Arithmetic theory of E-operators

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Abstract

In [*Séries Gevrey de type arithmétique I. Théorèmes de pureté et de dualité*, Annals of Math. **151** (2000), 705–740], André has introduced E-operators, a class of differential operators intimately related to E-functions, and constructed local bases of solutions for these operators. In this paper we investigate the arithmetical nature of connection constants of E-operators at finite distance, and of Stokes constants at infinity. We prove that they involve values at algebraic points of E-functions in the former case, and in the latter one, values of G-functions and of derivatives of the Gamma function at rational points in a very precise way. As an application, we define and study a class of numbers having certain algebraic approximations defined in terms of E-functions. These types of approximations are motivated by the convergents to the number e, as well as by recent constructions of approximations to Euler's constant and values of the Gamma function. Our results and methods are completely different from those in our paper [*On the values of G-functions*, Commentarii Math. Helv. **29** (2014), 313–341], where we have studied similar questions for G-functions.

1 Introduction

In a seminal paper [1], André has introduced E-operators, a class of differential operators intimately related to E-functions, and constructed local bases of solutions for these operators. In this paper we investigate the arithmetical nature of connection constants of E-operators, and prove that they involve values at algebraic points of E-functions or G-functions, and values at rational points of derivatives of the Gamma function. As an application, we will focus on algebraic approximations to such numbers, in connection with Aptekarev's famous construction for Euler's constant γ.

To begin with, let us recall the following definition.

Definition 1. *An E-function E is a power series $E(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$ such that the coefficients a_n are algebraic numbers and there exists $C > 0$ such that:*

(i) *the maximum of the moduli of the Galois conjugates of a_n is $\leq C^{n+1}$ for any n.*

(ii) *there exists a sequence of non-zero rational integers d_n, with $|d_n| \leq C^{n+1}$, such that $d_n a_m$ is an algebraic integer for all $m \leq n$.*

(iii) $E(z)$ *satisfies a homogeneous linear differential equation with coefficients in* $\overline{\mathbb{Q}}(z)$.

A G-function is defined similarly, as $\sum_{n=0}^{\infty} a_n z^n$ with the same assumptions *(i)*, *(ii)*, *(iii)*; throughout the paper we fix a complex embedding of $\overline{\mathbb{Q}}$.

We refer to [1] for an overview of the main properties of E and G-functions. For the sake of precision, we mention that the class of E-functions was first defined by Siegel in a more general way, with bounds of the shape $n!^{\varepsilon}$ for any $\varepsilon > 0$ and any $n \gg_{\varepsilon} 1$, instead of C^{n+1} for all $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$. The functions covered by Definition 1 are called E^*-functions by Shidlovskii [23], and are the ones used in the recent litterature under the denomination E-functions (see [1, 6, 17]); it is believed that both classes coincide.

Examples of E-functions include $e^{\alpha z}$ with $\alpha \in \overline{\mathbb{Q}}$, hypergeometric series ${}_pF_p$ with rational parameters, and Bessel functions J_{α} with $\alpha \in \mathbb{N}$. Very precise transcendence (and even algebraic independence) results are known on values of E-functions, such as the Siegel-Shidlovskii theorem [23]. Beukers' refinement of this result enables one to deduce the following statement (see §3.1), whose analogue is false for G-functions (see [5] for interesting examples):

Theorem 1. *An E-function with coefficients in a number field $\mathbb{K}$ takes at an algebraic point α either a transcendental value or a value in $\mathbb{K}(\alpha)$.*

In this paper we consider the following set $\mathbf{E}$, which is analogous to the ring $\mathbf{G}$ of values at algebraic points of analytic continuations of G-functions studied in [11].

Definition 2. *The set $\mathbf{E}$ is defined as the set of all values taken by any E-function at any algebraic point.*

We recall that $\mathbf{G}$ might be equal to $\mathcal{P}[1/\pi]$, where $\mathcal{P}$ is the ring of periods (in the sense of Kontsevich-Zagier [16]: see §2.2 of [11]). On the other hand, it seems reasonable to imagine that $\mathbf{E}$ is contained in the ring generated by $1/\pi$ and exponential periods (see [12]).

Since E-functions are entire and $E(\alpha z)$ is an E-function for any E-function $E(z)$ and any $\alpha \in \overline{\mathbb{Q}}$, we may restrict in Definition 2 to values at $z = 1$. Moreover E-functions form a ring, so that $\mathbf{E}$ is a subring of $\mathbb{C}$. Its group of units contains $\overline{\mathbb{Q}}^*$ and $\exp(\overline{\mathbb{Q}})$ because algebraic numbers, $\exp(z)$ and $\exp(-z)$ are E-functions. Other elements of $\mathbf{E}$ include values at algebraic points of Bessel functions J_{α} with $\alpha \in \mathbb{N}$, and also of any arithmetic Gevrey series of negative order (see [1], Corollaire 1.3.2). It seems unlikely that $\mathbf{E}$ is a field and we don't know if we have a full description of its units.

A large part of our results is devoted to the arithmetic description of *connection constants* or *Stokes constants*. Any E-function $E(z)$ satisfies a differential equation $Ly = 0$, where L is an E-operator (see [1]); it is not necessarily minimal and its only possible singularities are 0 and ∞, the former beging regular and the latter irregular. André has proved [1] that a basis of solutions of L at $z = 0$ is of the form $(E_1(z), \ldots, E_{\mu}(z)) \cdot z^M$ where M is an upper triangular $\mu \times \mu$ matrix with coefficients in $\mathbb{Q}$ and the $E_j(z)$ are

E-functions. This implies that any local solution $F(z)$ of L at $z = 0$ is of the form

$$F(z) = \sum_{j=1}^{\mu} \Big(\sum_{s \in S_j} \sum_{k \in K_j} \phi_{j,s,k} z^s \log(z)^k \Big) E_j(z) \tag{1.1}$$

where $S_j \subset \mathbb{Q}, K_j \subset \mathbb{N}$ are finite sets and $\phi_{j,s,k} \in \mathbb{C}$. Our purpose is to study the connection constants of $F(z)$, assuming all coefficients $\phi_{j,s,k}$ to be algebraic (with a special focus on the special case where $F(z)$ itself is an E-function).

Any point $\alpha \in \overline{\mathbb{Q}} \setminus \{0\}$ is a regular point of L and there exists a basis of locally holomorphic solutions $G_1(z), \dots, G_\mu(z) \in \overline{\mathbb{Q}}[[z - \alpha]]$ such that, around $z = \alpha$,

$$F(z) = \omega_1 G_1(z) + \cdots + \omega_\mu G_\mu(z) \tag{1.2}$$

for some complex numbers $\omega_1, \dots, \omega_\mu$, called the connection constants (at finite distance).

Proposition 1. *If all coefficients $\phi_{j,s,k}$ in (1.1) are algebraic then the connection constants $\omega_1, \dots, \omega_\mu$ in (1.2) belong to $\mathbf{E}[\log \alpha]$, and even to $\mathbf{E}$ if $F(z)$ is an E-function.*

The situation is much more complicated around ∞, which is in general an irregular singularity of L; this part is therefore much more involved than the corresponding one for G-functions [11] (since ∞ is a regular singularity of G-operators, the connection constants of G-functions at any $\zeta \in \overline{\mathbb{Q}} \cup \{\infty\}$ always belong to $\mathbf{G}$). The local solutions at ∞ involve divergent series, which give rise to Stokes phenomenon: the expression of an E-function $E(z)$ on a given basis is valid on certain angular sectors, and the connection constants may change from one sector to another when crossing certain rays called anti-Stokes directions. For this reason, we speak of Stokes constants rather than connection constants. More precisely, let $\theta \in \mathbb{R}$ and assume that θ is not an anti-Stokes direction (which amounts to excluding finitely many values of $\theta \bmod 2\pi$). Then we compute explicitly the asymptotic expansion

$$E(z) \approx \sum_{\rho \in \Sigma} e^{\rho z} \sum_{\alpha \in S} \sum_{i \in T} \sum_{n=0}^{\infty} c_{\rho,\alpha,i,n} z^{-n-\alpha} \log(1/z)^i \tag{1.3}$$

as $|z| \to \infty$ in a large sector $\theta - \frac{\pi}{2} - \varepsilon \le \arg(z) \le \theta + \frac{\pi}{2} + \varepsilon$ for some $\varepsilon > 0$; in precise terms, $E(z)$ can be obtained from this expansion by Borel-Laplace summation (i.e., Ramis' 1-summation; see §4.1). Here $\Sigma \subset \overline{\mathbb{Q}}$, $S \subset \mathbb{Q}$ and $T \subset \mathbb{N}$ are finite subsets, and the coefficients $c_{\rho,\alpha,i,n}$ are complex numbers (that also depend on θ); all of them are constructed explicitly in terms of the Laplace transform $g(z)$ of $E(z)$, which is annihilated by a G-operator. In applying or studying (1.3) we shall always assume that the sets Σ, S and T have the least possible cardinality (so that $\alpha - \alpha' \notin \mathbb{Z}$ for any distinct $\alpha, \alpha' \in S$) and that for any α there exist ρ and i with $c_{\rho,\alpha,i,0} \ne 0$. Then the asymptotic expansion (1.3) is uniquely determined by $E(z)$ and θ (see §4.1).

The existence of an asymptotic expansion of the form (1.3) is *a priori* ensured by the theory of linear differential equations with meromorphic coefficients, see [13, page 582, Theorem VIII.7], but its explicit determination is a difficult task in general. One of our

main contributions is the value of $c_{\rho,\alpha,i,n}$, which is given in terms of derivatives of $1/\Gamma$ at $\alpha \in \mathbb{Q}$ and connection constants of $g(z)$ at its finite singularities ρ. André has constructed [1, Théorème 4.3 (v)] a basis $H_1(z), \ldots, H_\mu(z)$ of formal solutions at infinity of an E-operator that annihilates $E(z)$; these solutions involve divergent Gevrey series of order 1, and are of the same form as the right hand side of (1.3), with algebraic coefficients $c_{\rho,\alpha,i,n}$. The asymptotic expansion (1.3) of $E(z)$ in a large sector bisected by θ can be written on this basis as

$$\omega_{1,\theta} H_1(z) + \ldots + \omega_{\mu,\theta} H_\mu(z) \tag{1.4}$$

with Stokes constants $\omega_{i,\theta}$. To identify these constants, we first introduce another ring.

Definition 3. *We define* $\mathbf{S}$ *as the* $\mathbf{G}$*-module generated by all the values of derivatives of the Gamma function at rational points. It is also the* $\mathbf{G}[\gamma]$*-module generated by all the values of* Γ *at rational points, and it is a ring.*

We show in §2 why the two modules coincide, and why $\mathbf{S}$ is a ring. The Rohrlich-Lang conjecture (see [2] or [24]) implies that the values $\Gamma(s)$, for $s \in \mathbb{Q}$ with $0 < s \leq 1$, are $\overline{\mathbb{Q}}$-linearly independent. We conjecture that these numbers are in fact also $\mathbf{G}[\gamma]$-linearly independent, so that $\mathbf{S}$ is the free $\mathbf{G}[\gamma]$-module they generate.

We then have the following result. We recall that the coefficients $c_{\rho,\alpha,i,n}$ depend on θ.

Theorem 2. *Let* $E(z)$ *be an* E*-function, and* $\theta \in \mathbb{R}$ *be a direction which is not anti-Stokes for* $E(z)$*. Then:*

(i) *The Stokes constants* $\omega_{i,\theta}$ *belong to* $\mathbf{S}$*.*

(ii) *All coefficients* $c_{\rho,\alpha,i,n}$ *in* (1.3) *belong to* $\mathbf{S}$*.*

(iii) *Let* $\rho \in \Sigma$*,* $\alpha \in S$*, and* $n \geq 0$*; denote by* k *the largest* $i \in T$ *such that* $c_{\rho,\alpha,i,n} \neq 0$*. If* k *exists then for any* $i \in T$ *the coefficient* $c_{\rho,\alpha,i,n}$ *is a* $\mathbf{G}$*-linear combination of* $\Gamma(\alpha)$*,* $\Gamma'(\alpha)$*, ...,* $\Gamma^{(k-i)}(\alpha)$*. In particular,* $c_{\rho,\alpha,k,n} \in \Gamma(\alpha) \cdot \mathbf{G}$*. Here* $\Gamma^{(\ell)}(\alpha)$ *is understood as* $\Gamma^{(\ell)}(1)$ *if* $\alpha \in \mathbb{Z}_{\leq 0}$*.*

(iv) *Let* $F(z)$ *be a local solution at* $z = 0$ *of an* E*-operator, with coefficients* $\phi_{j,s,k} \in \mathbf{S}$ *in* (1.1)*. Then assertions* (i) *and* (ii) *hold with* $F(z)$ *instead of* $E(z)$*.*

Assertion (iv) applies to many special functions, including Bessel's functions J_α with $\alpha \in \mathbb{Q}$ and $\mathrm{Ai}(z^{2/3})$ where $\mathrm{Ai}(z)$ is Airy's oscillating integral (see [1]).

Assertions (i) and (iv) of Theorem 2 are consistent with André's remark in [1, p. 722]: "*Nous privilégierons une approche formelle, qui permettrait de travailler sur* $\overline{\mathbb{Q}}(\Gamma^{(k)}(a))_{k \in \mathbb{N}, a \in \mathbb{Q}}$ *plutôt que sur* $\mathbb{C}$ *si l'on voulait*". ([1])

Many examples of E-functions for which values of (derivatives of) Γ appear in Stokes constants are known (see for instance Section 16.41 of [25] for confluent hypergeometric

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[1] "We adopt a formal approach, which would enable one to work over $\overline{\mathbb{Q}}(\Gamma^{(k)}(a))_{k \in \mathbb{N}, a \in \mathbb{Q}}$ rather than $\mathbb{C}$ if one would prefer".

equations of order 2, or [10]). The point in Theorem 2 is that, in some sense, no other number can appear. Moreover, an important feature of assertion (iii) is that $\Gamma^{(k)}(\alpha)$, for $k \geq 1$, never appears in the coefficient of a leading term of (1.3), but only combined with higher powers of $\log(1/z)$. This motivates the logarithmic factor in (1.8) below, and explains an observation we had made on Euler's constant: it always appears through $\gamma - \log(1/z)$ (see Eq. (4.7) in §4.2). Moreover, in (iii), it follows from the remarks made in §2 that, alternatively, $c_{\rho,\alpha,i,n} = \Gamma(\alpha) \cdot P_{\rho,\alpha,i,n}(\gamma)$ for some polynomial $P_{\rho,\alpha,i,n}(X) \in \mathbf{G}[X]$ of degree $\leq k - i$.

The proof of Theorem 2 is based on Laplace transform, the André-Chudnovski-Katz Theorem on solutions of G-operators, and a specific complex integral (see [1], p. 735). At some point, we take advantage of the existence of André's basis $(H_1, \ldots, H_\mu)$ of the E-operator at infinity, not to increase the length of the paper. However, our approach also provides a new construction of this basis, from bases of microsolutions of the underlying G-operator (see [12]).

As an application of Proposition 1 and Theorem 2, we study sequences of algebraic (or rational) approximations of special interest related to E-functions. In [11] we have proved that a complex number α belongs to the fraction field Frac $\mathbf{G}$ of $\mathbf{G}$ if, and only if, there exist sequences $(P_n)_n$ and $(Q_n)_n$ of algebraic numbers such that $\lim_n P_n/Q_n = \alpha$ and $\sum_{n\geq 0} P_n z^n$, $\sum_{n\geq 0} Q_n z^n$ are G-functions. We have introduced this notion in order to give a general framework for irrationality proofs of values of G-functions such as zeta values. Such sequences are called G-approximations of α, when P_n and Q_n are rational numbers. We drop this last assumption in the context of E-functions (see §3.1), and consider the following definition.

Definition 4. *Sequences* $(P_n)_{n\geq 0}$ *and* $(Q_n)_{n\geq 0}$ *of algebraic numbers are said to be E-approximations of $\alpha \in \mathbb{C}$ if*

$$\lim_{n\to+\infty} \frac{P_n}{Q_n} = \alpha$$

and

$$\sum_{n=0}^{\infty} P_n z^n = A(z) \cdot E\big(B(z)\big), \qquad \sum_{n=0}^{\infty} Q_n z^n = C(z) \cdot F\big(D(z)\big)$$

where E and F are E-functions, A, B, C, D are algebraic functions in $\overline{\mathbb{Q}}[[z]]$ with $B(0) = D(0) = 0$.

