Arithmetic theory of E and G-operators

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E and G-functions

We fix an embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}$.

Definition 1

A G-function is a formal power series $G(z) = \sum_{n=0}^{\infty} a_n z^n$ such that $a_n \in \overline{\mathbb{Q}}$ and there exists $C > 0$ such that:

(i) *the maximum of the moduli of the conjugates of a_n is $\leq C^{n+1}$ for any n.*

(ii) *there exists a sequence of rational integers $d_n \neq 0$, with $|d_n| \leq C^{n+1}$, such that $d_n a_m$ is an algebraic integer for all $m \leq n$.*

(iii) *$G(z)$ satisfies a homogeneous linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$.*

An E-function $E(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$ is defined similarly.

Properties of E and G-functions

A G-function is not entire, unless it is a polynomial, but it is always holomorphic at $z = 0$. The set of G-functions is a ring (for the Cauchy product), stable by derivation and integration, it contains algebraic functions (over $\overline{\mathbb{Q}}(z)$) holomorphic at $z = 0$ and $\log(1 - z)$ for instance. Its group of units is formed by the algebraic functions holomorphic and non zero at $z = 0$ (André).

An E-function is an entire function. The set of E-functions is a ring (for the Cauchy product), stable by derivation and integration, it contains the exponential function and the Bessel functions for instance. Its units are of the form $\alpha \exp(\beta z)$, where $\alpha \in \overline{\mathbb{Q}}^*$ and $\beta \in \overline{\mathbb{Q}}$ (André).

The intersection of both classes is reduced to polynomial functions.

Three sets of numbers related to E and G-functions

Definition 2

(*i*) *The set* **E** *is the set of all the values taken at algebraic points by E-functions.*

It is a ring. Its group of units contains $\overline{\mathbb{Q}}^* \exp(\overline{\mathbb{Q}})$.

(*ii*) *The set* **G** *is the set of all the values taken at algebraic points by (analytic continuation of) G-functions.*

It is a ring. Its group of units contains $\overline{\mathbb{Q}}^*$ *and the Beta values* $B(\mathbb{Q}, \mathbb{Q})$.

(*iii*) *The set* **S** *is the module generated over* **G** *by all the values of derivatives of the Gamma function at rational points.*

It is also the module generated over $\mathbf{G}[\gamma]$ *by all the values of* Γ *at rational points, where* γ *is Euler's constant.*

It is a ring.

André-Chudnovski-Katz Theorem, G-operator

Given a G-function $G(z)$, consider the minimal linear differential equation $My = 0$ of order η and with coefficients in $\overline{\mathbb{Q}}[z]$, of which $G(z)$ is a solution. Let $\xi_1, \dots, \xi_p$ denote the singularities of the operator M at finite distance. Then,

• M is globally fuchsian, with rational exponents at each ξ_j and at ∞.

• In $\mathbb{C}$ minus (fixed) cuts with the ξ_j's for origin, M has a local basis of solutions $F_1(z), \dots, F_\eta(z)$ at $z = \xi \in \overline{\mathbb{Q}}$ such that

$$F_k(z) = \sum_{s \in S_k} \sum_{t \in T_k} \alpha_{s,t,k} \log(z-\xi)^s (z-\xi)^t G_{s,t,k}(z-\xi)$$

where $S_k \subset \mathbb{N}$ and $T_k \subset \mathbb{Q}$ are finite, $\alpha_{s,t,k} \in \overline{\mathbb{Q}}$, and if $\xi \neq \xi_k$, $S_k = T_k = \{0\}$.

$G_{s,t,k}(z)$ are holomorphic at $z = 0$; and they are G-functions.

• If $\xi = \infty$, the same result holds provided we replace $z - \xi$ by $1/z$ everywhere.

M is called a G-operator.

Connection constants for G-functions, Structure of $\mathbf{G}$

Let $G(z)$ be a G-function solution of the minimal differential equation $My(z) = 0$ of order η.

