SIMULTANEOUS GENERATION OF KOECHER AND ALMKVIST-GRANVILLE'S APÉRY-LIKE FORMULAE

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Abstract. We prove a very general identity, conjectured by Henri Cohen, involving the generating function of the familly $(\zeta(2r+4s+3))_{r,s\geq 0}$: it unifies two identities, proved by Koecher in 1980 and Almkvist & Granville in 1999, for the generating functions of $(\zeta(2r+3))_{r\geq 0}$ and $(\zeta(4s+3))_{s\geq 0}$ respectively. As a consequence, we obtain that, for any integer $j\geq 0$, there exist at least $[j/2]+1$ Apéry-like formulae for $\zeta(2j+3)$.

1. Introduction

In proving that $\zeta(3)=\sum_{k=1}^{\infty}1/k^3$ is irrational, Apéry [2] noted that

$$\zeta(3)=\frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{\binom{2k}{k}k^3}. \tag{1.1}$$

Although the series on the right hand side converges much faster than the defining series for $\zeta(3)$, formula (1.1) is not essential in Apéry's proof since truncations of this series are not diophantine approximations to $\zeta(3)$. On the other hand, it is very likely that (1.1) was a source of inspiration for Apéry[1] and many authors have looked for similar identities, in the hope that they might give some idea of how to prove the irrationality of $\zeta(2s+1)=\sum_{k=1}^{\infty}1/k^{2s+1}$ for any integer $s\geq 2$: see for example [4, 6, 8, 10, 13]. This problem is far from being solved[2], but many beautiful Apéry-like formulae have been proved. In fact, two apparently unrelated families of such formulae for $\zeta(2s+3)$ and $\zeta(4s+3)$ have emerged, both of which are more easily explained with the help of the generating functions

$$\sum_{s=0}^{\infty}\zeta(2s+3)\,a^{2s}=\sum_{n=1}^{\infty}\frac{1}{n(n^2-a^2)}\quad\text{and}\quad\sum_{s=0}^{\infty}\zeta(4s+3)\,b^{4s}=\sum_{n=1}^{\infty}\frac{n}{n^4-b^4}.$$

(The series on the left hand sides converge only for $|a|<1$ and $|b|<1$, whereas the right hand sides converge on much larger domains.) Koecher [8] (and independently

2000 *Mathematics Subject Classification.* Primary 11M06; Secondary 05A15, 11J72.

Key words and phrases. Riemann zeta function, Apéry-like series, generating functions.

[1]See [5, 12] for a detailed explanation of Apéry's original method.

[2]We now know that infinitely many of the values $\zeta(2s+1)$ $(s\geq 1)$ are $\mathbb{Q}$–linearly independent [3, 11] and that at least one amongst $\zeta(5),\zeta(7),\zeta(9),\zeta(11)$ is irrational [14].

Leshchiner [10] in an expanded form) proved that

$$\sum_{n=1}^{\infty} \frac{1}{n(n^2-a^2)} = \frac{1}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\binom{2k}{k}k^3} \frac{5k^2-a^2}{k^2-a^2} \prod_{n=1}^{k-1}\left(1-\frac{a^2}{n^2}\right), \tag{1.2}$$

for any complex number a such that $|a|<1$, and, more recently, Almkvist & Granville [1] proved another identity, first conjectured by Borwein & Bradley [4]:

$$\sum_{n=1}^{\infty} \frac{n}{n^4-b^4} = \frac{1}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\binom{2k}{k}} \frac{5k}{k^4-b^4} \prod_{n=1}^{k-1}\left(\frac{n^4+4b^4}{n^4-b^4}\right), \tag{1.3}$$

for any complex number b such that $|b|<1$. For $a=b=0$, these identities reduce to (1.1), but otherwise produce different identities for the values of the zeta function at odd integers. For example, Borwein & Bradley note that (1.2) implies

$$\zeta(7) = 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\binom{2k}{k}k^7} - 2\sum_{k>j\geq 1} \frac{(-1)^{k+1}}{\binom{2k}{k}k^5 j^2} + \frac{5}{2}\sum_{k>j>i\geq 1} \frac{(-1)^{k+1}}{\binom{2k}{k}k^3 j^2 i^2}$$

while (1.3) implies

$$\zeta(7) = \frac{5}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\binom{2k}{k}k^7} + \frac{25}{2}\sum_{k>j\geq 1} \frac{(-1)^{k+1}}{\binom{2k}{k}k^3 j^4}.$$