This definition is motivated by the fact that many sequences of approximations to classical numbers are E-approximations, for instance diagonal Padé approximants to e^z. Because of the asymptotic nature of the notion, a more flexible definition would be that the generating series of $(P_{n+k})_{n\geq 0}$ and $(Q_{n+k})_{n\geq 0}$ are of the desired form for some given integer k; however by changing the name of the sequences one may assume that $k = 0$. We also prove that the convergents of the respective continued fraction expansions of e and $\frac{e-1}{e+1}$ define E-approximations (see §6.1). The classical proof that $\sum_{n=1}^{\infty} \frac{1}{(an)!^b c^n}$ is irrational (for positive integers a, b, c) is based on sequences of rational approximations

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that are E-approximations, as in the special case of e. We hope that focusing at E-approximations may be helpful in finding irrationality proofs for new interesting numbers. Elements in Frac $\mathbf{G}$ also have E-approximations, since G-approximations $(P_n)_n$ and $(Q_n)_n$ of a complex number always provide E-approximations $P_n/n!$ and $Q_n/n!$ of the same number. In §6.1, we construct E-approximations to $\Gamma(\alpha)$ for any $\alpha \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$, $\alpha < 1$, by letting $E_\alpha(z) = \sum_{n=0}^\infty \frac{z^n}{n!(n+\alpha)}$, $Q_n(\alpha) = 1$, and defining $P_n(\alpha)$ by the series expansion (for $|z| < 1$)

$$\frac{1}{(1-z)^{\alpha+1}} E_\alpha \left(-\frac{z}{1-z} \right) = \sum_{n=0}^\infty P_n(\alpha) z^n \in \mathbb{Q}[[z]];$$

then $\lim_n P_n(\alpha) = \Gamma(\alpha)$. The number $\Gamma(\alpha)$ appears in this setting as a Stokes constant. The condition $\alpha < 1$ is harmless because we readily deduce E-approximations to $\Gamma(\alpha)$ for any $\alpha \in \mathbb{Q}$, $\alpha > 1$, by means of the functional equation $\Gamma(s+1) = s\Gamma(s)$. Moreover, since $\frac{1}{(1-z)^{\alpha+1}} E_\alpha \left(-\frac{z}{1-z} \right)$ is holonomic, the sequence $(P_n(\alpha))$ satisfies a linear recurrence, of order 3 with polynomial coefficients in $\mathbb{Z}[n, \alpha]$ of total degree 2 in n and α; see §6.1. This construction yields a new sequence of rational approximations to $\Gamma(\alpha)$; it is simpler than that in [21] but the convergence to $\Gamma(\alpha)$ is slower.

Definition 4 enables us to consider an interesting class of numbers: those having E-approximations. Of course this is a countable subset of $\mathbb{C}$. We have seen that it contains all values of the Gamma function at rational points s, which are conjectured to be irrational if $s \notin \mathbb{Z}$; very few results are known in this direction (see [24]), and using suitable E-approximations may lead to prove new ones.

However we conjecture that Euler's constant γ does not have E-approximations: all approximations we have thought of seem to have generating functions not as in Definition 4. This is a reasonable conjecture in view of Theorem 3 we are going to state now.

Given two subsets X and Y of $\mathbb{C}$, we set

$$X \cdot Y = \left\{ xy \,\middle|\, x \in X, y \in Y \right\}, \quad \frac{X}{Y} = \left\{ \frac{x}{y} \,\middle|\, x \in X, y \in Y \setminus \{0\} \right\}.$$

We also set $\Gamma(\mathbb{Q}) = \{\Gamma(x) | x \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}\}$. If X is a ring then we denote by Frac $X = \frac{X}{X}$ its field of fractions. We define Euler's Beta function by $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$. We recall [11] that $B(x,y)$ belongs to the group of units $\mathbf{G}^*$ of $\mathbf{G}$ for any $x, y \in \mathbb{Q}$, so that Γ induces a group homomorphism $\mathbb{Q} \to \mathbb{C}^*/\mathbf{G}^*$ (by letting $\Gamma(x) = 1$ for $x \in \mathbb{Z}_{\leq 0}$). Therefore $\Gamma(\mathbb{Q}) \cdot \mathbf{G}^*$ is a subgroup of $\mathbb{C}^*$, and so is $\Gamma(\mathbb{Q}) \cdot \exp(\overline{\mathbb{Q}}) \cdot \text{Frac}\,\mathbf{G}$; for future reference we write

$$\Gamma(\mathbb{Q}) \cdot \Gamma(\mathbb{Q}) \subset \Gamma(\mathbb{Q}) \cdot \mathbf{G} \quad \text{and} \quad \frac{\Gamma(\mathbb{Q})}{\Gamma(\mathbb{Q})} \subset \Gamma(\mathbb{Q}) \cdot \mathbf{G}. \tag{1.5}$$

Theorem 3. *The set of numbers having E-approximations contains*

$$\frac{\mathbf{E} \cup \Gamma(\mathbb{Q})}{\mathbf{E} \cup \Gamma(\mathbb{Q})} \cup \text{Frac}\,\mathbf{G} \tag{1.6}$$

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and it is contained in

$$\frac{\mathbf{E} \cup (\Gamma(\mathbb{Q}) \cdot \mathbf{G})}{\mathbf{E} \cup (\Gamma(\mathbb{Q}) \cdot \mathbf{G})} \cup \left(\Gamma(\mathbb{Q}) \cdot \exp(\overline{\mathbb{Q}}) \cdot \mathrm{Frac}\, \mathbf{G} \right). \tag{1.7}$$

The proof of (1.6) is constructive; the one of (1.7) is based on an explicit determination of the asymptotically dominating term of a sequence $(P_n)_n$ as in Definition 4. This determination is based on analysis of singularities, the saddle point method, asymptotic expansions (1.3) of $E(z)$, Proposition 1, and Theorem 2; it is of independent interest (see Theorem 6 in §5). The dominating term comes from the local behaviour of $E(z)$ at some $z_0 \in \mathbb{C}$ (providing elements of $\mathbf{E}$, in connection with Proposition 1) or at infinity (providing elements of $\Gamma(\mathbb{Q}) \cdot \mathbf{G}$; Theorem 2 is used in this case). This dichotomy leads to the unions in (1.6) and (1.7); it makes it unlikely for the set of numbers having E-approximations to be a field, or even a ring. We could have obtained a field by restricting Definition 4 to the case where $B(z) = D(z) = z$ and $A(z), C(z)$ are not polynomials, since in this case the behavior of $E(z)$ at ∞ would not come into the play; this field would be simply $\mathrm{Frac}\, \mathbf{E}$.

It seems likely that there exist numbers having E-approximations but no G-approximations, because conjecturally $\mathrm{Frac}\, \mathbf{E} \cap \mathrm{Frac}\, \mathbf{G} = \overline{\mathbb{Q}}$ and $\Gamma(\mathbb{Q}) \cap \mathrm{Frac}\, \mathbf{G} = \mathbb{Q}$. It is also an open question to prove that the number $\Gamma^{(n)}(s)$ does not have E-approximations, for $n \geq 1$ and $s \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$. To obtain approximations to these numbers, one can consider the following generalization of Definition 4: we replace $A(z) \cdot E(B(z))$ (and also $C(z) \cdot F(D(z))$) with a finite sum

$$\sum_{i,j,k,\ell} \alpha_{i,j,k,\ell} \log(1 - A_i(z))^j \cdot B_k(z) \cdot E_\ell\big(C(z)\big) \tag{1.8}$$

where $\alpha_{i,j,k,\ell} \in \overline{\mathbb{Q}}$, $A_i(z), B_k(z), C(z)$ are algebraic functions in $\overline{\mathbb{Q}}[[z]]$, $A_i(0) = C(0) = 0$, and $E_\ell(z)$ are E-functions. For instance, let us consider the E-function $E(z) = \sum_{n=1}^{\infty} \frac{z^n}{n!n}$ and define P_n by the series expansion (for $|z| < 1$)

$$\frac{\log(1 - z)}{1 - z} - \frac{1}{1 - z}E\left(-\frac{z}{1 - z}\right) = \sum_{n=0}^{\infty} P_n z^n \in \mathbb{Q}[[z]]. \tag{1.9}$$

Then we prove in §6.4 that $\lim_n P_n = \gamma$, so that letting $Q_n = 1$ we obtain E-approximations of Euler's constant in this extended sense. Since $\frac{\log(1-z)}{1-z} - \frac{1}{1-z}E\left(-\frac{z}{1-z}\right)$ is holonomic, the sequence $(P_n)_n$ satisfies a linear recurrence, of order 3 with polynomial coefficients in $\mathbb{Z}[n]$ of degree 2; see §6.4. Again, this construction is new and much simpler than those in [4, 15, 20] but the convergence to γ is slower. A construction similar to (1.9), based on an immediate generalization of the final equation for $\Gamma^{(n)}(1)$ in [22], shows that the numbers $\Gamma^{(n)}(s)$ have E-approximations in the extended sense of (1.8) for any integer $n \geq 0$ and any rational number $s \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$.

The set of numbers having such approximations is still countable, and we prove in §6.4 that it is contained in

$$\frac{(\mathbf{E} \cdot \log(\overline{\mathbb{Q}}^*)) \cup \mathbf{S}}{(\mathbf{E} \cdot \log(\overline{\mathbb{Q}}^*)) \cup \mathbf{S}} \cup \left(\exp(\overline{\mathbb{Q}}) \cdot \mathrm{Frac}\, \mathbf{S} \right)$$

where $\log(\overline{\mathbb{Q}}^*) = \exp^{-1}(\overline{\mathbb{Q}}^*)$.

The generalization (1.8) does not cover all interesting constructions of approximations to derivatives of Gamma values in the literature. For instance, it does not seem that Aptekarev's or the second author's approximations to γ (in [4] and [20] respectively) can be described by (1.8). This is also not the case of Hessami-Pilehrood's approximations to $\Gamma^{(n)}(1)$ in [14, 15] but in certain cases their generating functions involve sums of products of E-functions at various algebraic functions, rather linear forms in E-functions at one algebraic function as in (1.8). Another possible generalization of (1.8) is to let $\alpha_{i,j,k,\ell} \in \mathbf{E}$; we describe such an example in §6.4, related to the continued fraction $[0; 1, 2, 3, 4, \ldots]$ whose partial quotients are the consecutive positive integers.

The structure of this paper is as follows. In §2, we discuss the properties of $\mathbf{S}$. In §3 we prove our results at finite distance, namely Theorem 1 and Proposition 1. Then we discuss in §4.1 the definition and basic properties of asymptotic expansions. This allows us to prove Theorem 2 in §4, and to determine in §5 the asymptotic behavior of sequences (P_n) as in Definition 4. Finally, we gather in §6 all results related to E-approximations.

2   Structure of S

In this short section, we discuss the structural properties of the $\mathbf{G}$-module $\mathbf{S}$ generated by the numbers $\Gamma^{(n)}(s)$, for $n \geq 0$, $s \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$. It is not used in the proof of our theorems.

The Digamma function Ψ is defined as the logarithmic derivative of the Gamma function. We have

$$\Psi(x) = -\gamma + \sum_{k=0}^{\infty} \left(\frac{1}{k+1} - \frac{1}{k+x} \right) \quad \text{and} \quad \Psi^{(n)}(x) = \sum_{k=0}^{\infty} \frac{(-1)^{n+1} n!}{(k+x)^{n+1}} \quad (n \geq 1).$$

From the relation $\Gamma'(x) = \Psi(x)\Gamma(x)$, we can prove by induction on the integer $n \geq 0$ that

$$\Gamma^{(n)}(x) = \Gamma(x) \cdot P_n\big(\Psi(x), \Psi^{(1)}(x), \ldots, \Psi^{(n-1)}(x)\big)$$

where $P_n(X_1, X_2, \ldots, X_n)$ is a polynomial with integer coefficients. Moreover, the term of maximal degree in X_1 is X_1^n.

It is well-known that $\Psi(s) \in -\gamma + \mathbf{G}$ (Gauss' formula, [3, p. 13, Theorem 1.2.7]) and that $\Psi^{(n)}(s) \in \mathbf{G}$ for any $n \geq 1$ and any $s \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$. It follows that

$$\Gamma^{(n)}(s) = \Gamma(s) \cdot P_n\big(\Psi(s), \Psi^{(1)}(s), \ldots, \Psi^{(n-1)}(s)\big) = \Gamma(s) \cdot Q_{n,s}(\gamma) \qquad (2.1)$$

where $Q_{n,s}(X)$ is a polynomial with coefficients in $\mathbf{G}$, of degree n and leading coefficient equal to $(-1)^n$.

Proposition 2. *The set $\mathbf{S}$ coincides with the $\mathbf{G}[\gamma]$-module $\widehat{\mathbf{S}}$ generated by the numbers $\Gamma(s)$, for $s \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$. It is a ring.*

Proof. Eq. (2.1) shows immediately that $\mathbf{S} \subset \widehat{\mathbf{S}}$. For the converse inclusion $\widehat{\mathbf{S}} \subset \mathbf{S}$, it is enough to show that $\Gamma(s)\gamma^n \in \mathbf{S}$ for any $n \geq 0$, $s \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$. This can be proved by induction on n from (2.1) because we can rewrite it as

$$\Gamma(s)\gamma^n = (-1)^n \Gamma^{(n)}(s) + \Gamma(s) \cdot \widehat{Q}_{n,s}(\gamma)$$

for some polynomial $\widehat{Q}_{n,s}(X)$ with coefficients in $\mathbf{G}$ and degree $\leq n - 1$.

Let us now prove that $\widehat{\mathbf{S}}$ is a ring. For any $x, y \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ such that $x + y \notin \mathbb{Z}_{\leq 0}$, we have $\Gamma(x)\Gamma(y) = \Gamma(x + y)B(x, y) \in \widehat{\mathbf{S}}$ because $B(x, y) \in \mathbf{G}$ in this case (see [11]). If $x, y \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ but $x + y \in \mathbb{Z}_{\leq 0}$, then by the reflection formula $\Gamma(x)\Gamma(y) \in \pi\overline{\mathbb{Q}} \subset \widehat{\mathbf{S}}$. $\qquad\square$

Remark. The fact that $\mathbf{S}$ is a ring can also be proved directly from the definition of $\mathbf{S}$. For any $x, y \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ such that $x + y \notin \mathbb{Z}_{\leq 0}$, we have

$$
\begin{aligned}
\Gamma^{(m)}(x)\Gamma^{(n)}(y) &= \frac{\partial^{m+n}}{\partial x^m \partial y^n}\Gamma(x+y)B(x,y) \\
&= \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}\Gamma^{(i+j)}(x+y)\frac{\partial^{m+n-i-j}}{\partial x^{m-i}\partial y^{n-j}}B(x,y) \in \mathbf{S}
\end{aligned}
$$

because $\frac{\partial^{m+n-i-j}}{\partial x^{m-i}\partial y^{n-j}}B(x,y) \in \mathbf{G}$, arguing as in [11] for the special case $m - i = n - j = 0$. If $x, y \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ and $x + y \in \mathbb{Z}_{\leq 0}$, we argue as above using the reflection formula.

3 First results on values of E-functions

3.1 Around Siegel-Shidlovskii and Beukers' theorems

To begin with, let us mention the following result. It is proved in [11] (and due to the referee of that paper) in the case $\mathbb{K} = \mathbb{Q}(i)$; actually the same proof, which relies on Beukers' version [6] of the Siegel-Shidlovskii theorem, works for any number field $\mathbb{K}$.

Theorem 4. *Let $E(z)$ be an E-function with coefficients in some number field $\mathbb{K}$, and $\alpha, \beta \in \overline{\mathbb{Q}}$ be such that $E(\alpha) = \beta$ or $E(\alpha) = e^\beta$. Then $\beta \in \mathbb{K}(\alpha)$.*

This result implies Theorem 1 stated in the introduction; without further hypotheses $E(\alpha)$ may really belong to $\mathbb{K}(\alpha)$, because if $E(z)$ is an E-function then so is $(z - \alpha)E(z)$.

Theorem 4 shows that if we restrict the coefficients of E-functions to a given number field then the set of values we obtain is a proper subset of $\mathbf{E}$. In this respect the situation is completely different from the one with G-functions, since any element of $\mathbf{G}$ can be written [11] as $f(1)$ for some G-function f with Taylor coefficients in $\mathbb{Q}(i)$. This is also the reason why we did not restrict to rational numbers P_n, Q_n in Definition 4.