Locally around $\alpha \in \overline{\mathbb{Q}} \cup \{\infty\}$, we have

$$G(z) = \omega_1 F_1(z) + \cdots + \omega_\eta F_\eta(z).$$

where $F_1(z), \ldots, F_\eta(z)$ are given by the André-Chudnovski-Katz theorem, and $\omega_1, \ldots, \omega_\eta$ are certain complex numbers.

Theorem 1 (Fischler-R, 2012)

(i) *The connection constants $\omega_1, \ldots, \omega_\eta$ belong to $\mathbf{G}$.*

(ii) *A number ξ is in $\mathbf{G}$ if and only if $\xi = G(1)$, where G is a G-function with coefficients in $\mathbb{Q}(i)$, whose radius of convergence can be as large as a priori wished.*

Corollary 1

$\mathbf{G}$ *is a ring.*

Theorem 1(*ii*) for E-functions?

Given $\xi \in \mathbf{E}$, can we alway find an E-function $E(z)$ with coefficients in $\mathbb{Q}(i)$ such that $\xi = E(1)$?

No.

Theorem 2

An E-function with coefficients in a number field $\mathbb{K}$ takes at an algebraic point α either a transcendental value or a value in $\mathbb{K}(\alpha)$.

In particular, there is no E-function $E(z) \in \mathbb{Q}[[z]]$ such that $E(1) = \sqrt{2}$.

This theorem is due to the referee of our 2012 paper in the case $\mathbb{K} = \mathbb{Q}(i)$ and $\alpha = 1$, but his proof can be easily generalized. It is based on Beukers' refinement of the Siegel-Shidlovskii theorem.

Aparté

Let $Y(z) = {}^t(E_1(z), \ldots, E_n(z))$ be a vector of E-functions solution of a differential system $Y'(z) = M(z)Y(z)$ where $M(z) \in M_n(\overline{\mathbb{Q}}(z))$. Let $T(z)$ be the least common denominator of the entries of $M(z)$.

• **Siegel-Shidlovskii** (1929, 1956). For any $\alpha \in \overline{\mathbb{Q}}$ such that $\alpha T(\alpha) \neq 0$

$$\mathrm{degtr}_{\overline{\mathbb{Q}}(z)}(E_1(z), \ldots, E_n(z)) = \mathrm{degtr}_{\overline{\mathbb{Q}}}(E_1(\alpha), \ldots, E_n(\alpha)).$$

• **Nesterenko-Shidlovskii** (1996). There exists a finite set S such that for any $\alpha \in \overline{\mathbb{Q}}$, $\alpha \notin S$, the following holds. For any $P \in \overline{\mathbb{Q}}[X_1, \ldots, X_n]$ such that $P(E_1(\alpha), \ldots, E_n(\alpha)) = 0$, there exists $Q \in \overline{\mathbb{Q}}[Z, X_1, \ldots, X_n]$ such that $Q(\alpha, X_1, \ldots, X_n) = P(X_1, \ldots, X_n)$ and $Q(z, E_1(z), \ldots, E_n(z)) = 0$.

• **Beukers** (2006). We have $S \subset \{\alpha \in \overline{\mathbb{Q}} : \alpha T(\alpha) \neq 0\}$.

• The analogue of the Siegel-Shidlovskii theorem for G-functions is false in general (André, Beukers, $n = 2$). It is believed that the polynomial relations between values of G-functions are described by the "Period Conjecture" of Grothendieck, through the Bombieri-Dwork Conjecture (ie, G-functions come from geometry).

E-operators

Definition 3 (André, 2000)

A differential operator $L \in \overline{\mathbb{Q}}[x, \frac{d}{dx}]$ is an E-operator if the operator $M \in \overline{\mathbb{Q}}[z, \frac{d}{dz}]$ obtained from L by formally changing

$$x \to -\frac{d}{dz}, \qquad \frac{d}{dx} \to z \qquad \textbf{(Fourier-Laplace transform of } L\textbf{)}$$

is a G-operator, i.e. $My(z) = 0$ has at least one G-function solution for which it is minimal.