The purpose of this article is to prove the following very general generating function identity, which was conjectured by H. Cohen on the basis of computations in Pari.

Theorem 1. *Let a and b be complex numbers such that $|a|^2+|b|^4<1$. Then*

$$\sum_{n=1}^{\infty} \frac{n}{n^4-a^2n^2-b^4} = \frac{1}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\binom{2k}{k}k} \frac{5k^2-a^2}{k^4-a^2k^2-b^4} \prod_{n=1}^{k-1}\left(\frac{(n^2-a^2)^2+4b^4}{n^4-a^2n^2-b^4}\right). \tag{1.4}$$

We remark that Identity (1.4) unifies (1.2) (case $b=0$) and (1.3) (case $a=0$); consequently, it should yield new Apéry-like formulae. This is indeed true since

$$\sum_{n=1}^{\infty} \frac{n}{n^4-a^2n^2-b^4} = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \binom{r+s}{r} \zeta(2r+4s+3)\, a^{2r}\, b^{4s}$$

and since the number of representations of an integer $j\geq 0$ as $j=r+2s$ with integers $r,s\geq 0$ is $[j/2]+1$. Hence, (1.4) produces $[j/2]+1$ different identities for $\zeta(2j+3)$ for any integer $j\geq 0$, obtained by differentiating the right hand side of (1.4) r, resp. s, times with respect to a^2, resp. b^4, with $j=r+2s$, and then by letting $a=b=0$.

For $0\leq j\leq 2$, one of r,s is 0 and we only obtain identities resulting from (1.2) or (1.3). This is also the case for $j=3$, $(r,s)=(3,0)$ and the first apparently new identity is for

$j = 3$, $(r,s) = (1,1)$:

$$\zeta(9) = \frac{9}{4}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{\binom{2k}{k}k^9} + 5\sum_{k>j\geq 1}\frac{(-1)^{k+1}}{\binom{2k}{k}k^5j^4} + 5\sum_{k>j\geq 1}\frac{(-1)^{k+1}}{\binom{2k}{k}k^3j^6}$$
$$- \frac{5}{4}\sum_{k>j\geq 1}\frac{(-1)^{k+1}}{\binom{2k}{k}k^7j^2} - \frac{25}{4}\sum_{k>j>i\geq 1}\frac{(-1)^{k+1}}{\binom{2k}{k}k^3j^4i^2} - \frac{25}{4}\sum_{k>j>i\geq 1}\frac{(-1)^{k+1}}{\binom{2k}{k}k^3j^2i^4}.$$

To prove Theorem 1, we will use Borwein & Bradley's method in which the proof of (1.4) was reduced in several steps to the proof of a finite combinatorial identity (the last step in [4] is due to Wenchang Chu), which was finally proved by Almkvist & Granville. In our case, we will show that Theorem 1 follows from the identity

$$\sum_{k=1}^{n}\frac{2}{k^2-a^2}\,\frac{\prod_{j=1}^{n-1}(k^2+(j-k)^2-a^2)(k^2+(j+k)^2-a^2)}{\prod_{j=1,\,j\neq k}^{n}(k^2-j^2)(k^2+j^2-a^2)} = \frac{1}{n^2-a^2}\binom{2n}{n},$$

for any integer $n \geq 1$, which we will then prove as corollary of the following result.