3.2 Connection constants at finite distance

Let us prove Proposition 1 stated in the introduction, which we state again here in a slightly more general version; the strategy is analogous to the corresponding one with G-functions [11], and even easier because E-functions are entire.

Proposition 3. *Let*

$$F(z) = \sum_{j=1}^{\mu} \left(\sum_{s \in S_j} \sum_{k \in K_j} \phi_{j,s,k} z^s \log(z)^k \right) E_j(z) \tag{3.1}$$

where $S_j \subset \mathbb{Q}, K_j \subset \mathbb{N}$ are finite sets, $\phi_{j,s,k} \in \overline{\mathbb{Q}}$ and $E_1, \ldots, E_\mu$ are E-functions. Let $\alpha \in \overline{\mathbb{Q}} \setminus \{0\}$, and $G_1(z), \ldots, G_\mu(z) \in \overline{\mathbb{Q}}[[z - \alpha]]$ be a local basis of solutions of an E-operator L such that $LF = 0$. Let $\omega_1, \ldots, \omega_\mu \in \mathbb{C}$ be such that around $z = \alpha$,

$$F(z) = \omega_1 G_1(z) + \cdots + \omega_\mu G_\mu(z). \tag{3.2}$$

Then $\omega_1, \ldots, \omega_\mu \in \mathbf{E}[\log \alpha]$; moreover if $F(z)$ is an E-function then $\omega_1, \ldots, \omega_\mu \in \mathbf{E}$.

Proof. We denote by $W_G(z)$ the wronskian built on the functions $G_1(z), \ldots, G_\mu(z)$:

$$W_G(z) = \begin{vmatrix} G_1(z) & \cdots & G_\mu(z) \\ G_1^{(1)}(z) & \cdots & G_\mu^{(1)}(z) \\ \vdots & \cdots & \vdots \\ G_1^{(\mu-1)}(z) & \cdots & G_\mu^{(\mu-1)}(z) \end{vmatrix}.$$

All functions $G_j^{(k)}(z)$ are holomorphic at $z = \alpha$ with Taylor coefficients in $\overline{\mathbb{Q}}$, so that $W_G(\alpha) \in \overline{\mathbb{Q}}$. On the other hand, let us write

$$L = \frac{d^\mu}{dz^\mu} + a_{\mu-1}(z)\frac{d^{\mu-1}}{dz^{\mu-1}} + \cdots + a_1(z)\frac{d}{dz} + a_0(z),$$

where $a_j \in \overline{\mathbb{Q}}(z)$. Then $z = 0$ is the only singularity at finite distance of L, and it is a regular singularity with rational exponents (see [1]): we have $z^i a_{\mu-i}(z) \in \overline{\mathbb{Q}}[z]$ for any i. Since $W_G(z)$ is a solution of the differential equation $y'(z) + a_{\mu-1}(z)y(z) = 0$, it is of the form $W(z) = cz^\rho e^{q(z)}$ with $c \in \mathbb{C}$, $\rho \in \mathbb{Q}$ and $q(z) \in \overline{\mathbb{Q}}[z]$ (in fact, q has degree ≤ 1 here). Moreover the G_j's form a basis of solutions of L, so that $c \neq 0$ and $W_G(\alpha) \in \overline{\mathbb{Q}} \setminus \{0\}$.

We now differentiate (3.2) to obtain the relations

$$F^{(k)}(z) = \sum_{j=1}^{\mu} \omega_j G_j^{(k)}(z), \quad k = 0, \ldots, \mu - 1$$

for any z in some open disk $\mathcal{D}$ centered at $z = \alpha$. We interpret these equations (with $z = \alpha$) as a linear system with unknowns ω_j, and solve it using Cramer's rule. We obtain

in this way that

$$
\omega_j = \frac{1}{W_G(\alpha)}
\begin{vmatrix}
G_1(\alpha) & \cdots & G_{j-1}(\alpha) & F(\alpha) & G_{j+1}(\alpha) & \cdots & G_\mu(\alpha) \\
G_1^{(1)}(\alpha) & \cdots & G_{j-1}^{(1)}(\alpha) & F^{(1)}(\alpha) & G_{j+1}^{(1)}(\alpha) & \cdots & G_\mu^{(1)}(\alpha) \\
\vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
G_1^{(\mu-1)}(\alpha) & \cdots & G_{j-1}^{(\mu-1)}(\alpha) & F^{(\mu-1)}(\alpha) & G_{j+1}^{(\mu-1)}(\alpha) & \cdots & G_\mu^{(\mu-1)}(\alpha)
\end{vmatrix}
\tag{3.3}
$$

since $W_G(\alpha) \neq 0$.

Now recall that $1/W_G(\alpha)$ and $G_j^{(k)}(\alpha)$ belong to $\overline{\mathbb{Q}} \subset \mathbf{E}$. If we assume that $F(z)$ is an E-function, this is also the case of its derivatives, so that $F^{(k)}(\alpha) \in \mathbf{E}$ for all $k \geq 0$ and (3.3) implies that $\omega_j \in \mathbf{E}$. To prove the general case, we simply observe that if $F(z)$ is given by (3.1) with algebraic coefficients $\phi_{j,s,k}$ then all values at $z = \alpha$ of derivatives of $F(z)$ belong to $\mathbf{E}[\log(\alpha)]$. $\qquad\square$

4 Stokes constants of E-functions

In this section we construct explicitly the asymptotic expansion of an E-function: our main result is Theorem 5, stated in §4.2 and proved in §4.3. Before that we discuss in §4.1 the asymptotic expansions used in this paper. Finally we show in §4.4 that Theorem 5 implies Theorem 2.

Throughout this section, we let $\widehat{\Gamma} := 1/\Gamma$ for simplicity.

4.1 Asymptotic expansions

The asymptotic expansions used throughout this paper are defined as follows.

Definition 5. *Let $\theta \in \mathbb{R}$, and $\Sigma \subset \mathbb{C}$, $S \subset \mathbb{Q}$, $T \subset \mathbb{N}$ be finite subsets. Given complex numbers $c_{\rho,\alpha,i,n}$, we write*

$$
f(x) \approx \sum_{\rho \in \Sigma} e^{\rho x} \sum_{\alpha \in S} \sum_{i \in T} \sum_{n=0}^{\infty} c_{\rho,\alpha,i,n} x^{-n-\alpha} (\log(1/x))^i
\tag{4.1}
$$

and say that the right hand side is the asymptotic expansion of $f(x)$ in a large sector bisected by the direction θ, if there exist $\varepsilon, R, B, C > 0$ and, for any $\rho \in \Sigma$, a function $f_\rho(x)$ holomorphic on

$$
U = \left\{ x \in \mathbb{C}, \ |x| \geq R, \ \theta - \frac{\pi}{2} - \varepsilon \leq \arg(x) \leq \theta + \frac{\pi}{2} + \varepsilon \right\},
$$

such that

$$
f(x) = \sum_{\rho \in \Sigma} e^{\rho x} f_\rho(x)
$$

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and

$$\left| f_\rho(x) - \sum_{\alpha \in S} \sum_{i \in T} \sum_{n=0}^{N-1} c_{\rho,\alpha,i,n} x^{-n-\alpha} (\log(1/x))^i \right| \leq C^N N! |x|^{B-N}$$

for any $x \in U$ and any $N \geq 1$.

This means exactly (see [19, §§2.1 and 2.3]) that for any $\rho \in \Sigma$,

$$\sum_{\alpha \in S} \sum_{i \in T} \sum_{n=0}^{N-1} c_{\rho,\alpha,i,n} x^{-n-\alpha} (\log(1/x))^i \tag{4.2}$$

is 1-summable in the direction θ and its sum is $f_\rho(x)$. In particular, using a result of Watson (see [19, §2.3]), the sum $f_\rho(x)$ is determined by its expansion (4.2). Therefore the asymptotic expansion on the right hand side of (4.1) determines the function $f(x)$ (up to analytic continuation). The converse is also true, as the following lemma shows.

Lemma 1. *A given function $f(x)$ can have at most one asymptotic expansion in the sense of Definition 5.*

Of course we assume implicitly in Lemma 1 (and very often in this paper) that Σ, S and T in (4.1) cannot trivially be made smaller, and that for any α there exist ρ and i with $c_{\rho,\alpha,i,0} \neq 0$.

Proof. We proceed by induction on the cardinality of Σ. If the result holds for proper subsets of Σ, we choose θ' very close to θ such that the complex numbers $\rho e^{i\theta'}$, $\rho \in \Sigma$, have pairwise distinct real parts and we denote by ρ_0 the element of Σ for which $\mathrm{Re}\,(\rho_0 e^{i\theta'})$ is maximal. Then the asymptotic expansion (4.2) of $f_{\rho_0}(x)$ is also an asymptotic expansion of $e^{-\rho_0 x} f(x)$ as $|x| \to \infty$ with $\arg(x) = \theta'$, in the usual sense (see for instance [9, p. 182]); accordingly it is uniquely determined by f, so that its 1-sum $f_{\rho_0}(x)$ is also uniquely determined by f. Applying the induction procedure to $f(x) - e^{\rho_0 x} f_{\rho_0}(x)$ with $\Sigma \setminus \{\rho_0\}$ concludes the proof of Lemma 1. $\qquad \square$

4.2 Notation and statement of Theorem 5

We consider a non-polynomial E-function $E(x)$ such that $E(0) = 0$, and write

$$E(x) = \sum_{n=1}^{\infty} \frac{a_n}{n!} x^n.$$

Its associated G-function is

$$G(z) = \sum_{n=1}^{\infty} a_n z^n.$$

We denote by $\mathcal{D}$ a G-operator such that $\overline{\mathcal{F}} \mathcal{D} E = 0$, where $\overline{\mathcal{F}} : \mathbb{C}[z, \frac{\mathrm{d}}{\mathrm{d}z}] \to \mathbb{C}[x, \frac{\mathrm{d}}{\mathrm{d}x}]$ is the Fourier transform of differential operators, i.e. the morphism of $\mathbb{C}$-algebras defined by

$\overline{\mathcal{F}}(z) = \frac{\mathrm{d}}{\mathrm{d}x}$ and $\overline{\mathcal{F}}(\frac{\mathrm{d}}{\mathrm{d}z}) = -x$. Recall that such a $\mathcal{D}$ exists because E is annihilated by an E-operator, and any E-operator can be written as $\overline{\mathcal{F}}\mathcal{D}$ for some G-operator $\mathcal{D}$.

We let $g(z) = \frac{1}{z}G(\frac{1}{z})$, so that $(\frac{\mathrm{d}}{\mathrm{d}z})^{\delta}\mathcal{D}g = 0$ where δ is the degree of $\mathcal{D}$ (i.e. the order of $\overline{\mathcal{F}}\mathcal{D}$; see [1], p. 716). This function is the Laplace transform of $E(x)$: for $\mathrm{Re}\,(z) > C$, where $C > 0$ is such that $|a_n| \ll C^n$, we have

$$g(z) = \int_0^{\infty} E(x)e^{-xz}\mathrm{d}x.$$

From the definition of $g(z)$ and the assumption $E(0) = 0$ we deduce that $g(z) = \mathcal{O}(1/|z|^2)$ as $z \to \infty$.

We denote by Σ the set of all finite singularities ρ of $\mathcal{D} = \sum_{j=0}^{d} u_j(z)(\frac{\mathrm{d}}{\mathrm{d}z})^j$, i.e. the zeros of the leading polynomial $u_d(z)$. Observe that $(\frac{\mathrm{d}}{\mathrm{d}z})^{\delta}\mathcal{D}$ has the same singularities as $\mathcal{D}$. We also let

$$\mathcal{S} = \mathbb{R} \setminus \{\arg(\rho - \rho'), \rho, \rho' \in \Sigma, \rho \neq \rho'\} \tag{4.3}$$

where all the values modulo 2π of the argument of $\rho - \rho'$ are considered, so that $\mathcal{S} + \pi = \mathcal{S}$.

The directions $\theta \in \mathbb{R} \setminus (-\mathcal{S})$ (i.e., such that $(\rho - \rho')e^{i\theta}$ is real for some $\rho \neq \rho'$ in Σ) may be *anti-Stokes* (or *singular*, see for instance [18, p. 79]): when crossing such a direction, the renormalized sum of a formal solution at infinity of $\mathcal{D}$ may change. In the statement and proof of Theorem 5 we fix a direction $\theta \in -\mathcal{S}$.

For any $\rho \in \Sigma$ we denote by $\Delta_{\rho} = \rho - e^{-i\theta}\mathbb{R}_+$ the half-line of angle $-\theta + \pi \mod 2\pi$ starting at ρ. Since $-\theta \in \mathcal{S}$, no singularity $\rho' \neq \rho$ of $\mathcal{D}$ lies on Δ_{ρ}: these half-lines are pairwise disjoint. We shall work in the simply connected cut plane obtained from $\mathbb{C}$ by removing the union of these closed half-lines. We agree that for $\rho \in \Sigma$ and z in the cut plane, $\arg(z - \rho)$ will be chosen in the open interval $(-\theta - \pi, -\theta + \pi)$. This enables one to define $\log(z - \rho)$ and $(z - \rho)^{\alpha}$ for any $\alpha \in \mathbb{Q}$.

Now let us fix $\rho \in \Sigma$. Combining theorems of André, Chudnovski and Katz (see [1, p. 719]), there exist (non necessarily distinct) rational numbers $t_1^{\rho}, \ldots, t_{J(\rho)}^{\rho}$, with $J(\rho) \geq 1$, and G-functions $g_{j,k}^{\rho}$, for $1 \leq j \leq J(\rho)$ and $0 \leq k \leq K(\rho, j)$, such that a basis of local solutions of $(\frac{\mathrm{d}}{\mathrm{d}z})^{\delta}\mathcal{D}$ around ρ (in the above-mentioned cut plane) is given by the functions

$$f_{j,k}^{\rho}(z - \rho) = (z - \rho)^{t_j^{\rho}} \sum_{k'=0}^{k} g_{j,k-k'}^{\rho}(z - \rho)\frac{(\log(z - \rho))^{k'}}{k'!} \tag{4.4}$$

for $1 \leq j \leq J(\rho)$ and $0 \leq k \leq K(\rho, j)$. Since $(\frac{\mathrm{d}}{\mathrm{d}z})^{\delta}\mathcal{D}g = 0$ we can expand g in this basis:

$$g(z) = \sum_{j=1}^{J(\rho)} \sum_{k=0}^{K(\rho,j)} \varpi_{j,k}^{\rho} f_{j,k}^{\rho}(z - \rho) \tag{4.5}$$

with connection constants $\varpi_{j,k}^{\rho}$; Theorem 2 of [11] yields $\varpi_{j,k}^{\rho} \in \mathbf{G}$.

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We denote by $\{u\} \in [0,1)$ the fractional part of a real number u, and agree that all derivatives of this or related functions taken at integers will be right-derivatives. We also denote by $\star$ the Hadamard (coefficientwise) product of formal power series in z, and we let

$$y_{\alpha,i}(z) = \sum_{n=0}^{\infty} \frac{1}{i!} \frac{d^i}{dy^i} \left(\frac{\Gamma(1-\{y\})}{\Gamma(-y-n)} \right)_{|y=\alpha} z^n \in \mathbb{Q}[[z]]$$

for $\alpha \in \mathbb{Q}$ and $i \in \mathbb{N}$. To compute the coefficients of $y_{\alpha,i}(z)$, we may restrict to values of y with the same integer part as α, denoted by $\lfloor \alpha \rfloor$. Then

$$\frac{\Gamma(1-\{y\})}{\Gamma(-y-n)} = \frac{\Gamma(-y+\lfloor\alpha\rfloor+1)}{\Gamma(-y-n)} = \begin{cases} (-y-n)_{n+\lfloor\alpha\rfloor+1} & \text{if } n \geq -\lfloor\alpha\rfloor \\[2mm] \frac{1}{(-y+\lfloor\alpha\rfloor+1)_{-n-\lfloor\alpha\rfloor-1}} & \text{if } n \leq -1-\lfloor\alpha\rfloor \end{cases} \tag{4.6}$$

is a rational function of y with rational coefficients, so that $y_{\alpha,i}(z) \in \mathbb{Q}[[z]]$; here $(x)_k = x(x+1)\dots(x+k-1)$ is Pochhammer's symbol. Even though this won't be used in the present paper, we mention that $y_{\alpha,i}(z)$ is an arithmetic Gevrey series of order 1 (see [1]); in particular it is divergent for any $z \neq 0$ (unless it is a polynomial, namely if $i = 0$ and $\alpha \in \mathbb{Z}$).

