Arithmetic properties of the Taylor coefficients of differentially algebraic power series

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Conference *Équations différentielles et aux différences : Analyse, arithmétique et approches galoisiennes*, april 2025, Lille.

Setting

$\overline{\mathbb{Q}}$ is the field of algebraic numbers, $\mathcal{O}_{\overline{\mathbb{Q}}}$ is the ring of algebraic integers.

Let us consider a non-trivial algebraic differential equation (ADE)

$$Q(x, y(x), \ldots, y^{(k)}(x)) = 0, \tag{1}$$

where $Q \in \overline{\mathbb{Q}}[X, Y_0, \ldots, Y_k] \setminus \{0\}$.

Assume that

$$f(x) := \sum_{n=0}^{\infty} f_n x^n \in \overline{\mathbb{Q}}[[x]]$$

is a solution of (1).

Let d_n be the denominator of f_n, *i.e.*, the least positive integer such that $d_n f_n \in \mathcal{O}_{\overline{\mathbb{Q}}}$.

What can be said of the Archimedean and non-Archimedean growth of the sequences f_n and d_n?

Classical results

- $\overline{\mathbb{Q}}$ embedded into $\mathbb{C}$ with the usual Archimedean absolute value $|\cdot|$.

Maillet (1903): $|f_n| \leq n!^{\mathcal{O}(1)}$.

Pólya (1916) over $\mathbb{Q}$, Popken (1935) over $\overline{\mathbb{Q}}$: $d_n \leq n!^{\mathcal{O}(\log(n))}$ in the general case, and $d_n \leq n!^{\mathcal{O}(1)}$ if Q is a linear form in $Y_0, \ldots, Y_k$.

Popken (1935) using Maillet for the "Galoisian conjugates" of f: either $f_n = 0$ or $|f_n| \geq 1/n!^{\mathcal{O}(\log(n))}$.

- $\overline{\mathbb{Q}}$ with any non-Archimedean absolute value $|\cdot|_v$, v a finite place.

Mahler (1973, 1976): $|f_n|_v \leq n!^{\mathcal{O}(1)}$, and either $f_n = 0$ or $|f_n|_v \geq 1/n!^{\mathcal{O}(\log(n))}$.

Sibuya-Sperber (1981): $|f_n|_v \leq e^{\mathcal{O}(n)}$ (Dwork's conjecture).

- All the implicit constants in the $\mathcal{O}$ depend on the differential equation and f. They are all effective except possibly in Sibuya-Sperber's result.

- Mahler's conjecture (1976): the exponent $\mathcal{O}(\log(n))$ in Popken's lower bound *"can probably be improved to something like"* $\mathcal{O}(\log\log(n))$.

This would follow from the same improvement for the exponent in Pólya-Popken's upper bound for d_n.

Riccati equations

- $\frac{x}{\log(1+x)} = \sum_{n=0}^{\infty} g_n x^n$ is solution of $x(1+x)y' + y^2 - (1+x)y = 0$.

$$g_{n+1} = -\frac{1}{n+2}\Big((n-1)g_n + \sum_{j=1}^{n} g_j g_{n+1-j}\Big), \quad n \geq 0,\ g_0 = 1.$$

$\operatorname{lcm}\{1, 2, \ldots, n+1\} n!\, g_n \in \mathbb{Z}$ for all $n \geq 0$. Hence, $n!(n+1)!\, g_n \in \mathbb{Z}$.

- $\frac{x}{e^x-1} = \sum_{n=0}^{\infty} b_n x^n$ is solution of $xy' + y^2 + (x-1)y = 0$. We have $b_0 = 0, b_1 = -1/2,\ b_{2n+1} = 0$ for $n \geq 2$, and

$$b_{n+1} = -\frac{1}{n+2}\Big(b_n + \sum_{j=1}^{n} b_j b_{n+1-j}\Big), \quad n \geq 0,\ b_0 = 1.$$

Clausen–von Staudt Theorem: the denominator of $(2n)! b_{2n}$ is the product of the primes p such that $p-1$ divides $2n$.

$\operatorname{lcm}\{1, 2, \ldots, 2n+1\}(2n)! b_{2n} \in \mathbb{Z}$ for all $n \geq 0$. Hence, $n!(n+1)! b_n \in \mathbb{Z}$.