Theorem 2. *Let $g(X) \in \mathbb{C}[X]$ be of degree at most 2. For any integer $n \geq 1$ and any complex numbers a and t, with $a \notin \{\pm 1, \pm 2, \ldots, \pm n\}$, we have that*

$$\sum_{k=1}^{n}(-1)^{n-k}\binom{2n}{n-k}\frac{4k^2}{k^2-a^2}\left(\prod_{\substack{0\leq j<n-k\\ \text{or } n<j<n+k}}\big(t(k^2-a^2)+g(j)\big) - \prod_{\substack{0\leq j<n-k\\ \text{or } n<j<n+k}} g(j)\right) = 0. \quad (1.5)$$

For the special case $a = 0$, we obtain the key identity proved in [1].

2. First step

We transform the right hand side of (1.4) by a partial fraction decomposition, with respect to b^4:

$$\frac{1}{k^4-a^2k^2-b^4}\prod_{n=1}^{k-1}\frac{(n^2-a^2)^2+4b^4}{n^4-a^2n^2-b^4} = \sum_{n=1}^{k}\frac{C_{n,k}(a)}{n^4-a^2n^2-b^4}, \quad (2.1)$$

where

$$C_{n,k}(a) = \frac{\prod_{j=1}^{k-1}(n^2+(j-n)^2-a^2)(n^2+(j+n)^2-a^2)}{\prod_{j=1,\,j\neq n}^{k}(j^2-n^2)(j^2+n^2-a^2)}. \quad (2.2)$$

Inserting (2.1) in the right hand side of (1.4) and inverting the summations, we see that it will be enough to show that (and in fact, this is equivalent)

$$\sum_{n=1}^{\infty}\frac{n}{n^4-a^2n^2-b^4} = \sum_{n=1}^{\infty}\frac{1}{n^4-a^2n^2-b^4}\sum_{k=n}^{\infty}\frac{(-1)^{k+1}}{\binom{2k}{k}}\frac{5k^2-a^2}{2k}\,C_{n,k}(a).$$

Clearly, it is enough to show that, for any integer $n \geq 1$ and any complex a with $|a| < 1$,

$$\sum_{k=n}^{\infty} \frac{(-1)^{k+1}}{\binom{2k}{k}} \frac{5k^2 - a^2}{2k} C_{n,k}(a) = n. \tag{2.3}$$

From now on, and otherwise specified, we assume that $|a| < 1$.

3. Second step

We define $t_n(k)$ to be the summand of the series in (2.3) and δ to be $\sqrt{n^2 - a^2}$ (for any fixed branch of the logarithm). We observe that $t_n(k)$ can be extended to a meromorphic function of the complex variable k:

$$t_n(k) = \frac{(-1)^n e^{i\pi k} n^2 \Gamma(1 \pm i\delta)(5k^2 - a^2)\Gamma(k+1)^2 \Gamma(k \pm n \pm i\delta)}{\Gamma(1 - n \pm i\delta)\Gamma(n \pm i\delta))\, k\, \Gamma(2k+1)\Gamma(k+1 \pm n)\Gamma(k+1 \pm i\delta)}, \tag{3.1}$$

where $\Gamma(x \pm y \pm z)$ is defined to be $\Gamma(x+y+z)\Gamma(x+y-z)\Gamma(x-y+z)\Gamma(x-y-z)$, etc.

We note that, as a result of the factor $\Gamma(k+1-n)$ in the denominator of (3.1), we have $t_n(k) = 0$ for $k = 1, \ldots, n-1$. Furthemore, simple computations give that $t_n(0) = a^2 n/(2n^2 - a^2)$ and, for $k \in \{1, \ldots, n\}$,

$$t_n(-k) = -\frac{n^3(n^2 - a^2)}{2n^2 - a^2} \binom{2k}{k} \frac{5k^2 - a^2}{(n^2 + (k-n)^2 - a^2)(n^2 + (k+n)^2 - a^2)} \cdot \prod_{j=1}^{k-1} \frac{(n^2 - j^2)(j^2 + n^2 - a^2)}{(n^2 + (j-n)^2 - a^2)(n^2 + (j+n)^2 - a^2)}. \tag{3.2}$$

We are now ready to prove our second step.