Finally, we define

$$\eta_{j,k}^{\rho}(1/x) = \sum_{m=0}^{k} (y_{t_j^{\rho},m} \star g_{j,k-m}^{\rho})(1/x) \in \overline{\mathbb{Q}}[[1/x]]$$

for any $1 \leq j \leq J(\rho)$ and $0 \leq k \leq K(j,\rho)$; this is also an arithmetic Gevrey series of order 1. It is not difficult to see that $\eta_{j,k}^{\rho}(1/x) = 0$ if $f_{j,k}^{\rho}(z-\rho)$ is holomorphic at ρ. Indeed in this case $k = 0$ and $t_j^{\rho} \in \mathbb{Z}$; if $t_j^{\rho} \geq 0$ then $y_{t_j^{\rho},0}$ is identically zero, and if $t_j^{\rho} \leq -1$ then $y_{t_j^{\rho},0}$ is a polynomial in z of degree $-1-t_j^{\rho}$ whereas $g_{j,0}^{\rho}$ has valuation at least $-t_j^{\rho}$.

The main result of this section is the following asymptotic expansion, valid in the setting of Definition 5 for $\theta \in -\mathcal{S}$. It is at the heart of Theorem 2; recall that we assume here $E(0) = 0$, and that we let $\widehat{\Gamma} = 1/\Gamma$.

Theorem 5. *We have*

$$E(x) \approx \sum_{\rho \in \Sigma} e^{\rho x} \sum_{j=1}^{J(\rho)} \sum_{k=0}^{K(j,\rho)} \varpi_{j,k}^{\rho} x^{-t_j^{\rho}-1} \sum_{i=0}^{k} \left(\sum_{\ell=0}^{k-i} \frac{(-1)^{\ell}}{\ell!} \widehat{\Gamma}^{(\ell)}(1-\{t_j^{\rho}\}) \eta_{j,k-\ell-i}^{\rho}(1/x) \right) \frac{(\log(1/x))^i}{i!}.$$

We observe that the coefficients are naturally expressed in terms of $\widehat{\Gamma}^{(\ell)}$. Let us write Theorem 5 in a slightly different way. For $t \in \mathbb{Q}$ and $s \in \mathbb{N}$, let

$$\lambda_{t,s}(1/x) = \sum_{\nu=0}^{s} \frac{(-1)^{s-\nu}}{(s-\nu)!} \widehat{\Gamma}^{(s-\nu)}(1-\{t\}) \frac{(\log(1/x))^{\nu}}{\nu!}.$$

In particular, $\lambda_{t,0}(1/x) = \widehat{\Gamma}(1 - \{t\})$ and $\lambda_{t,1}(1/x) = \widehat{\Gamma}(1 - \{t\})\log(1/x) - \widehat{\Gamma}^{(1)}(1 - \{t\})$; for $t \in \mathbb{Z}$ we have $\lambda_{t,1}(1/x) = \log(1/x) - \gamma$.

Then Theorem 5 reads (by letting $s = i + \ell$):

$$E(x) \approx \sum_{\rho \in \Sigma} e^{\rho x} \sum_{j=1}^{J(\rho)} \sum_{k=0}^{K(j,\rho)} \varpi_{j,k}^{\rho} x^{-t_j^{\rho}-1} \sum_{s=0}^{k} \lambda_{t_j^{\rho},s}(1/x) \eta_{j,k-s}^{\rho}(1/x). \tag{4.7}$$

Here we see that the derivatives of $1/\Gamma$ do not appear in an arbitrary way, but always through these sums $\lambda_{t,s}(1/x)$. In particular γ appears through $\lambda_{t,1}(1/x) = \log(1/x) - \gamma$, as mentioned in the introduction.

In the asymptotic expansion of Theorem 5, and in (4.7), the singularities $\rho \in \Sigma$ at which $g(z)$ is holomorphic have a zero contribution because for any (j,k), either $\varpi_{j,k}^{\rho} = 0$ or $f_{j,k}^{\rho}(z - \rho)$ is holomorphic at ρ (and in the latter case, $k = 0$ and $\eta_{j,0}^{\rho}(1/x) = 0$, as mentioned before the statement of Theorem 5). Moreover, as the proof shows (see §4.3), it is not really necessary to assume that the functions $f_{j,k}^{\rho}(z - \rho)$ form a basis of local solutions of $(\frac{d}{dz})^{\delta}\mathcal{D}$ around ρ. Instead, it is enough to consider rational numbers t_j^{ρ} and G-functions $g_{j,k}^{\rho}$ such that all singularities of $g_{j,k}^{\rho}(z - \rho)$ belong to Σ and, upon defining $f_{j,k}^{\rho}$ by Eq. (4.4), Eq. (4.5) holds with some complex numbers $\varpi_{j,k}^{\rho}$. In this way, to compute the asymptotic expansion of $E(x)$ it is not necessary to determine $\mathcal{D}$ explicitly. The finite set Σ is used simply to control the singularities of the functions which appear, and prevent θ from being a possibly singular direction. This remark makes it easier to apply Theorem 5 to specific E-functions, for instance to obtain the expansions (6.2) and (6.5) used in §6.

4.3 Proof of Theorem 5

We fix an oriented line d such that the angle between $\mathbb{R}_+$ and d is equal to $-\theta + \frac{\pi}{2} \bmod 2\pi$, and all singularities of $\mathcal{D}$ lie on the left of d. Let $R > 0$ be sufficiently large (in terms of d and Σ). Then the circle $\mathcal{C}(0,R)$ centered at 0 of radius R intersects d at two distinct points a and b, with $\arg(b - a) = -\theta + \frac{\pi}{2} \bmod 2\pi$, and

$$E(x) = \lim_{R \to \infty} \frac{1}{2i\pi} \int_a^b g(z)e^{zx}\mathrm{d}z \tag{4.8}$$

where the integral is taken along the line segment ab contained in d.

For any $\rho \in \Sigma$ the circle $\mathcal{C}(0,R)$ intersects Δ_ρ at one point $z_\rho = \rho - A_\rho e^{-i\theta}$, with $A_\rho > 0$, which corresponds to two points at the border of the cut plane, namely $\rho + A_\rho e^{i(-\theta \pm \pi)}$ with values $-\theta \pm \pi$ of the argument. We consider the following path $\Gamma_{\rho,R}$: a straight line from $\rho + A_\rho e^{i(-\theta-\pi)}$ to ρ (on one bank of the cut plane), then a circle around ρ with essentially zero radius and $\arg(z - \rho)$ going up from $-\theta - \pi$ to $-\theta + \pi$, and finally a straight line from ρ to $\rho + A_\rho e^{i(-\theta+\pi)}$ on the other bank of the cut plane. We denote by Γ_R the closed loop obtained by concatenation of the line segment ba, the arc az_{ρ_1} of the circle $\mathcal{C}(0,R)$, the path $\Gamma_{\rho_1,R}$, the arc $z_{\rho_1}z_{\rho_2}$, the path $\Gamma_{\rho_2,R}$, ..., and the arc $z_{\rho_p}b$ (where $\rho_1, \ldots, \rho_p$ are the distinct elements of Σ, ordered so that $z_{\rho_1}, z_{\rho_2}, \ldots, z_{\rho_p}$ are met successively when
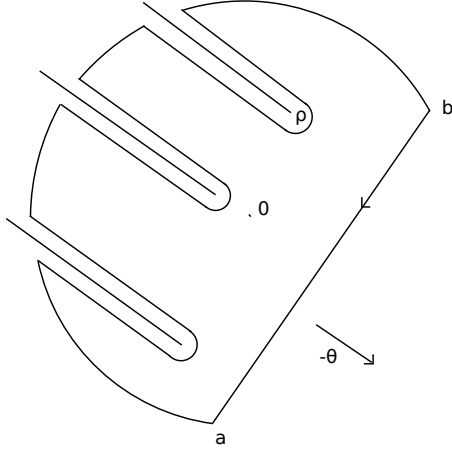
Figure 1: The contour Γ_R

going along $\mathcal{C}(0, R)$ from a to b in the negative direction); see Figure 1. We refer to [9, pp. 183–192] for a similar computation.

We observe that

$$\frac{1}{2i\pi} \int_{\Gamma_R} g(z)e^{zx}\mathrm{d}z = 0$$

for any $x \in \mathbb{C}$, because Γ_R is a closed simple curve inside which the integrand has no singularity.

Now assume that $\theta - \frac{\pi}{2} < \arg(x) < \theta + \frac{\pi}{2}$. As $R \to \infty$, the integral of $g(z)e^{zx}$ over the line segment ba tends to $-E(x)$, using Eq. (4.8). Moreover, as z describes $\Gamma_{\rho,R}$ (except maybe in a bounded neighborhood of ρ) we have $\mathrm{Re}\,(zx) < 0$ and $g(z) = \mathcal{O}(1/|z^2|)$, so that letting $R \to \infty$ one obtains (as in [9])

$$E(x) = \sum_{\rho \in \Sigma} \frac{1}{2i\pi} \int_{\Gamma_\rho} g(z)e^{zx}\mathrm{d}z, \tag{4.9}$$

where Γ_ρ is the extension of $\Gamma_{\rho,R}$ as $R \to \infty$.

Plugging Eq. (4.5) into Eq. (4.9) yields

$$E(x) = \sum_{\rho \in \Sigma}\sum_{j=1}^{J(\rho)}\sum_{k=0}^{K(j,\rho)} \varpi_{j,k}^\rho \frac{1}{2i\pi} \int_{\Gamma_\rho} f_{j,k}^\rho(z - \rho)e^{zx}\mathrm{d}z. \tag{4.10}$$

To study the integrals on the right hand side we shall prove the following general claim (see [12, §2.5]). *Let $\rho \in \Sigma$, and φ be a G-function such that $\varphi(z - \rho)$ is holomorphic on the cut plane. For any $\alpha \in \mathbb{Q}$ and any $k \in \mathbb{N}$, let*

$$\varphi_{\alpha,k}(z - \rho) = \varphi(z - \rho)(z - \rho)^\alpha \frac{(\log(z - \rho))^k}{k!}.$$

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Then

$$\frac{1}{2i\pi}\int_{\Gamma_\rho}\varphi_{\alpha,k}(z-\rho)e^{zx}\mathrm{d}z$$

admits the following asymptotic expansion in a large sector bisected by θ (with $\widehat{\Gamma}:=1/\Gamma$):

$$e^{\rho x}x^{-\alpha-1}\sum_{\ell=0}^{k}\frac{(-1)^\ell}{\ell!}\widehat{\Gamma}^{(\ell)}(1-\{\alpha\})\sum_{i=0}^{k-\ell}\Big(y_{\alpha,k-\ell-i}\star\varphi\Big)(1/x)\frac{(\log(1/x))^i}{i!}.$$

To prove this claim, we first observe that

$$\int_{\Gamma_\rho}\varphi_{\alpha,k}(z-\rho)e^{zx}\mathrm{d}z=\frac{1}{k!}\frac{\partial^k}{\partial\alpha^k}\Big[\int_{\Gamma_\rho}\varphi_{\alpha,0}(z-\rho)e^{zx}\mathrm{d}z\Big]$$

where the k-th derivative is taken at α; this relation enables us to deduce the general case from the special case $k=0$ considered in [9]. We write also

$$\varphi(z-\rho)=\sum_{n=0}^{\infty}c_n(z-\rho)^n.$$

Following [9, pp. 185-191], given $\varepsilon>0$ we obtain $R,C,\kappa>0$ such that, for any $n\geq1$ and any x with $|x|\geq R$ and $\theta-\frac{\pi}{2}+\varepsilon<\arg(x)<\theta+\frac{\pi}{2}-\varepsilon$, we have

$$\left|\frac{x^{-\alpha-n-1}}{\Gamma(-\alpha-n)}-\frac{1}{2i\pi}e^{-\rho x}\int_{\Gamma_\rho}(z-\rho)^{\alpha+n}e^{zx}\mathrm{d}z\right|\leq C^n n!|x|^{-\alpha-n-1}e^{-\kappa|x|\sin(\varepsilon)}.$$

Then following the proof of [9, pp. 191-192] and using the fact that $\limsup|c_n|^{1/n}<\infty$, for any $\varepsilon>0$ we obtain $R,B,C>0$ such that, for any $N\geq1$ and any x with $|x|\geq R$ and $\theta-\frac{\pi}{2}+\varepsilon<\arg(x)<\theta+\frac{\pi}{2}-\varepsilon$, we have

$$\left|e^{-\rho x}\frac{1}{2i\pi}\int_{\Gamma_\rho}\varphi_{\alpha,k}(z-\rho)e^{zx}\mathrm{d}z-\sum_{n=0}^{N-1}\frac{c_n}{k!}\frac{\partial^k}{\partial\alpha^k}\Big[\frac{x^{-\alpha-n-1}}{\Gamma(-\alpha-n)}\Big]\right|\leq C^N N!|x|^{B-N}.\qquad(4.11)$$

Now observe that $\mathcal{S}$ is a union of open intervals, so that θ can be made slightly larger or slightly smaller while remaining in the same open interval. In this process, the cut plane changes but the left handside of (4.11) remains the same (by the residue theorem, since $\varphi(z-\rho)$ is holomorphic on the cut plane). The asymptotic expansion (4.11) remains valid as $|x|\to\infty$ in the new sector $\theta-\frac{\pi}{2}+\varepsilon<\arg(x)<\theta+\frac{\pi}{2}-\varepsilon$, so that finally it is valid in a large sector $\theta-\frac{\pi}{2}-\varepsilon\leq\arg(x)\leq\theta+\frac{\pi}{2}+\varepsilon$ for some $\varepsilon>0$.

Now Leibniz' formula yields the following equality between functions of α:

$$\Big(\frac{x^{-\alpha-n-1}}{\Gamma(-\alpha-n)}\Big)^{(k)}=\sum_{\ell=0}^{k}\sum_{i=0}^{k-\ell}\frac{k!}{\ell!i!(k-\ell-i)!}\big(\widehat{\Gamma}(1-\{\alpha\})\big)^{(\ell)}\Big(\frac{\Gamma(1-\{\alpha\})}{\Gamma(-\alpha-n)}\Big)^{(k-\ell-i)}$$

$$\times(\log(1/x))^i x^{-\alpha-n-1}$$

$$=\sum_{\ell=0}^{k}\frac{k!}{\ell!}\big(\widehat{\Gamma}(1-\{\alpha\})^{(\ell)}\sum_{i=0}^{k-\ell}\Big(y_{\alpha,k-\ell-i}\star z^n\Big)(1/x)x^{-\alpha-1}\frac{(\log(1/x))^i}{i!}$$

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so that

$$\sum_{n=0}^{\infty} \frac{c_n}{k!} \left(\frac{x^{-\alpha-n-1}}{\Gamma(-\alpha-n)} \right)^{(k)} = \sum_{\ell=0}^{k} \frac{1}{\ell!} \left(\widehat{\Gamma}(1-\{\alpha\}) \right)^{(\ell)} \sum_{i=0}^{k-\ell} \left(y_{\alpha,k-\ell-i} \star \varphi \right)(1/x) x^{-\alpha-1} \frac{(\log(1/x))^i}{i!}.$$

Using (4.11) this concludes the proof of the claim.

Now we apply the claim to the G-functions $g_{j,k}^{\rho}$, since all singularities of $g_{j,k}^{\rho}(z-\rho)$ are singularities of $(\frac{\mathrm{d}}{\mathrm{d}z})^{\delta}\mathcal{D}$ and therefore belong to Σ. Combining this result with Eqns. (4.4) and (4.10) yields:

$$E(x) = \sum_{\rho,j,k,k'} \varpi_{j,k}^{\rho} \frac{1}{2i\pi} \int_{\Gamma_\rho} g_{j,k-k'}^{\rho}(z-\rho)(z-\rho)^{t_j^{\rho}} \frac{(\log(z-\rho))^{k'}}{k'!} e^{zx} \mathrm{d}z$$

$$\approx \sum_{\rho,j,k,k'} \varpi_{j,k}^{\rho} e^{\rho x} x^{-t_j^{\rho}-1} \sum_{\ell=0}^{k'} \frac{(-1)^{\ell}}{\ell!} \widehat{\Gamma}^{(\ell)}(1-\{t_j^{\rho}\}) \sum_{i=0}^{k'-\ell} \left(y_{t_j^{\rho},k'-\ell-i} \star g_{j,k-k'}^{\rho} \right)(1/x) \frac{(\log(1/x))^i}{i!}$$

$$= \sum_{\rho,j,k} \varpi_{j,k}^{\rho} e^{\rho x} x^{-t_j^{\rho}-1} \sum_{\ell=0}^{k} \frac{(-1)^{\ell}}{\ell!} \widehat{\Gamma}^{(\ell)}(1-\{t_j^{\rho}\}) \sum_{i=0}^{k-\ell} \eta_{j,k-\ell-i}(1/x) \frac{(\log(1/x))^i}{i!}.$$

This concludes the proof of Theorem 5.