Elliptic differential equations

- *Weierstraß elliptic function* $\wp$ with modular invariants $g_2, g_3 \in \overline{\mathbb{Q}}$ (such that $g_2^3 \neq 27g_3^2$) satisfies

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3 \quad \text{and} \quad 12\wp^2 - 2\wp'' - g_2 = 0.$$

We have $\wp(x) = 1/x^2 + \sum_{n=2}^{\infty} p_n x^{2n-2}$, where for $n \geq 4$,

$$p_n = \frac{3}{(2n+1)(n-3)} \sum_{j=2}^{n-2} p_j p_{n-j}, \quad p_2 = \frac{g_2}{20}, \ p_3 = \frac{g_3}{28}.$$

Apparently, not much is known about the denominator of p_n in general beyond Pólya-Popken's upper bound.

- Lemniscate case: $g_2 = 4, g_3 = 0$.

$$\wp(x) = \frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{2^{4n} E_n}{4n} \cdot \frac{x^{4n-2}}{(4n-2)!} \in \mathbb{Q}[[x]],$$

Hurwitz (1898): *Let D_n denote the denominator of E_n.* (1) *no prime* $\equiv 3 \mod 4$ *divides D_n.* (2) *if a prime $p \equiv 1 \mod 4$ divides D_n, then $p-1$ divides $4n$ and p^2 does not divide D_n.* (3) $v_2(E_n) = -1$.

Consequently $(\prod_{p\,:\,p-1|4n} p)E_n \in \mathbb{Z}$, and $(4n+1)!E_n \in \mathbb{Z}$.

Non-linear recurrence for $(f_n)_{n\geq 0}$

In general, we have

$$f_{n+1} = \frac{1}{M(n)} \sum_{\sigma=\sigma_1}^{\sigma_2} \sum_{k=1}^{k_0} \sum_{\substack{j_1+\cdots+j_k=n-\sigma \\ 0\leq j_1,\ldots,j_k\leq n}} P_{\sigma,k}(n, j_1, j_2, \ldots, j_k) f_{j_1} f_{j_2} \cdots f_{j_k}, \quad n \geq N, \quad (2)$$

where N is some non-negative integer, $M(X) \in \mathcal{O}_{\overline{\mathbb{Q}}}[X]$ vanishes for no $n \geq N$, the coefficients $P_{\sigma,k}(n, j_1, j_2, \ldots, j_k)$ are in $\mathcal{O}_{\overline{\mathbb{Q}}}$. More precisely, the $P_{\sigma,k}(n, j_1, j_2 \ldots, j_k)$ are piecewise polynomials in $n, j_1, j_2, \ldots, j_k$ with coefficients in $\mathcal{O}_{\overline{\mathbb{Q}}}$.

The constants σ_1, σ_2 are integers, k_0 is a positive integer.

The initial values $f_0, f_1, \ldots, f_N$ are algebraic numbers with common denominator D.

The restriction $j_1, \ldots, j_k \leq n$ can be ignored for *non-negative* σ, while it has an effect for *negative* σ.

Interpretation of the polynomial M: Sibuya-Sperber's Lemma

Let $f(x) = \sum_{n=0}^{\infty} f_n x^n \in \overline{\mathbb{Q}}[[x]]$ be a solution of $Q(x, y, \ldots, y^{(k)}) = 0$ with coefficients in $\overline{\mathbb{Q}}$.

Sibuya-Sperber then consider an ADE $\widetilde{Q}(x, y, \ldots, y^{(\ell)}) = 0$ with $\widetilde{Q} \in \overline{\mathbb{Q}}[X, Y_0, \ldots, Y_\ell] \setminus \{0\}$ of which f is still a solution, where $\ell \geq 0$ is minimal amongst all ADEs satisfied by f, and the degree of $\widetilde{Q}$ in Y_ℓ is also minimal.

ℓ is the transcendence degree of the field generated over $\overline{\mathbb{Q}}(x)$ by f and all its derivatives.

This in particular ensures the crucial fact that

$$\frac{\partial \widetilde{Q}}{\partial Y_\ell}(x, f, \ldots, f^{(\ell)}) \neq 0.$$

They attach to $\widetilde{Q}$ and f a linear differential operator $L_0 \in \overline{\mathbb{Q}}[[x]][\frac{d}{dx}]$ of order ℓ.

Let P_0 be the indicial polynomial at the origin of L_0.