Motivation: Given an E-function $E(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$, there exists an E-operator L, of order μ say, such that $LE(x) = 0$. Moreover, let

$$g(z) = \int_0^{\infty} E(x) e^{-xz} dx = \sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}} \qquad \textbf{(Laplace transform of } E\textbf{)}.$$

Then

$$\left(\left(\frac{d}{dz}\right)^{\mu} \circ M\right) g(z) = 0.$$

Basis of solutions of L at $z = 0$

Theorem 3 (André, 2000)

(*i*) *An E-operator has at most 0 and ∞ as singularities: 0 is always a regular singularity, while ∞ is an irregular one in general.*

(*ii*) *An E-operator L of order μ has a basis of solutions at $z = 0$ of the form*

$$(E_1(z), \ldots, E_\mu(z)) \cdot z^M$$

where M is an upper triangular $\mu \times \mu$ matrix with coefficients in $\mathbb{Q}$ and the $E_j(z)$ are E-functions.

Any local solution $F(z)$ of $Ly(z) = 0$ at $z = 0$ is of the form

$$F(z) = \sum_{j=1}^{\mu} \Big(\sum_{s \in S_j} \sum_{k \in K_j} \phi_{j,s,k} z^s \log(z)^k \Big) E_j(z) \qquad (1)$$

where $S_j \subset \mathbb{Q}$, $K_j \subset \mathbb{N}$ are finite and $\phi_{j,s,k} \in \mathbb{C}$.

Interesting case for us: $\phi_{j,s,k} \in \overline{\mathbb{Q}}$.

Connection constants at finite distance

Let $F(z)$ be a local solution of $Ly(z) = 0$ at $z = 0$, of the form given in (1).

Any point $\alpha \in \overline{\mathbb{Q}} \setminus \{0\}$ is a regular point of L.

There exists a basis of local solutions $F_1(z), \ldots, F_\mu(z) \in \overline{\mathbb{Q}}[[z - \alpha]]$, holomorphic around $z = \alpha$, such that

$$F(z) = \omega_1 F_1(z) + \cdots + \omega_\mu F_\mu(z) \tag{2}$$

where $\omega_1, \ldots, \omega_\mu$ are connection constants.

Theorem 4 (F-R, 2014)

If $\phi_{j,s,k} \in \overline{\mathbb{Q}}$ in (1), then $\omega_1, \ldots, \omega_\mu$ belong to $\mathbf{E}[\log \alpha]$, and even to $\mathbf{E}$ if $F(z)$ is an E-function.

Proof: Differentiate $\mu - 1$ times (2) to construct a $\mu \times \mu$ linear system with the ω_j's as unknown. Solve it at $z = \alpha$ using the wronskian built on the F_j's (Cramer's rule). Use in particular the fact that, by André's result on singularities of E-operators, the wronskian $= cz^\rho e^{\beta z}$ with $c \in \overline{\mathbb{Q}}^*$, $\rho \in \mathbb{Q}$ and $\beta \in \overline{\mathbb{Q}}$.

Basis of solutions of L at $z = \infty$

The situation is more complicated because of divergent asymptotic series and of Stokes' phenomenon.

Let $\theta \in [0, 2\pi)$ not in some explicit finite set which contains the anti-Stokes directions. We have a generalized asymptotic expansion

$$E(z) \sim \sum_{\rho \in \Sigma} e^{\rho z} \sum_{\alpha \in S} z^{\alpha} \sum_{i \in T} \log(z)^i \sum_{n=0}^{\infty} \frac{c_{\theta,\rho,\alpha,i}(n)}{z^n} \qquad (3)$$

as $|z| \to \infty$ in a large angular sector bisected by $\{z : \arg(z) = \theta\}$.

The sets $\Sigma \subset \overline{\mathbb{Q}}$, $S \subset \mathbb{Q}$ and $T \subset \mathbb{N}$ are finite, and $c_{\theta,\rho,\alpha,i}(n) \in \mathbb{C}$.

We have found a new explicit construction of (3) by deforming the integral

$$E(x) = \frac{1}{2i\pi} \int_L g(z) e^{zx} dz \qquad (L \text{ “vertical”}).$$

The series $\sum_{n=0}^{\infty} c_{\theta,\rho,\alpha,i}(n)z^{-n}$ in (3) are divergent, but

$$\sum_{n=0}^{\infty} \frac{1}{n!} c_{\theta,\rho,\alpha,i}(n)z^{n}$$

are finite linear combinations of G-functions.