4.4 Proof of Theorem 2

To begin with, let us prove assertions (ii) and (iii). Changing θ slightly if necessary, we may assume $\theta \in -\mathcal{S}$. Adding the constant term $E(0) \in \overline{\mathbb{Q}} \subset \mathbf{G}$ to (1.3) if necessary, we may assume that $E(0) = 0$. Then Theorem 5 applies; moreover, in the setting of §4.2 we may assume that the rational numbers t_j^{ρ} have different integer parts as soon as they are distinct. Then letting S denote the set of all $t_j^{\rho} + 1$, for $\rho \in \Sigma$ and $1 \leq j \leq J(\rho)$, and denoting by T the set of non-negative integers less than or equal to $\max_{j,\rho} K(j,\rho)$, the asymptotic expansion of Theorem 5 is exactly (1.3) with coefficients

$$c_{\rho,\alpha,i,n} = \sum_{\substack{1 \leq j \leq J(\rho) \\ \text{with } \alpha = t_j^{\rho}+1}} \sum_{k=i}^{K(j,\rho)} \varpi_{j,k}^{\rho} \sum_{\ell=0}^{k-i} \frac{(-1)^{\ell}}{\ell!} \widehat{\Gamma}^{(\ell)}(1-\{\alpha\})$$

$$\sum_{m=0}^{k-\ell-i} \frac{1}{m!} \frac{\mathrm{d}^m}{\mathrm{d}y^m} \left(\frac{\Gamma(1-\{y\})}{\Gamma(-y-n)} \right)_{|y=\alpha-1} g_{j,k-\ell-i-m,n}^{\rho}$$

where $g_{j,k-\ell-i-m}^{\rho}(z-\rho) = \sum_{n=0}^{\infty} g_{j,k-\ell-i-m,n}^{\rho}(z-\rho)^n$.

Assertion (ii) of Theorem 2 is $c_{\rho,\alpha,i,n} \in \mathbf{S}$; let us prove this now. The coefficients $g_{j,k-\ell-i-m,n}^{\rho}$ are algebraic because $g_{j,k-\ell-i-m}^{\rho}$ is a G-function, and $\frac{\mathrm{d}^m}{\mathrm{d}y^m} \left(\frac{\Gamma(1-\{y\})}{\Gamma(-y-n)} \right)_{|y=\alpha-1}$ is a rational number. Since $\varpi_{j,k}^{\rho} \in \mathbf{G}$ and $\overline{\mathbb{Q}} \subset \mathbf{G}$, the coefficient $c_{\rho,\alpha,i,n}$ is a $\mathbf{G}$-linear

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combination of derivatives of $\widehat{\Gamma} = 1/\Gamma$ taken at the rational point $1 - \{\alpha\}$. By the reflection formula, $\widehat{\Gamma}(z) = \frac{\sin(\pi z)}{\pi} \Gamma(1-z)$: applying Leibniz' formula we see that $\widehat{\Gamma}^{(k)}(z)$ is a $\mathbf{G}$-linear combination of derivatives of Γ at $1 - z$ up to order k, provided $z \in \mathbb{Q} \setminus \mathbb{Z}$ (using the fact [11] that $\mathbf{G}$ contains π, $1/\pi$, and the algebraic numbers $\sin(\pi z)$ and $\cos(\pi z)$). When $z = 1$, we use (at $x = 0$) the identity

$$\Gamma(x + 1) = \exp\left(-\gamma x + \sum_{k=2}^{\infty} \frac{(-1)^k \zeta(k)}{k} x^k\right)$$

(see [3, p. 3, Theorem 1.1.2]) and the properties of Bell polynomials (see for instance [8, Chap. III, §3]). Since $\zeta(k) \in \mathbf{G}$ for any $k \geq 2$ (because polylogarithms are G-functions), it follows that both $\Gamma^{(k)}(1)$ and $\widehat{\Gamma}^{(k)}(1)$ are polynomials of degree k in Euler's constant γ, with coefficients in $\mathbf{G}$; moreover the leading coefficients of these polynomials are rational numbers. This implies that $\widehat{\Gamma}^{(k)}(1)$ is a $\mathbf{G}$-linear combination of derivatives of Γ at 1 up to order k, and concludes the proof that all coefficients $c_{\rho,\alpha,i,n}$ in the expansion (1.3) provided by Theorem 5 belong to $\mathbf{S}$.

To prove (iii), we fix ρ and α and denote by K the maximal value of $K(j, \rho)$ among integers j such that $\alpha = t_j^\rho + 1$. Then

$$c_{\rho,\alpha,i,n} = \sum_{\ell=0}^{K-i} \frac{(-1)^\ell}{\ell!} \widehat{\Gamma}^{(\ell)}(1 - \{\alpha\}) g'_{\ell+i,n}$$

where

$$g'_{\lambda,n} = \sum_{j} \sum_{k=\lambda}^{K(j,\rho)} \varpi_{j,k}^\rho \sum_{m=0}^{k-\lambda} \frac{1}{m!} \frac{\mathrm{d}^m}{\mathrm{d}y^m}\left(\frac{\Gamma(1 - \{y\})}{\Gamma(-y - n)}\right)_{|y=\alpha-1} g_{j,k-\lambda-m,n}^\rho \in \mathbf{G};$$

here $0 \leq \lambda \leq K$ and the first sum is on $j \in \{1, \ldots, J(\rho)\}$ such that $\alpha = t_j^\rho + 1$ and $K(j, \rho) \geq \lambda$. If n is fixed and $g'_{\lambda,n} \neq 0$ for some λ, then denoting by λ_0 the largest such integer λ we have $c_{\rho,\alpha,\lambda_0,n} \in \widehat{\Gamma}(1 - \{\alpha\}) \cdot \mathbf{G} \setminus \{0\} = \Gamma(\alpha) \cdot \mathbf{G} \setminus \{0\}$ and assertion (iii) follows since λ_0 is the integer denoted by k in (iii).

To prove (i) and (iv), we first observe that if $F(z)$ is given by (1.1) with coefficients $\phi_{j,s,k} \in \mathbf{S}$, the asymptotic expansions of $F_j(z)$ we have just obtained can be multiplied by $\phi_{j,s,k} z^s \log(z)^k$ and summed up, thereby proving (ii) for $F(z)$ since $\mathbf{S}$ is a ring. To deduce (i) from (ii) for any solution $F(z)$ of an E-operator L, we recall that any formal solution f of L at ∞ can be written as (1.3) with complex coefficients $c_{\rho,\alpha,i,n}(f)$, and denote by $\Phi(f)$ the family of all these coefficients. The linear map Φ is injective, so that there exists a finite subset X of the set of indices (ρ, α, i, n) such that $\Psi : f \mapsto (c_{\rho,\alpha,i,n}(f))_{(\rho,\alpha,i,n)\in X}$ is a bijective linear map. Denoting by F_θ the asymptotic expansion of $F(z)$ in a large sector bisected by θ, we have

$$\Psi(F_\theta) = \omega_{1,\theta} \Psi(H_1) + \ldots + \omega_{\mu,\theta} \Psi(H_\mu)$$

with the notation of (1.4). Now $\Psi(H_1)$, ..., $\Psi(H_\mu)$ are linearly independent elements of $\overline{\mathbb{Q}}^X$ and $\omega_{1,\theta}, \ldots, \omega_{\mu,\theta}$ can be obtained by Cramer's rule (this is the same kind of argument as in Section 3.2), so that they are linear combinations of the components of $\Psi(F_\theta)$ with coefficients in $\overline{\mathbb{Q}} \subset \mathbf{G}$: using (ii) this concludes the proof of (i).

5 Asymptotics of the coefficients of $A(z) \cdot E\big(B(z)\big)$

In this section we deduce from Theorem 2 the following result, of independent interest, which is the main step in the proof of Theorem 3 (see §6.3). Its proof decomposes in many cases, some of which involve the saddle point method. In these cases, we do not write down all the details of the derivation which is classical and because this involves lengthy technicalities. Instead, we refer the reader to [26, 27] for detailed asymptotic computations, which we slightly generalize here to get only the leading terms, and to [13, Chapter VIII] for a general overview of this method. The existence of such asymptotics is ensured *a priori* by the Birkhoff-Trjitzinsky theory because all functions $A(z) \cdot E\big(B(z)\big)$ considered in this section are holonomic.

Theorem 6. *Let $E(z)$ be an E-function, and $A(z), B(z) \in \overline{\mathbb{Q}}[[z]]$ be algebraic functions; assume that $P(z) = A(z) \cdot E\big(B(z)\big) = \sum_{n=0}^{\infty} P_n z^n$ is not a polynomial. Then either*

$$P_n = \frac{(2\pi)^{(1-d)/(2d)}}{n!^{1/d}} q^n n^{-u-1} (\log n)^v \Big(\sum_\theta \Gamma(-u_\theta) g_\theta e^{in\theta} + o(1) \Big) \qquad (5.1)$$

or

$$P_n = q^n e^{\sum_{\ell=1}^{d-1} \kappa_\ell n^{\ell/d}} n^{-u-1} (\log n)^v \Big(\sum_{\theta_1,\ldots,\theta_d} \omega_{\theta_1,\ldots,\theta_d} e^{\sum_{\ell=1}^{d} i\theta_\ell n^{\ell/d}} + o(1) \Big) \qquad (5.2)$$

where $q \in \overline{\mathbb{Q}}$, $u \in \mathbb{Q}$, $u_\theta \in \mathbb{Q} \setminus \mathbb{N}$, $d, v \in \mathbb{N}$, $d \geq 1$, $q > 0$, $g_\theta \in \mathbf{G} \setminus \{0\}$, $\kappa_1, \ldots, \kappa_{d-1} \in \mathbb{R}$, $\theta, \theta_1, \ldots, \theta_d \in [-\pi, \pi)$, the sums on θ and $\theta_1, \ldots, \theta_d$ are finite and non-empty, and

$$\begin{cases} \omega_{\theta_1,\ldots,\theta_d} = \frac{\xi}{\Gamma(-u)} \text{ with } \xi \in (\mathbf{E} \cup (\Gamma(\mathbb{Q}) \cdot \mathbf{G})) \setminus \{0\} \\ \qquad\qquad\qquad\qquad \text{if } v = \kappa_1 = \ldots = \kappa_{d-1} = \theta_1 = \ldots = \theta_{d-1} = 0, \qquad (5.3) \\ \omega_{\theta_1,\ldots,\theta_d} \in \Gamma(\mathbb{Q}) \cdot \exp(\overline{\mathbb{Q}}) \cdot \mathbf{G} \setminus \{0\} \text{ otherwise.} \end{cases}$$

As in the introduction, in (5.3) we let $\Gamma(-u) = 1$ if $u \in \mathbb{N}$. In the special case where

$$P(z) = (1-z)^\alpha \exp\Big(\sum_{i=1}^{k} \frac{b_i}{(1-z)^{\alpha_i}} \Big)$$

with $\alpha, \alpha_1, \ldots, \alpha_k \in \mathbb{Q}$, $b_1, \ldots, b_k \in \overline{\mathbb{Q}}$, $\alpha_1 > 0$ and $b_1 \neq 0$, Theorem 6 is consistent with Wright's asymptotic formulas [27] for P_n.

We shall now prove Theorem 6; we distinguish between two cases (see §5.1 and 5.2), which lead to Eqns. (5.1) and (5.2) respectively. This distinction, based on the growth of

P_n, is different from the one mentioned in the introduction (namely whether $E(z)$ plays a role as $z \to z_0 \in \mathbb{C}$ or as $z \to \infty$, providing elements of $\mathbf{E}$ or $\Gamma(\mathbb{Q}) \cdot \mathbf{G}$ respectively). We start with the following consequence of Theorem 2, which is useful to study $E(z)$ as $z \to \infty$, in both §5.1 and §5.2.3.

Lemma 2. *For any E-function $E(z)$ there exist $K \geq 1$, $u_1, \ldots, u_K \in \mathbb{Q}$, $v_1, \ldots, v_K \in \mathbb{N}$, and pairwise distinct $\alpha_1, \ldots, \alpha_K \in \overline{\mathbb{Q}}$ such that*

$$E(z) = \sum_{k=1}^{K} \omega_k e^{\alpha_k z} z^{u_k} \log(z)^{v_k} (1 + o(1)) \tag{5.4}$$

as $|z| \to \infty$, uniformly with respect to $\arg(z)$, where $\omega_k \in \Gamma(-u_k) \cdot \mathbf{G} \setminus \{0\}$ with $\Gamma(-u_k) = 1$ if $u_k \in \mathbb{N}$.

If $K = 1$, the proof below shows that $v_1 = 0$ and $u_1 \in \mathbb{Z}$: $\log(z)$ does not appear in Eq. (5.4). Otherwise for any $k \in \{1, \ldots, K\}$ there exist $k' \neq k$ and $\theta \in \mathbb{R}$ such that $e^{\alpha_k z}$ is much smaller than $e^{\alpha_{k'} z}$ as $|z| \to \infty$ with $\arg(z) = \theta$; we choose a determination of $\log(z)$ with a cut at $\arg(z) = \theta \bmod 2\pi$, and use it in the term corresponding to k in Eq. (5.4). In this way, the cut of $\log(z)$ in Eq. (5.4) never occurs in a leading term.

Proof. For any $\alpha \in \mathbb{C}$, let I_α denote the set of all directions $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ such that $E(z)$ has an asymptotic expansion (1.3) in a large sector bisected by θ, with Σ having the least possible cardinality, $\alpha \in \Sigma$, and $\mathrm{Re}\,(\alpha' e^{i\theta}) \leq \mathrm{Re}\,(\alpha e^{i\theta})$ for any $\alpha' \in \Sigma$. This implies that in the direction θ, the growth of $E(z)$ is comparable to that of $e^{\alpha z}$. Then the closure J_α of I_α is the union of I_α and a set of anti-Stokes directions; it is either empty or of the form $[R_\alpha, S_\alpha] \bmod 2\pi$ with $R_\alpha \leq S_\alpha$. We denote by Σ_0 the set of all $\alpha \in \mathbb{C}$ such that $J_\alpha \neq \emptyset$; then Σ_0 is a subset of the finite set $\Sigma \subset \overline{\mathbb{Q}}$ constructed in §4.2, so that Σ_0 is finite: we denote by $\alpha_1, \ldots, \alpha_K$ its elements, with $K \geq 1$.

If $K = 1$ then $J_{\alpha_1} = \mathbb{R}/2\pi\mathbb{Z}$ and the asymptotic expansion (1.3) is the same in any direction: $e^{-\alpha_1 z} E(z)$ has (at most) a pole at ∞, and Lemma 2 holds with $u_1 \in \mathbb{Z}$, $v_1 = 0$, and $\omega_1 \in \mathbf{G}$ (using Theorem 2).