Lemma 1 (Sibuya-Sperber, 1981)

For any $c \geq 0$, there exist integers $N \geq 0, N' \geq N, N'' \geq c$ such that $u(x) := \sum_{n=N'}^{\infty} f_n x^{n-N} \in \overline{\mathbb{Q}}[[x]]$ satisfies

$$L(u) = x^{N''} F(x, u, u', \ldots, u^{(\ell)}), \tag{3}$$

where $F \in \mathcal{O}_{\overline{\mathbb{Q}}}[X, Y_0, \ldots, Y_\ell]$, $L \in \mathcal{O}_{\overline{\mathbb{Q}}}[x, \frac{d}{dx}]$ is of order ℓ and the order at $x = 0$ of leading coefficient of L is bounded independently of c.

The indicial polynomial at the origin of L is $P_0(X + N)$.

From (3), Sibuya and Sperber deduce a recurrence for $(f_n)_{n \geq 0}$ of the form (2) with $\sigma_1 \geq 0$ (because c can be arbitrarily large) and with

$$M(X) = P_0(X + N + 1).$$

This gives an interpretation of M in terms of the indicial polynomial P_0 of L_0 attached to $\widetilde{Q}$ and f.

P_0 and M may be different for another solution in $\overline{\mathbb{Q}}[[x]]$ of the ADE $Q(x, y, \ldots, y^{(k)}) = 0$.

Our main result

Theorem 1 (Krattenthaler-R., 2025)

Let $f \in \overline{\mathbb{Q}}[[x]]$ be a solution of a non-trivial algebraic differential equation $Q(x, y, \dots, y^{(k)}) = 0$, where $Q \in \overline{\mathbb{Q}}[X, Y_0, \dots, Y_k]$.

Assume the sequence $(f_n)_{n\geq 0}$ of the Taylor coefficients of f satisfies a recurrence of the form (2) with M split over $\mathbb{Q}$.

Then there exist δ and $\nu \in \mathbb{N}$ such that the denominator d_n of f_n divides $\delta^{n+1}(\nu n + \nu)!^{2s}$ for all $n \geq 0$, where s is the degree of M.

This proves a strong form of Mahler's conjecture in the split case. It is a completely effective result, but not always sharp due to its generality.

Corollary 1 (Effective version of Sibuya-Sperber's theorem in the split case)

In the setting of Theorem 1, for all finite places v of $\overline{\mathbb{Q}}$ over any given rational prime number p,

$$|f_n|_v \leq p^{\left(v_p(\delta)+\frac{2s\nu}{p-1}\right)(n+1)}, \quad n \geq 0,$$

with the standard normalization $|p|_v := 1/p$.

Ideas of the proof of Theorem 1

• Our strategy: Experiment with simple examples, Guess patterns, Prove them, Repeat this process with less simple examples, etc, untill full generality is achieved.

Let $f(x) = 1 + \sum_{n=1}^{\infty} f_n x^n$ be a solution of the Riccati equation $xf'(x) - xf(x)^2 + af(x) - a = 0$, where $a \geq 1$ is a fixed integer.

$$f_{n+1} = \frac{1}{n+a+1} \sum_{j=0}^{n} f_j f_{n-j}, \qquad n \geq 0, \ f_0 := 1.$$

Numerical observations: $n!\,(n+a)!\,f_n \in \mathbb{Z}$ for all $n \geq 0$.

Our first thought: this must be easy to prove with $\varphi_n := n!\,(n+a)!\,f_n$:

$$\varphi_{n+1} = \sum_{j=0}^{n} \frac{n+1}{(j+1)(j+2)\cdots(j+a)} \binom{n}{j} \binom{n+a}{j} \varphi_j \varphi_{n-j}, \qquad n \geq 0, \ \varphi_0 := a!,$$

The binomial coefficients are an important gain but this comes with a new "big" denominator $(j+1)(j+2)\cdots(j+a)$.

By a p-adic analysis of $\frac{n+1}{(j+1)(j+2)\cdots(j+a)} \binom{n}{j} \binom{n+a}{j}$: there exists $\delta \in \mathbb{N}$ such that

$$\delta^{n+1} n!\,(n+a)!\,f_n \in \mathbb{Z}.$$

• Consider a non-linear recurrence with M split over $\mathbb{Q}$:

$$f_{n+1} = \frac{1}{C\prod_{i=1}^{s}(a_i n + b_i)} \times \sum_{\sigma=\sigma_1}^{\sigma_2} \sum_{k=1}^{k_0} \sum_{\substack{j_1+\cdots+j_k=n-\sigma \\ 0\le j_1,\ldots,j_k\le n}} P_{\sigma,k}(n, j_1, j_2, \ldots, j_k) f_{j_1} f_{j_2} \cdots f_{j_k}, \quad \text{for } n \geq N, \quad (4)$$