Proposition 1. *For any given $n \geq 1$, Equation (2.3) is equivalent to the following finite combinatorial identity:*

$$\sum_{k=1}^{n} \binom{2k}{k} \frac{5k^2 - a^2}{(n^2 + (k-n)^2 - a^2)(n^2 + (k+n)^2 - a^2)} \cdot \prod_{j=1}^{k-1} \frac{(n^2 - j^2)(j^2 + n^2 - a^2)}{(n^2 + (j-n)^2 - a^2)(n^2 + (j+n)^2 - a^2)} = \frac{2}{n^2 - a^2}. \tag{3.3}$$

Remark. Given any integer $n \geq 1$, if (3.3) is true for $|a| < 1$, it is true for any complex number a such that a^2 can not be written $a^2 = n^2 + m^2$ with an integer $m \in \{0, \pm 1, \ldots, \pm n\}$.

Proof. We will prove below that

$$\sum_{k=-n}^{+\infty} t_n(k) = 0. \tag{3.4}$$

Equation (3.4) can be written

$$\sum_{k=1}^{n} t_n(-k) = -t_n(0) - \sum_{k=1}^{n-1} t_n(k) - \sum_{k=n}^{\infty} t_n(k) = -\frac{a^2 n}{2n^2 - a^2} - \sum_{k=n}^{\infty} t_n(k)$$

and $\sum_{k=n}^{\infty} t_n(k) = n$ is clearly equivalent to $\sum_{k=1}^{n} t_n(-k) = -2n^3/(2n^2 - a^2)$, which, given (3.2), is exactly (3.3).

We now prove (3.4), and for that we closely follow Borwein & Bradley, whose method is based on Gosper's hypergeometric summation algorithm (see [7, p. 225–227] for details). We note that

$$\frac{t_n(k+1)}{t_n(k)} = -\frac{1}{2}\frac{5(k+1)^2 - a^2}{5k^2 - a^2}\frac{k}{2k+1}\frac{(k \pm n \pm i\delta)}{(k+1 \pm n)(k+1 \pm i\delta)} = \frac{p_n(k+1)q_n(k)}{p_n(k)r_n(k+1)},$$

is a rational function of k, with $q_n(k) = (k - n \pm i\delta)$, $r_n(k) = -2(2k-1)(k+n)$ and

$$p_n(k) = (5k^2 - a^2)\prod_{j=1}^{n-1}(k-j)(k+j \pm i\delta).$$

Since q_n and r_n do not have roots differing by integers[3], Gosper's algorithm ensures that there exists a polynomial s_n of degree at most $\deg(p_n) - \deg(q_n - r_n) = 3n - 3$ such that $p_n(k) = s_n(k+1)q_n(k) - r_n(k)s_n(k)$. We now define

$$T_n(k) = \frac{r_n(k)s_n(k)t_n(k)}{p_n(k)},$$

which satisfies $T_n(k+1) - T_n(k) = t_n(k)$. Since $t_n(-n)$ is finite and $p_n(-n) \neq 0 = r_n(-n)$, we have $T_n(-n) = 0$. Hence, for any $k \geq 1 - n$, $T_n(k) = \sum_{j=-n}^{k-1} t_n(k)$. Since $\deg(r_n s_n) = \deg(p_n)$, we have $T_n(k) = O(t_n(k))$ as $k \to +\infty$, which implies that $T_n(k)$ tends to 0 as $k \to +\infty$. It follows that (3.4) holds. □

4. Third step

Here, we generalise the last reduction step of [4] (due to Wenchang Chu).