Let us assume now that $K \geq 2$. Then $S_{\alpha_k} - R_{\alpha_k} \leq \pi$ for any k, so that $E(z)$ admits an asymptotic expansion (1.3) in a large sector that contains all directions $\theta \in J_{\alpha_k}$. Among all terms corresponding to $e^{\alpha_k z}$ in this expansion, we denote the leading one by

$$\omega_k e^{\alpha_k z} z^{u_k} (\log z)^{v_k} \tag{5.5}$$

with $u_k \in \mathbb{Q}$, $v_k \in \mathbb{N}$, and $\omega_k \in \Gamma(-u_k) \cdot \mathbf{G} \setminus \{0\}$ (using assertion (iii) of Theorem 2), where $\Gamma(-u_k)$ is understood as 1 if u_k is a non-negative integer. These parameters are the ones in (5.4). To conclude the proof of Lemma 2, we may assume that $\arg(z)$ remains in a small segment I, and consider the asymptotic expansion (1.3) in a large sector containing I. Keeping only the dominant term corresponding to each $\alpha \in \Sigma$ in this expansion, we obtain

$$E(z) = \sum_{\alpha \in \Sigma} \omega'_\alpha e^{\alpha z} z^{u'_\alpha} (\log z)^{v'_\alpha} (1 + o(1)). \tag{5.6}$$

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To prove that (5.6) is equivalent to (5.4) as $|z| \to \infty$ with $\arg(z) \in I$, we may remove from both equations all terms corresponding to values α_k (resp. $\alpha \in \Sigma$) such that $I_{\alpha_k} \cap I = \emptyset$ (resp. $I_\alpha \cap I = \emptyset$), since they fall into error terms. Now for any $\alpha = \alpha_k$ such that $I_\alpha \cap I \neq \emptyset$, $E(z)$ admits an asymptotic expansion in a large sector containing $I_\alpha \cup I$ (since I_α has length at most π, and the length of I can be assumed to be sufficiently small in terms of E). Comparing the dominating exponential term of this expansion in a direction $\theta \in I_\alpha \cap I$ with the ones of (5.5) and (5.6), we obtain $\omega'_\alpha = \omega_k$, $u'_\alpha = u_k$, and $v'_\alpha = v_k$. This concludes the proof of Lemma 2. $\qquad\square$

5.1 $P(z)$ is an entire function

If $P(z)$ is an entire function then $A(z)$ and $B(z)$ are polynomials; we denote by $\delta \geq 0$ and $d \geq 1$ their degrees, and by A_δ and B_d their leading coefficients. We shall estimate the growth of the Taylor coefficients of $P(z)$ by the saddle point method. For any circle C_R of center 0 and radius R, Lemma 2 yields

$$P_n = \frac{1}{2i\pi} \int_{C_R} \frac{A(z) \cdot E(B(z))}{z^{n+1}} \mathrm{d}z$$

$$= \frac{1}{2i\pi} \sum_{k=1}^{K} \omega_k A_\delta B_d^{u_k} d^{v_k} \int_{C_R} e^{\alpha_k B(z)} \cdot z^{\delta + du_k - n - 1} (\log z)^{v_k} \cdot (1 + o(1)) \mathrm{d}z$$

where the $o(1)$ is with respect to $R \to +\infty$ and is uniform in n; here $\log(z)$ is a fixed determination which depends on k (see the remark after Lemma 2). We have to distinguish between the cases $\alpha_k = 0$ and $\alpha_k \neq 0$. In the former case, the integral

$$\frac{\omega_k}{2i\pi} \int_{C_R} z^{\delta + du_k - n - 1} (\log z)^{v_k} \cdot (1 + o(1)) \mathrm{d}z$$

tends to 0 as $R \to +\infty$ (provided n is sufficiently large) and there is no contribution coming from this case.

Now $E(z)$ is not a polynomial (otherwise $P(z)$ would be a polynomial too), so that if $\alpha_k = 0$ for some k then $K \geq 2$: there is always at least one integer k such that $\alpha_k \neq 0$. For any such k, the function

$$e^{\alpha_k B(z)} z^{\delta + du_k - n - 1} (\log z)^{v_k}$$

is smooth on C_R (except on the cut of $\log z$) and the integral can be estimated as $n \to \infty$ by finding the critical points of $\alpha_k B(z) - n \log(z)$, i.e. the solutions $z_{1,k}(n), \ldots, z_{d,k}(n)$ of $zB'(z) = n/\alpha_k$. As $n \to \infty$, we have $z_{j,k}(n) \sim (dB_d\alpha_k)^{-1/d} e^{2i\pi j/d} n^{1/d} \to \infty$, so that $\alpha_k B(z_{j,k}(n)) \sim n/d$.

Moreover, denoting by $\Delta_{j,k}(n)$ the second derivative of $\alpha_k B(z) - n \log(z)$ at $z = z_{j,k}(n)$, we see that asymptotically

$$\Delta_{j,k}(n) = \alpha_k B''(z_{j,k}(n)) + \frac{n}{z_{j,k}(n)^2} \sim d(dB_d\alpha_k)^{2/d} e^{-4i\pi j/d} n^{1 - 2/d}.$$

Then the saddle point method yields:

$$P_n = \sum_{\alpha_k \neq 0} \omega_k' \sum_{j=0}^{d-1} \frac{1}{\sqrt{2\pi\Delta_{j,k}(n)}} e^{\alpha_k B(z_{j,k}(n))} z_{j,k}(n)^{\delta+du_k-n-1} (\log z_{j,k}(n))^{v_k} (1+o(1))$$

with $\omega_k' = \omega_k A_\delta B_d^{u_k} d^{v_k} \in \overline{\mathbb{Q}}^* \omega_k$. This relation yields

$$P_n = \sum_{\alpha_k \neq 0} \frac{\omega_k''}{\sqrt{2\pi}} n^{-n/d} (edB_d\alpha_k)^{n/d} n^{\frac{\delta}{d}+u_k-\frac{1}{2}} (\log n)^{v_k} \left(\sum_{j=0}^{d-1} e^{2i\pi jn/d} + o(1) \right)$$

with $\omega_k'' \in \overline{\mathbb{Q}}^* \omega_k$. Now let $\widetilde{\alpha} = \max(|\alpha_1|, \ldots, |\alpha_K|)$ and consider the set $\mathcal{K}$ of all k such that $|\alpha_k| = \widetilde{\alpha}$. For each $k \in \mathcal{K}$ we write $\alpha_k^{1/d} = \widetilde{\alpha}^{1/d} e^{i\theta_k}$; then Stirling's formula yields

$$P_n = (2\pi)^{(1-d)/(2d)} n!^{-1/d} (dB_d\widetilde{\alpha})^{n/d} \sum_{k \in \mathcal{K}} \omega_k'' n^{\frac{\delta}{d}+u_k-\frac{1}{2}+\frac{1}{2d}} (\log n)^{v_k} \sum_{j=0}^{d-1} e^{i(\theta_k+\frac{2\pi j}{d})n} (1+o(1)).$$

Keeping only the dominant terms provides Eq. (5.1).

5.2 $P(z)$ is not an entire function

Let us move now to the case where $P(z)$ is not entire, and prove Eq. (5.2). Let $q > 0$ and $\Theta \subset [0, 2\pi]$ be such that the singularities of $P(z)$ of minimal modulus are the $q^{-1}e^{-i\theta_d}$ with $\theta_d \in \Theta$; then Θ is finite and non-empty. As usual the contributions of these singularities add up to determine the asymptotic behavior of P_n; this corresponds to the sum over θ_d in Eq. (5.2). For simplicity we shall restrict in the proof to the case of a unique singularity $\rho = q^{-1}e^{-i\theta_d}$ of minimal modulus q^{-1}. We consider first two special cases, and then the most difficult one.

5.2.1 $B(z)$ has a finite limit at ρ

Let us assume that $B(z)$ admits a finite limit as $z \to \rho$, denoted by $B(\rho)$; ρ can be a singularity of B or not. In both cases, as $z \to \rho$ we have

$$B(z) = B(\rho) + \mathfrak{B}(z-\rho)^t (1+o(1))$$

with $t \in \mathbb{Q}$, $t \geq 0$, and $\mathfrak{B} \in \overline{\mathbb{Q}}^*$ (unless B is a constant; in this case the proof is even easier). Now all Taylor coefficients of $E(z)$ at $B(\rho)$ belong to $\mathbf{E}$, so that

$$E(B(z)) \sim \eta(z-\rho)^{t'}$$

as $z \to \rho$, with $t' \in \mathbb{Q}$, $t' \geq 0$, and $\eta \in \mathbf{E} \setminus \{0\}$. On the other hand, if ρ is a singularity of the algebraic function $A(z)$ then its Puiseux expansion yields $s \in \mathbb{Q} \setminus \mathbb{N}$, $\mathfrak{A} \in \overline{\mathbb{Q}}^*$ and a polynomial $\widetilde{A}$ such that

$$A(z) = \widetilde{A}(z-\rho) + \mathfrak{A}(z-\rho)^s (1+o(1))$$

as $z \to \rho$; if ρ is not a singularity of A we have the same expression with $s \in \mathbb{N}$ and $\widetilde{A} = 0$. In both cases we obtain finally $p \in \mathbb{Q} \setminus \mathbb{N}$, $\mathfrak{P} \in \mathbf{E} \setminus \{0\}$ and a polynomial $\widetilde{P}$ such that

$$P(z) = \widetilde{P}(z - \rho) + \mathfrak{P}(z - \rho)^p(1 + o(1)).$$

Using standard transfer results (see [13, p. 393]) this implies

$$P_n \sim \frac{(-\rho)^{-p}\mathfrak{P}}{\Gamma(-p)} \rho^{-n} n^{-p-1}.$$

Therefore the singularity contributes to (5.2) through a term in which $v = \kappa_1 = \ldots = \kappa_{d-1} = \theta_1 = \ldots = \theta_{d-1} = 0$ and $\rho^{-1} = qe^{i\theta_d}$.

5.2.2 E is a polynomial

In this case, $P(z)$ is an algebraic function (and not a polynomial) so that

$$P_n \sim \frac{\omega}{\Gamma(-s)} \cdot n^{-s-1} \rho^{-n}$$

with $\omega \in \overline{\mathbb{Q}}^* \subset \mathbf{E} \setminus \{0\}$ and $s \in \mathbb{Q} \setminus \mathbb{N}$ determined by the Puiseux expansion of $P(z)$ around ρ (using the same transfer result as above). Therefore each singularity $\rho = q^{-1}e^{-i\theta_d}$ contributes to a term in (5.2) with $v = \kappa_1 = \ldots = \kappa_{d-1} = \theta_1 = \ldots = \theta_{d-1} = 0$.

5.2.3 The main part of the proof

Let us come now to the most difficult part of the proof, namely the contribution of a singularity ρ at which $B(z)$ does not have a finite limit (in the case where $E(z)$ is not a polynomial). As above we assume (for simplicity) that ρ is the unique singularity of $P(z)$ of minimal modulus q^{-1}. As $z \to \rho$, we have

$$A(z) \sim \mathfrak{A}(z - \rho)^{t/s} \text{ and } B(z) \sim \mathfrak{B}(z - \rho)^{-\tau/\sigma} \tag{5.7}$$

with $\mathfrak{A}, \mathfrak{B} \in \overline{\mathbb{Q}}^*$, $s, t, \sigma, \tau \in \mathbb{Z}$, $s, \sigma, \tau > 0$, and $\gcd(s, t) = \gcd(\sigma, \tau) = 1$. For any circle C_R of center 0 and radius $R < |\rho|$, we have (using Lemma 2 as in §5.1)

$$P_n = \frac{1}{2i\pi} \sum_{k=1}^{K} \omega_k \int_{C_R} \frac{e^{\alpha_k B(z)}}{z^{n+1}} \cdot A(z)B(z)^{u_k} \log(B(z))^{v_k} \cdot (1 + o(1))\mathrm{d}z \tag{5.8}$$

where $o(1)$ is with respect to $R \to |\rho|$ and is uniform in n.

If $\alpha_k = 0$ for some k, then the corresponding term in (5.8) has to be treated in a specific way, since the main contribution may come from the error term $o(1)$. For this reason we observe that in Lemma 2, the term corresponding to $\alpha_k = 0$ can be replaced with any truncation of the asymptotic expansion of $E(z)$, namely with

$$\sum_{u=-U_0}^{U_1} \sum_{v=0}^{V} \omega_{u,v} z^{u/d} (\log z)^v + o(z^{-U_0/d})$$

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where $d \geq 1$ and U_0 can be chosen arbitrarily large. Now the corresponding term in (5.8) becomes

$$\frac{1}{2i\pi} \int_{C_R} \frac{1}{z^{n+1}} \Big(\sum_{u=-U_0}^{U_1} \sum_{v=0}^{V} \omega_{u,v} A(z) B(z)^{u/d} (\log B(z))^v + o(A(z)B(z)^{-U_0/d}) \Big) dz. \qquad (5.9)$$

The point is that the function $\omega_{u,v} A(z) B(z)^{u/d} (\log B(z))^v$ may be holomorphic at $z = \rho$, because $\omega_{u,v} = 0$ or because the singularities at ρ of $A(z)$ and $B(z)^{u/d} (\log B(z))^v$ cancel out; in this case the corresponding integral over C_R is $o(q'^n)$ for some $q' < q = |\rho|^{-1}$ so that it falls into error terms. If this happens for any U_0, any u and any v, then the term corresponding to $\alpha_k = 0$ in (5.8) is $o(q^n n^{-U})$ for any $U > 0$, so that it falls into the error term of the expression (5.2) we are going to obtain for P_n. Otherwise we may consider the maximal pair (u, v) (with respect to lexicographic order) for which this function is not holomorphic; then (5.9) is equal to

$$\frac{\omega'_{u,v}}{2i\pi} \int_{C_R} \frac{(\rho - z)^T \log(\rho - z)^v}{z^{n+1}} \cdot (1 + o(1)) dz$$

for some $T \in \mathbb{Q}$ and $\omega'_{u,v} \in \overline{\mathbb{Q}}^* \omega_{u,v} \subset \Gamma(\mathbb{Q}) \cdot \mathbf{G}$ (using assertion (iii) of Theorem 2). We obtain finally the following formula for (5.9) (see [13, p. 387]):

$$\begin{cases} \dfrac{\omega'_{u,v}}{\Gamma(-T)} \rho^{T-n} n^{-T-1} \log(n)^v (1 + o(1)) & \text{if } T \notin \mathbb{N}, \\[2em] \omega'_{u,v} \rho^{T-n} n^{-T-1} \log(n)^{v-1} (1 + o(1)) & \text{if } T \in \mathbb{N} \text{ (so that } v \geq 1). \end{cases}$$

This contribution can either fall into the error term of (5.2), or give a term with $\kappa_1 = \ldots = \kappa_{d-1} = \theta_1 = \ldots = \theta_{d-1} = 0$.

Let us study now the terms in (5.8) for which $\alpha_k \neq 0$; since $E(z)$ is not a polynomial there is at least one such term. The function

$$\frac{e^{\alpha_k B(z)}}{z^{n+1}} \cdot A(z) B(z)^{u_k} \log(B(z))^{v_k}$$

is smooth on C_R (except on the cuts of $\log(B(z))$) and the integral can be estimated as $n \to \infty$ by finding the critical points of $\alpha_k B(z) - n \log(z)$, i.e. the solutions of $z B'(z) = n/\alpha_k$. For large n, any critical point z must be close to ρ (since $z B'(z)$ is bounded away from ρ for $|z| \leq |\rho|$). Now in a neighborhood of $z = \rho$ we have

$$z B'(z) \sim -\frac{\rho \tau \mathfrak{B}}{\sigma} \cdot \frac{1}{(z - \rho)^{1+\tau/\sigma}}$$

so that we have $\tau + \sigma$ critical points $z_{j,k}(n)$, for $j = 0, \ldots, \sigma + \tau - 1$, with

$$z_{j,k}(n) - \rho \sim e^{2i\pi j\sigma/(\sigma+\tau)} \cdot \Big(-\frac{\sigma n}{\rho \mathfrak{B} \tau \alpha_k} \Big)^{-\sigma/(\sigma+\tau)} .$$

Using (5.7) and letting $\kappa = t/s \in \mathbb{Q}$ we deduce that

$$A(z_{j,k}(n)) \sim \mathfrak{A} e^{2i\pi j \sigma \kappa/(\sigma+\tau)} \cdot \left(-\frac{\sigma n}{\rho \mathfrak{B} \tau \alpha_k}\right)^{-\sigma \kappa/(\sigma+\tau)} \neq 0.$$

Moreover we have

$$\alpha_k B(z_{j,k}(n)) \sim \frac{-\sigma}{\tau}(z_{j,k}(n) - \rho)\alpha_k B'(z_{j,k}(n)) \sim \frac{-\sigma n}{\rho \tau}(z_{j,k}(n) - \rho) \sim \mathfrak{D}_{j,k} n^{\tau/(\sigma+\tau)}$$

with

$$\mathfrak{D}_{j,k} = \left(\alpha_k \mathfrak{B} e^{2i\pi j}\right)^{\sigma/(\sigma+\tau)} \left(\frac{-\sigma}{\rho \tau}\right)^{\tau/(\sigma+\tau)} \neq 0. \tag{5.10}$$