where $N \geq 0$, $P_{\sigma,k}(n, j_1, j_2, \ldots, j_k) \in \mathcal{O}_{\overline{\mathbb{Q}}}$, $C \in \mathbb{Z}^*$, $\sigma_1, \sigma_2 \in \mathbb{Z}$, $s, k_0 \in \mathbb{N}$, $b_i \in \mathbb{Z}$, $a_i \in \mathbb{N}$ s.t. $\gcd(a_i, b_i) = 1$ and $a_i n + b_i \neq 0$ for all $i \in \{1, \ldots, s\}$ and all $n \geq N$.

The initial values $f_0, f_1, \ldots, f_N$ are algebraic numbers with common denominator D.

First step: we found an effective procedure to determine another recurrence for $(f_n)_{n\geq 0}$ where $\sigma_1 \geq 0$, with possibly N replaced by some $\widetilde{N} > N$. This is an alternative to the Sibuya-Sperber procedure.

Second step: we proved a completely explicit and more precise version of Theorem 1 when $\sigma_1 \geq 0$.

Theorem 2

Let us consider a recurrence of the form (4) with $\sigma_1 \geq 0$. Then, for all $n \geq N$, the denominator d_n of f_n divides

$$C^n D^{(k_0-1)n+1} E^n \prod_{i=1}^{s} (a_i n)!\,(a_i n + b_i - a_i)!,$$

where

$$E = \prod_{i=1}^{s} \Big(\big(\max\{b_i - a_i, 0\}\big)!^{k_0-1} \prod_{p<2\max_{1\leq j\leq s}(b_j - a_j)} p^{\lceil \log_p \max\{b_i - a_i, 1\} \rceil} \Big).$$

After a change of sequence

$$\varphi_n := \Big(\prod_{i=1}^{s} (a_i n)!\,(a_i n + b_i - a_i)! \Big) f_n,$$

the proof consists in a very complicated analysis of the p-adic valuation of

$$\prod_{i=1}^{s} \frac{(a_i n + a_i)!\,(a_i n + b_i - 1)!}{\prod_{\substack{t=1\\ j_t > N}}^{k} (a_i j_t)!\,(a_i j_t + b_i - a_i)!}. \tag{5}$$

The possible primes in the denominator of (5) divide E, with valuation at most n.

Examples again

- $\frac{x}{\log(1+x)} = \sum_{n=0}^{\infty} g_n x^n$ and $\frac{x}{e^x-1} = \sum_{n=0}^{\infty} b_n x^n$.

Theorem 2: $24^n(n-1)!(n+1)!\,\{g_n, b_n\} \in \mathbb{Z}$ for all $n \geq 1$.

- Weierstraß $\wp$ with $g_2, g_3 \in \overline{\mathbb{Q}}$, $g_2^3 \neq 27g_3^2$, and $\wp(x) = 1/x^2 + \sum_{n=2}^{\infty} p_n x^{2n-2}$.

Theorem 2: $D^{n+1}2520^n(n-1)!n!(2n)!(2n+5)!p_{n+2} \in \mathcal{O}_{\overline{\mathbb{Q}}}$ for all $n \geq 1$, where $D \geq 1$ is the least common denominator of $g_2/20$ and $g_3/28$.

- $u(x) := \sum_{n=0}^{\infty} u_n x^n$ solution of Painlevé PII'

$$y'' = \delta(2y^3 - 2xy) + \gamma(6y^2 + x) + \beta y + \alpha, \quad \alpha, \beta, \gamma, \delta \in \overline{\mathbb{Q}}.$$

$$u_{n+1} = \frac{2}{n(n+1)}\Big(\delta \sum_{\substack{j_1+j_2+j_3=n-1 \\ 0\leq j_1,j_2,j_3\leq n-1}} u_{j_1}u_{j_2}u_{j_3} + 3\gamma \sum_{\substack{j_1+j_2=n-1 \\ 0\leq j_1,j_2\leq n-1}} u_{j_1}u_{j_2} - 2\beta u_{n-1} - 2\delta u_{n-2}\Big), n \geq 2$$

with u_0, u_1 arbitrary in $\overline{\mathbb{Q}}$, and $u_2 = -\delta u_0^3 - 3\gamma u_0^2 - \beta u_0/2 - \alpha/2$.