Proposition 2. *Equation* (3.3) *for every integer* $n \geq 1$ *is equivalent to the following identity for every integer* $n \geq 1$*:*

$$\sum_{k=1}^{n} \frac{2}{k^2 - a^2} \frac{\prod_{j=1}^{n-1}(k^2 + (j-k)^2 - a^2)(k^2 + (j+k)^2 - a^2)}{\prod_{j=1,\, j\neq k}^{n}(k^2 - j^2)(k^2 + j^2 - a^2)} = \frac{1}{n^2 - a^2}\binom{2n}{n}. \tag{4.1}$$

Remark. The simplification (4.2) below shows that, given any integer $n \geq 1$, if (4.1) is true for $|a| < 1$, it is true for any complex number a such that $a \notin \{\pm 1, \ldots, \pm n\}$. Furthermore, it can also be written as

$$2\sum_{k=1}^{n} \frac{C_{k,n}(a)}{k^2 - a^2} = \frac{(-1)^{n+1}}{n^2 - a^2}\binom{2n}{n},$$

where $C_{k,n}(a)$ is defined in (2.2).

[3] Since $|a| < 1$ and $n \geq 1$, $i\delta$ can't be an integer.

Proof of Proposition 2. We use Krattenthaler's inversion formula [9]:

$$f(n) = \sum_{k=r}^{n} \frac{a_n d_n + b_n c_n}{d_k} \frac{\varphi(c_k/d_k; n)}{\psi_k(-c_k/d_k; n+1)}\, g(k) \quad \text{iff} \quad g(n) = \sum_{k=r}^{n} \frac{\psi(-c_n/d_n; k)}{\varphi(c_n/d_n; k+1)}\, f(k),$$

where

$$\varphi(x;k) = \prod_{j=0}^{k-1}(a_j + x b_j), \quad \psi(x;k) = \prod_{j=0}^{k-1}(c_j + x d_j) \quad \text{and} \quad \psi_m(x;k) = \prod_{\substack{j=0\\ j\neq m}}^{k-1}(c_j + x d_j).$$

Applied to (3.3), it yields the result with the choices $r = 1$, $a_j = (j^2 - a^2)^2$, $b_j = 4$, $c_j = j^4 - a^2 j^2$, $d_j = 1$,

$$f(k) = (-1)^k (5k^2 - a^2)\binom{2k}{k} \quad \text{and} \quad g(k) = \frac{2}{k^2 - a^2} \frac{4k^4 - 4a^2k^2 + (a^2-1)^2}{k^4 - a^2 k^2}.$$

□

Using the same trick as Almkvist & Granville, it is easy to write (4.1) in a more convenient form, that we will prove below: for any $n \geq 1$,

$$\sum_{k=1}^{n} (-1)^{n-k} \binom{2n}{n-k} \frac{4k^2}{k^2 - a^2} \prod_{\substack{0 \leq j < n-k \\ \text{or } n < j < n+k}} (k^2 + j^2 - a^2) = \frac{(2n)!}{n^2 - a^2}\binom{2n}{n}. \tag{4.2}$$

5. The final step

Note that (4.2) is simply Theorem 2 with $g(X) = X^2$ and $t = 1$: indeed, the first product in the left hand side of (1.5) corresponds exactly to the left hand side of (4.2) and (since only the n–th summand is non zero)

$$\sum_{k=1}^{n} (-1)^{n-k} \binom{2n}{n-k} \frac{4k^2}{k^2 - a^2} \prod_{\substack{0 \leq j < n-k \\ \text{or } n < j < n+k}} g(j)$$
$$= \frac{n^2}{n^2 - a^2} \prod_{n < j < 2n} j^2 = \frac{4n^2}{n^2 - a^2} \frac{(2n-1)!^2}{n!^2} = \frac{(2n)!}{n^2 - a^2}\binom{2n}{n}.$$

Hence Theorem 1 follows from Theorem 2.