To apply the saddle point method, we need to estimate the second derivative $\Delta_{j,k}(n)$ of $\alpha_k B(z) - n \log(z)$ at $z = z_{j,k}(n)$. We obtain

$$\Delta_{j,k}(n) = \alpha_k B''(z_{j,k}(n)) + \frac{n}{z_{j,k}(n)^2} \sim \frac{\tau(\sigma+\tau)}{\sigma^2}(\alpha_k \mathfrak{B})^{-\sigma/(\sigma+\tau)} e^{-2i\pi j \frac{2\sigma+\tau}{\sigma+\tau}} \left(-\frac{\sigma}{\rho\tau}\right)^{\frac{2\sigma+\tau}{\sigma+\tau}} n^{\frac{2\sigma+\tau}{\sigma+\tau}}.$$

Finally,

$$B(z_{j,k}(n))^{u_k} \sim (\mathfrak{D}_{j,k}/\alpha_k)^{u_k} n^{\tau u_k/(\sigma+\tau)}.$$

This enables us to apply the saddle point method. This yields a non-empty subset J_k of $\{0, \ldots, \sigma + \tau - 1\}$ such that the term corresponding to α_k in (5.8) is equal to

$$\sum_{j \in J_k} \frac{\omega_k}{\sqrt{2\pi \Delta_{j,k}(n)}} \frac{e^{\alpha_k B(z_{j,k}(n))}}{z_{j,k}(n)^{n+1}} A(z_{j,k}(n)) B(z_{j,k}(n))^{u_k} \log(B(z_{j,k}(n)))^{v_k}(1 + o(1)).$$

Now for any pair (j, k), $\alpha_k B(z_{j,k}(n))$ is an algebraic function of n so that it can be expanded as follows as $n \to \infty$:

$$\alpha_k B(z_{j,k}(n)) = \sum_{\ell=0}^{d'} \kappa_{j,k,\ell} n^{\ell/d} + o(1) \tag{5.11}$$

with $\kappa_{j,k,\ell} \in \overline{\mathbb{Q}}$, $0 < d' < d$ and $d'/d = \frac{\tau}{\sigma+\tau}$, $\kappa_{j,k,d'} = \mathfrak{D}_{j,k} \neq 0$. Increasing d and d' if necessary, we may assume that they are independent from (j, k). We denote by $(\kappa_{d'}, \ldots, \kappa_1)$ the family $(\mathrm{Re}\,\kappa_{j,k,d'}, \ldots, \mathrm{Re}\,\kappa_{j,k,1})$ which is maximal with respect to lexicographic order (as j and k vary with $\alpha_k \neq 0$ and $j \in J_k$), i.e. for which the real part of (5.11) has maximal growth as $n \to \infty$. Among the set of pairs (j, k) for which $\mathrm{Re}\,\kappa_{j,k,1} = \kappa_1, \ldots,$ $\mathrm{Re}\,\kappa_{j,k,d'} = \kappa_{d'}$, we define $\mathcal{K}$ to be the subset of those for which (u_k, v_k) is maximal (with respect to lexicographic order), and let (u, v) denote this maximal value. Then the total contribution to (5.8) of all terms with $\alpha_k \neq 0$ is equal to

$$\frac{n^{-\frac{\tau+2(1+\kappa)\sigma}{2\tau+2\sigma}}}{\sqrt{2\pi}} \rho^{-n} n^{\tau u/(\sigma+\tau)} \log(n)^v e^{\sum_{\ell=1}^{d'} \kappa_\ell n^{\ell/d}} \left(\sum_{(j,k) \in \mathcal{K}} \widehat{\omega}_{j,k} e^{\kappa_{j,k,0}} e^{\sum_{\ell=1}^{d'} i\mathrm{Im}\,\kappa_{j,k,\ell} n^{\ell/d}} + o(1)\right)$$

with $\widehat{\omega}_{j,k} \in \overline{\mathbb{Q}}^* \omega_k$. Since $\kappa_{d'} + i\mathrm{Im}\,\kappa_{j,k,d'} = \mathfrak{D}_{j,k} \neq 0$, this concludes the proof of Theorem 6.

6 Application to E-approximations

In this section we prove the results on E-approximations stated in the introduction. As a warm-up, we start in §6.1 with numbers related to the exponential function. Then we prove Theorem 3 in §§6.2 and 6.3. At last, we discuss in §6.4 the generalization involving (1.8).

6.1 E-approximations of exponential values

From the Taylor series $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, we can construct E-approximations of e^α for any algebraic number α. Indeed, let $A_n(\alpha) = \sum_{k=0}^{n} \frac{\alpha^n}{n!}$ and $B_n(\alpha) = 1$. Then, $\sum_{n=0}^{\infty} A_n(\alpha)x^n = \frac{\exp(\alpha x)}{1-x}$ and $\sum_{n=0}^{\infty} B_n(\alpha)x^n = \frac{1}{1-x}$, so that $A_n(\alpha)/B_n(\alpha)$ are E-approximations of e^α. This readily generalizes to any elements of **E**, see §6.2 below.

This is not the only way to produce E-approximations of the number e; in particular, we shall now prove that the convergents of its continued fraction expansion are E-approximations. In fact, this very property led us to the notion of E-approximations.

Proposition 4. *The sequence of convergents of the continued fraction expansion of e (resp. of $\frac{e-1}{e+1}$) defines E-approximations.*

Proof. We first provide an explicit expression for certain Padé approximants to $\exp(z)$. For any integer $n \geq 0$, the diagonal Padé approximant $[n/n]$ is given by $Q_n(z)e^z - P_n(z) = (-1)^n \frac{z^{2n+1}}{n!} \int_0^1 t^n(1-t)^n e^{zt}dt = \mathcal{O}(z^{2n+1})$ as $z \to 0$, with

$$Q_n(z) = \sum_{k=0}^{n}(-1)^k \binom{2n-k}{n} \frac{z^k}{k!} \quad \text{and} \quad P_n(z) = Q_n(-z).$$

The Padé approximant $[n-1/n]$ is given by $\widetilde{Q}_n(z)e^z - \widetilde{P}_n(z) = (-1)^n \frac{z^{2n}}{n!} \int_0^1 t^{n+1}(1-t)^n e^{zt}dt = \mathcal{O}(z^{2n})$ with

$$\widetilde{Q}_n(z) = \sum_{k=0}^{n}(-1)^k \binom{2n-k-1}{n-1} \frac{z^k}{k!} \quad \text{and} \quad \widetilde{P}_n(z) = \sum_{k=0}^{n-1} \binom{2n-k-1}{n} \frac{z^k}{k!}.$$

Finally, the Padé approximant $[n/n-1]$ is given by $\widehat{Q}_n(z)e^z - \widehat{P}_n(z) = (-1)^n \frac{z^{2n}}{n!} \int_0^1 t^n(1-t)^{n+1} e^{zt}dt = \mathcal{O}(z^{2n})$ with

$$\widehat{Q}_n(z) = \widetilde{P}_n(-z) \quad \text{and} \quad \widehat{P}_n(z) = \widetilde{Q}_n(-z).$$

We refer to [7] for a proof of these classical facts.

By changing the order of summations, we obtain that, for any $z \in \mathbb{C}$ and any x such that $|x| < 1/4$,

$$\sum_{n=0}^{\infty} Q_n(z)x^n = \frac{1}{\sqrt{1-4x}} e^{\frac{z}{2}(\sqrt{1-4x}-1)}, \quad \sum_{n=0}^{\infty} P_n(z)x^n = \frac{1}{\sqrt{1-4x}} e^{\frac{z}{2}(1-\sqrt{1-4x})}, \qquad (6.1)$$

$$\sum_{n=0}^{\infty} \widetilde{Q}_n(z)x^n = \frac{1+\sqrt{1-4x}}{2\sqrt{1-4x}}e^{\frac{z}{2}(\sqrt{1-4x}-1)}, \quad \sum_{n=0}^{\infty} \widetilde{P}_n(z)x^n = \frac{1-\sqrt{1-4x}}{2\sqrt{1-4x}}e^{\frac{z}{2}(1-\sqrt{1-4x})},$$

$$\sum_{n=0}^{\infty} \widehat{Q}_n(z)x^n = \frac{1-\sqrt{1-4x}}{2\sqrt{1-4x}}e^{\frac{z}{2}(\sqrt{1-4x}-1)}, \quad \sum_{n=0}^{\infty} \widehat{P}_n(z)x^n = \frac{1+\sqrt{1-4x}}{2\sqrt{1-4x}}e^{\frac{z}{2}(1-\sqrt{1-4x})}.$$

These identities will be used below.

We can now prove Proposition 4. We consider the numerator u_n and denominator v_n of the n-th convergent of the continued fraction $[0; 2, 6, 10, 14, 18, \ldots]$ of the number $\frac{e-1}{e+1}$, i.e., $u_n/v_n = [0; a_1, \ldots, a_n]$ with $a_k = 4k - 2$. It turns out that $u_n = n!(P_n(1) - Q_n(1))/2$ and $v_n = n!(P_n(1) + Q_n(1))/2$. This can be proved by computing the linear recurrence satisfied by $n!P_n(1)$ and $n!Q_n(1)$, using Zeilberger's algorithm for instance: it is $U_{n+1} = (4n+2)U_n + U_{n-1}$ for both sequences, which is exactly that satisfied by u_n and v_n (by definition), and the initial values coincide. It follows that

$$\sum_{n=0}^{\infty} \frac{u_n}{n!} = \frac{\sinh\left((1-\sqrt{1-4x})/2\right)}{\sqrt{1-4x}} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{v_n}{n!} = \frac{\cosh\left((1-\sqrt{1-4x})/2\right)}{\sqrt{1-4x}}.$$

These generating functions satisfy Definition 4, which proves that the sequence of the convergents $\left(\frac{u_n}{v_n}\right)_{n\geq 0}$ of $\frac{e-1}{e+1}$ defines E-approximations.

Finally, let us consider the case of e. Its continued fraction is $[2; 1, 2, 1, 1, 4, 1, 1, 6, \ldots]$, with a regular pattern after the second 2. As in [7], one may define the convergents of this continued fraction by $\frac{p_n}{q_n} = [2; b_1, \ldots, b_{n-2}]$ for $n \geq 3$, with $p_0 = q_0 = 1$, $p_1 = 1, q_1 = 0$ and $p_2 = 2, q_2 = 1$. Then for any $n \geq 0$ we have (see [7]):

$$p_{3n} = n!P_n(1), \quad p_{3n+1} = n!\widetilde{P}_n(1), \quad p_{3n+2} = n!\widehat{P}_n(1),$$
$$q_{3n} = n!Q_n(1), \quad q_{3n+1} = n!\widetilde{Q}_n(1), \quad q_{3n+2} = n!\widehat{Q}_n(1).$$

It follows that

$$\sum_{n=0}^{\infty} \frac{p_n}{\lfloor n/3 \rfloor!}x^n = \sum_{n=0}^{\infty} P_n(1)x^{3n} + \sum_{n=0}^{\infty} \widetilde{P}_n(1)x^{3n+1} + \sum_{n=0}^{\infty} \widehat{P}_n(1)x^{3n+2}$$
$$= \frac{2 + x + x^2 + x(x-1)\sqrt{1-4x^3}}{2\sqrt{1-4x^3}}e^{\frac{1}{2}(1-\sqrt{1-4x^3})}$$

and

$$\sum_{n=0}^{\infty} \frac{q_n}{\lfloor n/3 \rfloor!}x^n = \sum_{n=0}^{\infty} Q_n(1)x^{3n} + \sum_{n=0}^{\infty} \widetilde{Q}_n(1)x^{3n+1} + \sum_{n=0}^{\infty} \widehat{Q}_n(1)x^{3n+2}$$
$$= \frac{2 + x + x^2 + x(1-x)\sqrt{1-4x^3}}{2\sqrt{1-4x^3}}e^{\frac{1}{2}(\sqrt{1-4x^3}-1)}.$$

Again, the generating functions of $\left(\frac{p_n}{\lfloor n/3 \rfloor!}\right)_{n\geq 0}$ and $\left(\frac{q_n}{\lfloor n/3 \rfloor!}\right)_{n\geq 0}$ satisfy Definition 4, so that the sequence of the convergents $\left(\frac{p_n}{q_n}\right)_{n\geq 0}$ of e defines E-approximations. $\qquad\square$

From (6.1), we also deduce that the Padé approximants $[n/n]$ of $\exp(z)$ define E-approximations of e^α for any $\alpha \in \overline{\mathbb{Q}}$, because it is known that $\lim_n P_n(z)/Q_n(z) = \exp(z)$ for any $z \in \mathbb{C}$. We now give a proof of this fact which is an instance of the general asymptotic arguments presented in §5. The generating function for $Q_n(z)$ can be written as $\frac{e^{-z/2}}{\sqrt{1-4x}} f(z,x) + g(z,x)$, where $f(z,x)$ and $g(z,x)$ are entire functions of x, and $f(z,\frac{1}{4}) = f(-z,\frac{1}{4}) \neq 0$. Hence, by a standard transfer principle, the asymptotic behaviors as $n \to +\infty$ of $Q_n(z)$ and $P_n(z)$ are deduced from the behavior of their generating functions as $x \to \frac{1}{4}$: we get

$$Q_n(z) \sim e^{-z/2} f\left(z,\frac{1}{4}\right) 4^n \binom{-1/2}{n} \quad \text{and} \quad P_n(z) \sim e^{z/2} f\left(z,\frac{1}{4}\right) 4^n \binom{-1/2}{n}.$$

It follows that $\lim_n \frac{P_n(z)}{Q_n(z)} = e^z$.

6.2 Construction of numbers with E-approximations

In this section, we prove the first part of Theorem 3, namely that any element of

$$\frac{\mathbf{E} \cup \Gamma(\mathbb{Q})}{\mathbf{E} \cup \Gamma(\mathbb{Q})} \cup \operatorname{Frac} \mathbf{G}$$

has E-approximations. As mentioned in the introduction, this is true for any element of $\operatorname{Frac} \mathbf{G}$. To complete the proof, let us construct for any $\xi \in \mathbf{E} \cup \Gamma(\mathbb{Q})$ a sequence $(P_n)_n$ as in Definition 4 with $\lim_{n \to \infty} P_n = \xi$.

If $\xi = F(\alpha)$ where $\alpha \in \overline{\mathbb{Q}}$ and $F(z) = \sum_{n \geq 0} \frac{a_n}{n!} z^n$ is an E-function, we define $P_n \in \overline{\mathbb{Q}}$ by

$$\sum_{n=0}^{\infty} P_n z^n = \frac{1}{1-z} F(\alpha z).$$

Then, trivially,

$$P_n = \sum_{k=0}^{n} \frac{a_k}{k!} \alpha^k \longrightarrow F(\alpha) = \xi.$$

If $\xi = \Gamma(\alpha)$ with $\alpha \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$, we consider the E-function

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!(n+\alpha)}$$

and define $P_n(\alpha)$ as announced in the introduction, by the series expansion (for $|z| < 1$)

$$\frac{1}{(1-z)^{\alpha+1}} E_\alpha\left(-\frac{z}{1-z}\right) = \sum_{n \geq 0} P_n(\alpha) z^n \in \mathbb{Q}[[z]].$$

Then

$$P_n(\alpha) = \sum_{k=0}^{n} \binom{n+\alpha}{k+\alpha} \frac{(-1)^k}{k!(k+\alpha)}$$

(by direct manipulations) and, provided that $\alpha < 1$,

$$\lim_{n \to +\infty} P_n(\alpha) = \Gamma(\alpha) = \xi.$$

To see this, we start from the asymptotic expansion

$$E_\alpha(-z) \approx \frac{\Gamma(\alpha)}{z^\alpha} - e^{-z} \sum_{n=0}^{\infty} (-1)^n \frac{(1-\alpha)_n}{z^{n+1}} \tag{6.2}$$

in a large sector bisected by any $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, which is a special case of Theorem 5 (proved directly in [22, Proposition 1]). Since $\exp\left(-\frac{z}{1-z}\right) = \mathcal{O}(1)$ as $z \to 1$, $|z| < 1$, it follows that

$$\frac{1}{(1-z)^{\alpha+1}} E_\alpha\left(-\frac{z}{1-z}\right) = \frac{\Gamma(\alpha)}{1-z} + \mathcal{O}\left(\frac{1}{|1-z|^\alpha}\right) + \mathcal{O}(1)$$

as $z \to 1$, $|z| < 1$. The result follows by a standard transfer result since $\alpha < 1$; this example is of the type covered by §5.2.3 with $\alpha_1 = 0$.