Theorem 2: let $C \geq 1$ be the least common denominator of $2\delta, 6\gamma, 2\beta$, and let $D \geq 1$ be the least common denominator of u_0, u_1, u_2. Then

$$C^n D^{2n+1}(n-1)!\, n!^3\, u_n \in \mathcal{O}_{\overline{\mathbb{Q}}}, \quad n \geq 2.$$

- The dilogarithm $\mathrm{Li}_2(x) := \sum_{n=1}^{\infty} x^n/n^2$ admits as inverse for the composition

$$\ell(x) := \sum_{n=0}^{\infty} \ell_n x^n = x - \frac{1}{4}x^2 + \frac{1}{72}x^3 - \frac{1}{576}x^4 - \frac{31}{86400}x^5 - \frac{149}{1036800}x^6 - \cdots,$$

which satisfies

$$\ell''\ell - \ell''\ell^2 + \ell'^3 + \ell'^2\ell - \ell'^2 = 0$$

and

$$\ell_{n+1} = \frac{1}{(n+1)^2}\Big(- \sum_{\substack{i+j=n \\ i\le n-2, j\le n}} (i+1)(i+2)\ell_{i+2}\ell_j + \sum_{\substack{i+j+k=n \\ i\le n-2, j,k\le n}} (i+1)(i+2)\ell_{i+2}\ell_j\ell_k$$
$$- \sum_{\substack{i+j+k=n \\ i,j,k\le n-1}} (i+1)(j+1)(k+1)\ell_{i+1}\ell_{j+1}\ell_{k+1} - \sum_{\substack{i+j+k=n \\ i,j\le n-1, k\le n}} (i+1)(j+1)\ell_{i+1}\ell_{j+1}\ell_k$$
$$+ \sum_{\substack{i+j=n \\ i,j\le n-1}} (i+1)(j+1)\ell_{i+1}\ell_{j+1}\Big), \quad n \ge 0.$$

Theorem 1: for all $n \ge 0$, $\delta^{n+1}(\nu n + \nu)!^4 \ell_n \in \mathbb{Z}$.

• Various automorphic functions satisfy an ADE $S(y)(y')^2 = \{y; x\}$ of order 3, where

$$\{y; x\} := \frac{y'''}{y'} - \frac{3}{2}\left(\frac{y''}{y'}\right)^2$$

is the Schwarzian derivative with respect to x, and S is a rational function with poles of order at most 2.

Mahler (1969): the *elliptic modular invariant*

$$F(q) := J(q^2) = \frac{1}{q^2} + 744 + 196884q^2 + 21493760q^4 + 864299970q^6 \ldots \in \frac{1}{q^2}\mathbb{Z}[[q]],$$

satisfies

$$F''' = \frac{3q^2F''^2 - 4qF'F'' - F'^2}{2q^2F'} - F'^3\left(\frac{4}{9F^2} + \frac{3}{8(F-12^3)^2} - \frac{23}{72F(F-12^3)}\right)$$

and F cannot satisfy an ADE of order ≤ 2.

The indicial polynomial P_0 of $L_0 \in \overline{\mathbb{Q}}[[x]][\frac{d}{dx}]$ is irreducible over $\mathbb{Q}$:

$$P_0(X) = X^3 + 8X^2 - 10X + 64.$$

From Sibuya-Sperber's Lemma 1, we get a recurrence of type (2) for the coefficients of F with $M(X) = P_0(X + m)$ for some integer $m \geq 0$.

Theorem 1 cannot be applied. It is not known if there exists a recurrence with M split over $\mathbb{Q}$ for the coefficients of F.

Mahler's conjecture when M is not split over $\mathbb{Q}$

The equation $x^2y'' + (x-1)y' + y - xy^2 = 1$ has for solution the power series $\sum_{n=0}^{\infty} f_n x^n$ with

$$f_{n+1} = \frac{1}{n^2+1} \sum_{k=0}^{n} f_k f_{n-k}, \quad n \geq 0, \ f_0 := 1.$$

Numerical experiments for values of n up to 2000: $\log(d_n)/(n\log(n)^2)$ seems to converge to a constant close to 0.566.

Moreover,

$$\frac{\log(d_{2n})}{2n\log(2n)} - \frac{\log(d_n)}{n\log(n)}$$

seems to converge to a constant close to $0.39(\approx \log(2)\cdot 0.566)$.

This suggests that Mahler's conjecture does not hold in general.