Proof of Theorem 2. So far, we have been very lucky in that every step of [4] generalises without problems to this more general setting. But here, the general Theorem 1′ in [1] is apparently not strong enough to prove (4.2). Fortunately, we can adapt the method there used for our purpose. For any $k \geq 1$, we define the polynomial of degree $n - 1$

$$F_k(X) = \prod_{\substack{0 \leq j < n-k \\ \text{or } n < j < n+k}} (X - g(j)).$$

Proposition 1 in [1] establishes the existence of polynomials $Q_r(X)$ of degree $d_r \le r$ such that

$$F_k(X) - F_k(0) = \sum_{r=0}^{n-2} Q_r(k^2 - a^2)\, X^{n-1-r}. \tag{5.1}$$

The important point for us is the fact that since $F_k(X) - F_k(0)$ vanishes at $X = 0$, then the sum in (5.1) terminates at $n-2$. (In fact, $Q_r(X) = c_r(X + a^2)$ with the polynomials c_r given in [1].) We write $Q_r(X) = \sum_{i=0}^{d_r} q_{r,i}\, X^i$. Equation (1.5) can be expressed as

$$\begin{aligned}(-1)^{n-1} &\sum_{k=1}^{n} (-1)^{n-k} \binom{2n}{n-k} \frac{4k^2}{k^2 - a^2} \big(F_k(-t(k^2 - a^2)) - F_k(0)\big)\\ &= (-1)^{n-1} \sum_{r=0}^{n-2} \sum_{i=0}^{d_r} (-t)^{n-1-r} q_{r,i} \sum_{k=1}^{n} (-1)^{n-k} \binom{2n}{n-k} \frac{4k^2}{k^2 - a^2} (k^2 - a^2)^{i+n-1-r}. \end{aligned} \tag{5.2}$$

Since $i \ge 0$ et $r \le n-2$, we have

$$\frac{4k^2}{k^2 - a^2} (k^2 - a^2)^{i+n-1-r} = P(k^2),$$

where $P(X) = 4X(X - a^2)^{n+i-r-2}$ is a polynomial of degree $i+n-r-1 \le d_r+n-r-1 \le n-1$ such that $P(0) = 0$. Lemma 1 in [1], which reads

$$\sum_{k=1}^{n} (-1)^{n-k} \binom{2n}{n-k} k^{2\ell} = 0 \tag{5.3}$$

for any $1 \le \ell \le n-1$, then gives that

$$\sum_{k=1}^{n} (-1)^{n-k} \binom{2n}{n-k} \frac{4k^2}{k^2 - a^2} (k^2 - a^2)^{i+n-1-r} = \sum_{k=1}^{n} (-1)^{n-k} \binom{2n}{n-k} P(k^2) = 0.$$

This proves that the left hand side of (5.2) is 0 for all t and the proof of Theorem 2 is complete. $\square$

We conclude this section with the following remark. Almkvist & Granville proved (5.3) by expressing its left hand side as the 2ℓ-th Taylor coefficient of the function $e^{-nz}(e^z - 1)^{2n}$. Another proof is as follows: define $S(z) = z^\ell / z(z - 1^2) \cdots (z - n^2)$ for any integers $\ell \ge 0$ and $n \ge 0$. Then, by the residue theorem, for any closed direct contour Γ enclosing the poles of S, we have

$$-\mathrm{Res}_\infty(S) = \frac{1}{2i\pi} \int_\Gamma S(z)\, \mathrm{d}z = \sum_{k=0}^{n} \mathrm{Res}_{k^2}(S) = 2 \sum_{k=0}^{n} (-1)^{n-k} \frac{k^{2\ell}}{(n-k)!(n+k)!}.$$

If we assume that $\ell \le n-1$, then $\mathrm{Res}_\infty(S) = 0$ and if furthemore $\ell \ge 1$, then (5.3) follows after multiplication by $(2n)!/2$.

Acknowledgment

The main result (Theorem 1) of this paper was conjectured by Henri Cohen, and it was his encouragement that prompted me to work on this problem. I warmly thank him for this.

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