From the differential equation $zy''(z) + (\alpha + 1 - z)y'(z) - \alpha y(z) = 0$ satisfied by $E_\alpha(z)$, we easily get the differential equation satisfied by $\frac{1}{(1-z)^{\alpha+1}} E_\alpha\left(-\frac{z}{1-z}\right)$:

$$\left(3z^3 - z^4 - 3z^2 + z\right)y''(z) + \left(5z^2\alpha - 4z^3 - 2z^3\alpha + 8z^2 + 1 + \alpha - 5z - 4z\alpha\right)y'(z)$$
$$+ \left(-1 - 2z^2 - 3z^2\alpha + 2z - \alpha + 4z\alpha - \alpha^2 + 2z\alpha^2 - z^2\alpha^2\right)y(z) = 0. \tag{6.3}$$

This immediately translates into a linear recurrence satisfied by the sequence $(P_n(\alpha))$:

$$(n+3)(n+3+\alpha)P_{n+3}(\alpha) - (3n^2 + 4n\alpha + 14n + \alpha^2 + 9\alpha + 17)P_{n+2}(\alpha)$$
$$+ (3n + 5 + 2\alpha)(n+2+\alpha)P_{n+1}(\alpha) - (n+2+\alpha)(n+1+\alpha)P_n(\alpha) = 0 \tag{6.4}$$

with $P_0(\alpha) = \frac{1}{\alpha}$, $P_1(\alpha) = \frac{1+\alpha+\alpha^2}{\alpha(\alpha+1)}$ and $P_2(\alpha) = \frac{4+5\alpha+6\alpha^2+4\alpha^3+\alpha^4}{2\alpha(\alpha+1)(\alpha+2)}$.

6.3 Properties of numbers with E-approximations

Let us prove now the second part of Theorem 3, namely that any number $\xi \in \mathbb{C}^*$ with E-approximations belongs to

$$\frac{\mathbf{E} \cup (\Gamma(\mathbb{Q}) \cdot \mathbf{G})}{\mathbf{E} \cup (\Gamma(\mathbb{Q}) \cdot \mathbf{G})} \cup \left(\Gamma(\mathbb{Q}) \cdot \exp(\overline{\mathbb{Q}}) \cdot \operatorname{Frac} \mathbf{G}\right).$$

The proof is very similar to that of [11, §6.2] so we skip the details. Let (P_n, Q_n) be E-approximations of $\xi \in \mathbb{C}^*$. If (P_n) has the first asymptotic behavior (5.1) of Theorem 6,

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then so does (Q_n) with the same parameters d, q, u, v, and the sum is over the same non-empty finite set of θ. Therefore $\xi = \frac{g_\theta \Gamma(-u_\theta)}{g'_\theta \Gamma(-u'_\theta)} \in \Gamma(\mathbb{Q}) \cdot \operatorname{Frac} \mathbf{G}$, using Eq. (1.5).

Now if (P_n) satisfies (5.2) then so does (Q_n) with the same parameters q, u, v, κ_1, ..., κ_{d-1} (since we may assume that d is the same), and the same set of $(\theta_1, \ldots, \theta_d)$ in the sum. If $v = \kappa_1 = \ldots = \kappa_{d-1} = 0$ and a term in the sum corresponds to $\theta_1 = \ldots = \theta_{d-1} = 0$, then $\xi = \frac{\omega_{0,\ldots,0,\theta_d}}{\omega'_{0,\ldots,0,\theta_d}} \in \frac{\mathbf{E} \cup (\Gamma(\mathbb{Q}) \cdot \mathbf{G})}{\mathbf{E} \cup (\Gamma(\mathbb{Q}) \cdot \mathbf{G})}$, else $\xi \in \Gamma(\mathbb{Q}) \cdot \exp(\overline{\mathbb{Q}}) \cdot \operatorname{Frac} \mathbf{G}$ (using Eq. (1.5)).

6.4 Extended E-approximations

Let us consider the E-function
$$E(z) = \sum_{n=1}^{\infty} \frac{z^n}{n!n}.$$

We shall prove that the sequence (P_n) defined in the introduction by
$$\frac{\log(1-z)}{1-z} - \frac{1}{1-z} E\left(-\frac{z}{1-z}\right) = \sum_{n=0}^{\infty} P_n z^n \in \mathbb{Q}[[z]]$$

provides, together with $Q_n = 1$, a sequence of E-approximations of Euler's constant in the extended sense of (1.8). It is easy to see that
$$P_n = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{1}{k!k} - \sum_{k=1}^{n} \frac{1}{k} = \sum_{k=1}^{n} (-1)^k \binom{n}{k} \frac{1}{k}\left(1 - \frac{1}{k!}\right),$$

where the second equality is a consequence of the identity $\sum_{k=1}^{n} \frac{1}{k} = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{1}{k}$. We now observe that $E(z)$ has the asymptotic expansion
$$E(-z) \approx -\gamma - \log(z) - e^{-z} \sum_{n=0}^{\infty} (-1)^n \frac{n!}{z^{n+1}} \tag{6.5}$$

in a large sector bisected by any $\theta \in (-\pi, \pi)$ (see [22, Prop. 1]; this is also a special case of Theorem 5). Therefore, as $z \to 1$, $|z| < 1$,
$$-\frac{1}{1-z} E\left(-\frac{z}{1-z}\right) + \frac{\log(1-z)}{1-z} = \frac{\gamma}{1-z} + \mathcal{O}(1).$$

As in §6.1 in the case of $\Gamma(\alpha)$, a transfer principle readily shows that
$$\lim_{n \to +\infty} P_n = \gamma.$$

Since $E(z)$ is holonomic, this is also the case of $\frac{\log(1-z)}{1-z} - \frac{1}{1-z} E\left(-\frac{z}{1-z}\right)$. The latter function satisfies the differential equation
$$\left(3z^3 - z^4 - 3z^2 + z\right) y''(z) + \left(1 - 5z + 8z^2 - 4z^3\right) y'(z) + \left(-2z^2 + 2z - 1\right) y(z) = 0. \tag{6.6}$$

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This immediately translates into a linear recurrence satisfied by the sequence (P_n):

$$(n+3)^2 P_{n+3} - (3n^2 + 14n + 17)P_{n+2} + (n+2)(3n+5)P_{n+1} - (n+1)(n+2)P_n = 0 \quad (6.7)$$

with $P_0 = 0$, $P_1 = 0$, $P_2 = \frac{1}{4}$. The differential equation (6.6) and the recurrence relation (6.7) are the case $\alpha = 0$ of (6.3) and (6.4) respectively.

Let us now prove that any number with extended E-approximations belongs to

$$\frac{(\mathbf{E} \cdot \log(\overline{\mathbb{Q}}^*)) \cup \mathbf{S}}{(\mathbf{E} \cdot \log(\overline{\mathbb{Q}}^*)) \cup \mathbf{S}} \cup \left(\exp(\overline{\mathbb{Q}}) \cdot \mathrm{Frac}\,\mathbf{S} \right),$$

as stated in the introduction. Let $P(z) = \sum_{n=0}^{\infty} P_n z^n$ be given by a finite sum

$$P(z) = \sum_{i,j,k,\ell} \alpha_{i,j,k,\ell} \log(1 - A_i(z))^j \cdot B_k(z) \cdot E_\ell\big(C(z)\big) \quad (6.8)$$

where $\alpha_{i,j,k,\ell} \in \overline{\mathbb{Q}}$, $A_i(z), B_k(z), C(z)$ are algebraic functions in $\overline{\mathbb{Q}}[[z]]$, $A_i(0) = C(0) = 0$, and $E_\ell(z)$ are E-functions. If there is only one term in the sum, the conclusions of Theorems 3 and 6 hold and their proofs extend immediately, except that $\mathbf{E}$ has to be replaced with $\mathbf{E} \cdot \log(\overline{\mathbb{Q}}^*)$ in §5.2.1 and 5.2.2, and therefore in (1.7) and (5.3). Otherwise, we apply a variant of Lemma 2 to each E-function $E_\ell(z)$, obtaining exponential terms $e^{\alpha_{k,\ell} z}$: for each k we write sufficiently many terms in the asymptotic expansion before the error term $o(1)$ (and not only the dominant one as in §5). Theorem 2 asserts that all these terms are of the same form, but now the constants ω belong to $\mathbf{S}$. Combining these expressions yields

$$P(z) = \sum_{k=1}^{K} \omega_k e^{\alpha_k C(z)} U_k(z)(\log V_k(z))^{v_k}(1 + o(1))$$

as z tends to some point (possibly ∞) at which C is infinite; here U_k, V_k are algebraic functions, $v_k \in \mathbb{N}$, and $\omega_k \in \mathbf{S}$. However there is no reason why ω_k would belong to $\Gamma(\mathbb{Q}) \cdot \mathbf{G}$ in general, since it may come from non-dominant terms in the expansions of $E_\ell(z)$, due to compensations. Upon replacing $\Gamma(\mathbb{Q}) \cdot \mathbf{G}$ with $\mathbf{S}$ (and $\mathbf{E}$ with $\mathbf{E} \cdot \log(\overline{\mathbb{Q}}^*)$ as above), the proof of Theorems 3 and 6 extends immediately.

To conclude this section, we discuss another interesting example, which was also mentioned in the introduction. It corresponds to the more general notion of extended E-approximations where the coefficients of the linear form (6.8) are in $\mathbf{E}$ and not just in $\overline{\mathbb{Q}}$. Let us consider the E-function $F(z^2)$, where $F(z) = \sum_{n=0}^{\infty} z^n/n!^2$. It is solution of an E-operator L of order 2 with another solution of the form $G(z^2) + \log(z^2)F(z^2)$ where $G(z^2) = -2\sum_{n=0}^{\infty} \frac{1 + \frac{1}{2} + \cdots + \frac{1}{n}}{n!^2} z^{2n}$ is an E-function (in accordance with André's theory). Then,

$$F(1 - z) = \sum_{n=0}^{\infty} \frac{(1-z)^n}{n!^2} = \sum_{k=0}^{\infty} \frac{(-1)^k A_k}{k!} z^k$$

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with

$$A_k = (-1)^k \sum_{n=0}^{\infty} \frac{1}{n!(n+k)!}.$$

It is a remarkable (and known) fact that the sequence (A_k) satisfies the recurrence relation $A_{k+1} = kA_k + A_{k-1}$, $A_0 = F(1)$, $A_1 = -F'(1)$. This can be readily checked. It follows that $A_k = V_k F(1) - U_k F'(1)$ where the sequences of integers U_k, V_k are solutions of the same recurrence.

Hence, the sequence U_k/V_k is the sequence of convergents to $F(1)/F'(1)$ whose continued fraction is $[0; 1, 2, 3, 4, \ldots]$. Moreover, we have

$$\sum_{n=0}^{\infty} \frac{(-1)^k U_k}{k!} z^k = aF(1-z) + bG(1-z) + b\log(1-z)F(1-z)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^k V_k}{k!} z^k = cF(1-z) + dG(1-z) + d\log(1-z)F(1-z)$$

for some constants a, b, c, d, because both generating functions are solutions of an operator of order 2 obtained from L by changing z to $\sqrt{1-z}$. The conditions $V_0 = 1, U_0 = 0, V_1 = 0, U_1 = 1$ and $A_k = V_k F(1) - U_k F'(1)$ translate into a linear system in a, b, c, d with solutions given by

$$a = \frac{g}{gf' - f^2 - fg'} \in \mathbf{E}, \qquad b = -\frac{f}{gf' - f^2 - fg'} \in \mathbf{E},$$

$$c = -\frac{f+g'}{gf' - f^2 - fg'} \in \mathbf{E}, \qquad d = \frac{f'}{gf' - f^2 - fg'} \in \mathbf{E},$$

where $f = F(1), f' = F'(1), g = G(1), g' = G(1)$. We observe that $gf' - f^2 - fg' \in \overline{\mathbb{Q}}^*$ because it is twice the value at $z = 1$ of the wronskian built on the linearly independent solutions $F(z^2)$ and $G(z^2) + \log(z^2)F(z^2)$. It follows that U_k/V_k are extended E-approximations to the number $F(1)/F'(1)$ with "coefficients" in $\mathbf{E}$, but not in $\overline{\mathbb{Q}}$ (because the number f was proved to be transcendental by Siegel).

Bibliography

[1] Y. André, *Séries Gevrey de type arithmétique I. Théorèmes de pureté et de dualité*, Annals of Math. **151** (2000), 705–740.

[2] Y. André, *Une introduction aux motifs (motifs purs, motifs mixtes, périodes)*, Panoramas et Synthèses **17** (2004), Soc. Math. France, Paris.

[3] G. E. Andrews, R. A. Askey and R. Roy, *Special Functions*, The Encyclopaedia of Mathematics and its Applications, vol. 71, (G.-C. Rota, ed.), Cambridge University Press, Cambridge, 1999.

[4] A. I. Aptekarev (editor), *Rational approximants for Euler constant and recurrence relations*, Sovremennye Problemy Matematiki **9** ("Current Problems in Mathematics"), MIAN (Steklov Institute), Moscow, 2007.

[5] F. Beukers, *Algebraic values of G-functions*, J. Reine Angew. Math. **434** (1993), 45–65.

[6] F. Beukers, *A refined version of the Siegel-Shidlovskii theorem*, Annals of Math. **163** (2006), no. 1, 369–379.

[7] H. Cohn, *A short proof of the simple continued fraction expansion of e*, Am. Math. Monthly **113** (2006), 57–62.

[8] L. Comtet, *Analyse combinatoire*, tome 1, Presses Univ. de France, Coll. Le Mathématicien, 1970.

[9] V. Ditkine and A. Proudnikov, *Calcul Opérationnel*, Editions Mir, 1979.

[10] A. Duval and C. Mitschi, *Matrices de Stokes et groupe de Galois des équations hypergéométriques confluentes généralisées*, Pacific J. Math. **138** (1989), no. 1, 25–56.

[11] S. Fischler and T. Rivoal, *On the values of G-functions*, Commentarii Math. Helv. **29** (2014), no. 2, 313–341.

[12] S. Fischler and T. Rivoal, *Microsolutions of differential operators and values of arithmetic Gevrey series*, preprint arxiv 1506.03574 [math.NT], submitted.

[13] P. Flajolet and R. Sedgewick, *Analytic combinatorics*, Cambridge University Press, 2009.

[14] Kh. Hessami-Pilehrood and T. Hessami-Pilehrood, *Rational approximations to values of Bell polynomials at points involving Euler's constant and zeta values*, J. Aust. Math. Soc. **92** (2012), no. 1, 71–98.

[15] Kh. Hessami-Pilehrood and T. Hessami-Pilehrood, *On a continued fraction expansion for Euler's constant*, J. Number Theory **133** (2013), no. 2, 769–786.

[16] M. Kontsevich and D. Zagier, *Periods*, in: Mathematics Unlimited – 2001 and beyond, Springer, 2001, 771–808.

[17] J. Lagarias, *Euler's constant: Euler's work and modern developments*, Bull. Amer. Math. Soc. **50** (2013), 527–628.

[18] M. Loday-Richaud, *Séries formelles provenant de systèmes différentiels linéaires méromorphes*, in: Séries divergentes et procédés de resommation, Journées X-UPS, 1991, pp. 69–100.

[19] J. P. Ramis, *Séries Divergentes et Théories Asymptotiques*, Panoramas et Synthèses **21** (1993), Soc. Math. France, Paris.

[20] T. Rivoal, *Rational approximations for values of derivatives of the Gamma function*, Trans. Amer. Math. Soc. **361** (2009), 6115–6149.

[21] T. Rivoal, *Approximations rationnelles des valeurs de la fonction Gamma aux rationnels*, J. Number Theory **130** (2010), 944–955.

[22] T. Rivoal, *On the arithmetic nature of the values of the Gamma function, Euler's constant et Gompertz's constant*, Michigan Math. Journal **61** (2012), 239–254.

[23] A. B. Shidlovskii, *Transcendental Numbers*, de Gruyter Studies in Mathematics **12**, 1989.

[24] M. Waldschmidt, *Transcendance de périodes : état des connaissances*, Proceedings of the Tunisian Mathematical Society **11** (2007), 89–116.

[25] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Cambridge Univ. Press, 4th ed., 1927.

[26] E. M. Wright, *On the coefficients of power series having exponential singularities*, J. Lond. Math. Soc. **8** (1933), 71–80.

[27] E. M. Wright, *On the coefficients of power series having exponential singularities (second paper)*, J. Lond. Math. Soc. **24** (1949), 304–309.

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