André (2000): Construction of a special basis $H_1(z), \ldots, H_\mu(z)$ of formal solutions at infinity of the E-operator L that annihilates $E(z)$. Each H_k involves series like in (3) but with coefficients in $c_k\overline{\mathbb{Q}}$ for some c_k.

The asymptotic expansion (3) of $E(z)$ in a large sector bisected by $\{z : \arg(z) = \theta\}$ can be rewritten with this basis as

$$\omega_{\theta,1}H_1(z) + \cdots + \omega_{\theta,\mu}H_\mu(z) \tag{4}$$

with **Stokes' constants** $\omega_{\theta,k}$.

When θ "crosses" one of the anti-Stokes directions, the values of the $\omega_{\theta,k}$ may change . This is the Stokes phenomenon.

Stokes' constants at infinity

Setting:

$$E(z) \sim \omega_{\theta,1} H_1(z) + \cdots + \omega_{\theta,\mu} H_\mu(z)$$
$$\sim \sum_{\rho\in\Sigma} e^{\rho z} \sum_{\alpha\in S} z^\alpha \sum_{i\in T} \log(z)^i \sum_{n=0}^{\infty} \frac{c_{\theta,\rho,\alpha,i}(n)}{z^n}.$$

$\mathbf{S}$ is the module generated over $\mathbf{G}[\gamma]$ by all the values of Γ at rational points.

Theorem 5 (F-R, 2014)

Let $\theta \in [0, 2\pi)$ be a direction not in some explicit finite set. Then:

(i) *The Stokes constants $\omega_{\theta,k}$ belong to $\mathbf{S}$.*

(ii) *All the coefficients $c_{\theta,\rho,\alpha,i}(n)$ belong to $\mathbf{S}$.*

(iii) *Let $F(z)$ be a local solution at $z = 0$ of L, with $\phi_{j,s,k} \in \overline{\mathbb{Q}}$ in (1). Then Assertions (i) and (ii) hold with $F(z)$ instead of $E(z)$.*

G-approximations

Definition 4

Sequences (P_n) and (Q_n) of algebraic numbers are said to form G-approximations of $\alpha \in \mathbb{C}$ if

$$\lim_{n \to +\infty} \frac{P_n}{Q_n} = \alpha$$

and the generating functions $\sum_{n=0}^{\infty} P_n z^n$ and $\sum_{n=0}^{\infty} Q_n z^n$ are both G-functions.

Diophantine motivation: Many sequences of algebraic approximations of classical numbers are G-approximations. For instance, Apéry's approximations to $\zeta(2)$ and $\zeta(3)$.

Theorem 6 (F-R, 2012)

The set of numbers having G-approximations is $\mathrm{Frac}\,\mathbf{G}$.

Proof: We first show that a number α having G-approximations is the quotient of two connection constants of the G-operators related to the generating functions, and then we use Theorem 1(i).

E-approximations

Definition 5

Sequences (P_n) and (Q_n) of algebraic numbers are said to form E-approximations of $\alpha \in \mathbb{C}$ if

$$\lim_{n\to+\infty} \frac{P_n}{Q_n} = \alpha$$

and

$$\sum_{n=0}^{\infty} P_n z^n = a(z) \cdot E\big(b(z)\big), \quad \sum_{n=0}^{\infty} Q_n z^n = c(z) \cdot F\big(d(z)\big)$$

where E and F are E-functions, and a, b, c, d are algebraic functions in $\overline{\mathbb{Q}}[[z]]$ with $b(0) = d(0) = 0$.

Diophantine motivation: Many sequences of algebraic approximations of classical numbers are E-approximations. For instance diagonal Padé approximants to $\exp(z)$ evaluated at z algebraic, and in particular the convergents to e.

