pi) \setminus S$, ξ is a coefficient of the expansion of $F(z)$ in a large sector bisected by θ.

Assume that Siegel's question has a positive answer (in the case $p = q$).

There exist ${}_pF_p$-hypergeometric series $f_1, \ldots, f_n$ with rational parameters, algebraic numbers $\lambda_1, \ldots, \lambda_n$, and a polynomial $P \in \overline{\mathbb{Q}}[X_1, \ldots, X_n]$, such that

$$F(z) = P\big(f_1(\lambda_1 z), \ldots, f_n(\lambda_n z)\big).$$

Choose $\theta \in (-\pi, \pi) \setminus S$ such that $\theta + \arg(\lambda_i) \notin \pi\mathbb{Z}$ for every i. By Theorem 6, the expansion of each $f_i(\lambda_i z)$ in a large sector bisected by θ has coefficients in $\mathbf{H}$. The same holds for $F(z)$ because $\mathbf{H}$ is a $\overline{\mathbb{Q}}$-algebra.

Such an expansion being unique, the coefficient ξ belongs to $\mathbf{H}$.

A Siegel like problem for G-functions

The generalized hypergeometric series

$$ {}_{p+1}F_p\left[\begin{matrix} a_1, \dots, a_{p+1} \\ b_1, \dots, b_p \end{matrix}; z\right] := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_{p+1})_n z^n}{n!(b_1)_n \cdots (b_p)_n}, $$

is a G-function when $p \geq 0$, $a_j \in \mathbb{Q}$ and $b_j \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ for all j.

Question 2

Is it possible to write any G-function as a polynomial with coefficients in $\overline{\mathbb{Q}}$ of series of the form ${}_{p+1}F_p[a_1, \dots, a_{p+1}; b_1, \dots, b_p; \lambda(z)]$, with $a_j, b_j \in \mathbb{Q}$ and $\lambda(z)$ algebraic over $\overline{\mathbb{Q}}(z)$, regular at 0 and such that $\lambda(0) = 0$?

Theorem 7 (Fischler-R., 2019)

At least one of the following statements is true:

(i) $\mathbf{G} \subset \mathbf{H}$;

(ii) *Question 2 has a negative answer under the further assumption that the algebraic functions λ have a common singularity in $\overline{\mathbb{Q}}^* \cup \{\infty\}$ at which they all tend to ∞.*

If there exist $s \in \mathbb{N}^*$ and $\alpha \in \overline{\mathbb{Q}}$ such that $\mathrm{Li}_s(\alpha) \in \mathbf{G}$ is not in $\mathbf{H}$, then $\mathrm{Li}_s(\frac{\alpha z}{z-\alpha})$ is a counter-example of differential order $s+1$.

Why is the inclusion $\mathbf{G} \subset \mathbf{H}$ unlikely, according to Yves André

"The inclusion $\mathbf{G} \subset \mathbf{H}$ does not contradict Grothendieck's period conjecture but it contradicts its extension to exponential motives. In the description of $\mathbf{H}$ given in Proposition 1, we find

$1/\pi$, a period of the Tate motive,

$\mathrm{Li}_s(e^{2i\pi r})$, periods of a mixed Tate motive over $\mathbb{Z}[1/r]$,

$\log(q)$, a period of a 1-motive over $\mathbb{Q}$,

$\Gamma(r)$, whose suitable powers are periods of Abelian varieties with complex multiplication by $\mathbb{Q}(e^{2i\pi r})$,

γ, a period of an exponential motive, which is a non-classical extension of the Tate motives.

Let M be the Tannakian category of mixed motives over $\overline{\mathbb{Q}}$ generated by all these motives. Consider a non CM elliptic curve over $\overline{\mathbb{Q}}$ and E its motive. The periods of E are in $\mathbf{G}$.

If $\mathbf{G} \subset \mathbf{H}$, the periods of E are in $\mathbf{H}$. By the exponential period conjecture, E would be in M. This is impossible because the motivic Galois group of M is pro-solvable, while that of E is GL_2."