The set of E-approximable numbers

Given two subsets X and Y of $\mathbb{C}$, let

$$X \cdot Y = \left\{ xy \,\middle|\, x \in X, y \in Y \right\}, \quad \frac{X}{Y} = \left\{ \frac{x}{y} \,\middle|\, x \in X, y \in Y \setminus \{0\} \right\}.$$

Theorem 7 (F-R, 2014)

The set of numbers having E-approximations contains

$$\frac{\mathbf{E} \cup \Gamma(\mathbb{Q})}{\mathbf{E} \cup \Gamma(\mathbb{Q})} \cup \mathrm{Frac}\,\mathbf{G} \tag{5}$$

and it is contained in

$$\frac{\mathbf{E} \cup (\Gamma(\mathbb{Q}) \cdot \mathbf{G})}{\mathbf{E} \cup (\Gamma(\mathbb{Q}) \cdot \mathbf{G})} \cup \Big(\Gamma(\mathbb{Q}) \cdot \exp(\overline{\mathbb{Q}}) \cdot \mathrm{Frac}\,\mathbf{G} \Big). \tag{6}$$

Proof of (5): Explicit constructions.

Proof of (6): Saddle point method, singularity analysis, and Theorems 4 and 5 because E-approximable numbers appear either as connection constants or as Stokes' constants.

E-approximations of Gamma values

Let

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!(n+\alpha)}, \quad \alpha \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$$

and define $P_n(\alpha)$ by

$$\frac{1}{(1-z)^{\alpha+1}} E_\alpha\left(-\frac{z}{1-z}\right) = \sum_{n=0}^{\infty} P_n(\alpha) z^n \in \mathbb{Q}[[z]].$$

Then,

$$P_n(\alpha) = \sum_{k=0}^{n} \binom{n+\alpha}{k+\alpha} \frac{(-1)^k}{k!(k+\alpha)} \longrightarrow \Gamma(\alpha) \qquad \text{if } \alpha < 1.$$

The number $\Gamma(\alpha)$ appears as a Stokes constant in the expansion

$$E_\alpha(-z) \sim \frac{\Gamma(\alpha)}{z^\alpha} - e^{-z} \sum_{n=0}^{\infty} (-1)^n \frac{(1-\alpha)_n}{z^{n+1}}.$$

What about Euler's constant $\gamma = -\Gamma'(1)$?

We conjecture that γ does not have E-approximations, nor G-approximations. However, let

$$E(z) = \sum_{n=1}^{\infty} \frac{z^n}{n!n}$$

and define the sequence (P_n) by

$$-\frac{1}{1-z} E\left(-\frac{z}{1-z}\right) + \frac{\log(1-z)}{1-z} = \sum_{n=0}^{\infty} P_n z^n \in \mathbb{Q}[[z]].$$

Then

$$P_n = \sum_{k=1}^{n} (-1)^k \binom{n}{k} \frac{1}{k}\left(1 - \frac{1}{k!}\right) \longrightarrow \gamma.$$

Again, γ appears as a Stokes' constant in the asymptotic expansion

$$E(-z) \sim -\gamma - \log(z) - e^{-z} \sum_{n=0}^{\infty} (-1)^n \frac{n!}{z^{n+1}}.$$

Linear recurrences related to $\Gamma(\alpha)$ and γ

$$\begin{aligned}(n+3)(n+3+\alpha)P_{n+3}(\alpha) \\ - (3n^2+4n\alpha+14n+\alpha^2+9\alpha+17)P_{n+2}(\alpha) \\ + (3n+5+2\alpha)(n+2+\alpha)P_{n+1}(\alpha) \\ - (n+2+\alpha)(n+1+\alpha)P_n(\alpha) = 0\end{aligned}$$

with $P_0(\alpha) = \frac{1}{\alpha}$, $P_1(\alpha) = \frac{1+\alpha+\alpha^2}{\alpha(\alpha+1)}$ and $P_2(\alpha) = \frac{4+5\alpha+6\alpha^2+4\alpha^3+\alpha^4}{2\alpha(\alpha+1)(\alpha+2)}$.

$$\begin{aligned}(n+3)^2 P_{n+3} - (3n^2+14n+17)P_{n+2} \\ + (3n+5)(n+2)P_{n+1} - (n+2)(n+1)P_n = 0\end{aligned}$$

with $P_0 = 0$, $P_1 = 0$ and $P_2 = \frac{1}{4}$.